Quantum tunneling in dissipative media: Intermediate-coupling-strength results

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We consider the motion of a quantum particle placed in a metastable minima of a potential well and examine the effect of interactions with the thermal reservoir at temperature T. We use the dilute, bounce approximation for the Green's function. We find a particular value of the dissipative coupling strength for which we can solve the Euler-Lagrange equations exactly, in the range of temperatures between zero and the crossover to the regime where thermal hopping dominates. We obtain analytic expressions for the tunneling rate in this temperature regime.

I. INTRODUCTION

There has been a recent resurgence of interest in the effects that a thermodynamic heat bath has on the motion of a quantum-mechanical particle.[1-8] The effects of the coupling to the heat bath are most dramatic on particles which undergo quantum-mechanical tunneling. This phenomenon has long been recognized as playing an important role in many areas of physics.[9-12] It has been found that the coupling to the heat bath can exponentially suppress the rate at which particles tunnel out of a metastable state. Most of the recent investigations have utilized Feynman's functional-integral formulation of quantum mechanics,[13,14] as applied by Langer[15] and Coleman[16] to the decay of metastable states. Caldeira and Leggett[1,2] have extended this approach to tunneling in the presence of a thermodynamic heat bath. After integrating out the normal modes of the heat bath, the motion of the quantum particle is governed by an effective Lagrangian with a nonlocal term. They then apply the "bounce method" to calculate the tunneling rate, at zero temperature. The bounce trajectories are saddle points of the action which start in the local mininum of the potential at time $t=-T/2$ and return there at a later time $t=T/2$. The particle's Green's function can then be written as the sum of these trajectories together with small quantum fluctuations around these trajectories. Caldeira and Leggett[1,2] have shown that the multibounce interactions are small at low temperatures, which enabled them to derive a decay rate of the form

$$\Gamma=A\exp(-S_B/\hbar)\ , \qquad (1.1)$$

where S_B is the action evaluated along a single bounce trajectory. The prefactor contains the effects of quantum fluctuations about the bounce trajectory. Caldeira and Leggett[1,2] have obtained the bounce action S_B only in the limit of weak damping ($\eta\rightarrow 0$) and asymptotically strong damping ($\eta\rightarrow\infty$). This is possible since one can obtain exact analytic solutions of the Euler-Lagrange equations in these limits. Thus far, there have been no exact analytic solution for the bounce trajectories at intermediate damping strengths. In this note, we present an analytic solution for the bounce for a particular value of the heat-bath coupling η. We calculate the tunneling rate (1.1) at the temperature T_0, where the rate crosses over from thermal hopping to quantum tunneling.[7-9] We shall also present the zero-temperature rate. In Sec. II we outline the model problem and the formalism we shall utilize. We follow the method of Caldeira and Leggett very closely. In Sec. III we calculate the bounce trajectory. Due to the separation of the tunneling rate (1.1) into S_B and A, we also calculate S_B in Sec. III. The effect of quantum fluctuations about the bounce trajectory and their effect on the prefactor A is relegated to Sec. IV. In Sec. V we summarize our findings.

II. FORMULATION

The system under investigation consists of a quantum-mechanical particle which moves in a one-dimensional potential and is coupled to the normal modes of the heat bath. The particle is described by the variable q, has mass M, and moves in the potential $V(q)$. The particle is coupled to the heat bath through a bilinear interaction. The heat bath has degrees of freedom described by the set ϕ_n, $N\geq n\geq 1$. This set of variables describes the normal modes of the heat bath, which we also assume are harmonic oscillators. This model system was first introduced by Ullersma[17] and is exactly soluble for a quadratic potential $V(q)$. The properties of this type of heat bath are quite well known.[18,19]

Thus the system studied by us is governed by the Lagrangian

$$L=\frac{M}{2}\dot{q}^2-V(q)+\sum_{n=1}^{N}\left[\frac{m_n}{2}\dot{\phi}_n^2-\frac{m_n}{2}\omega_n^2\phi_n^2\right]-q\sum_{n=1}^{N}\lambda_n\phi_n-\tfrac{1}{2}q^2\sum_{n=1}^{N}\frac{\lambda_n^2}{m_n\omega_n^2}\ . \qquad (2.1)$$

The last term in (2.1), the counterterm, can be incorporated into the potential $V(q)$. The merits and demerits of this particular choice of definition for $V(q)$ have been discussed extensively by Caldeira and Leggett[1,2] as well as by Widom and Clark.[20] In order to maintain close correspondence with the work of Caldeira and Leggett[1,2] we shall use a cubic expression for the potential, sketched in Fig. 1,

$$V(q)=M\left[\frac{\omega_0^2}{2}q^2-\frac{u}{3}q^3\right] . \tag{2.2}$$

All the properties of the coupling of the heat bath to the particle is contained in the spectral density $J(\omega)$ given by

$$J(\omega)=\frac{\pi}{2}\sum_{n=1}^{N}\frac{\lambda_n^2}{m_n\omega_n}\delta(\omega-\omega_n) . \tag{2.3}$$

The normal modes of the heat bath can be integrated over in the functional-integral expression for the particle's Green's function.[21] The Green's function is given in terms of an effective action

$$S[q(t)]=\int_{-T/2}^{T/2}dt\left[\frac{M}{2}\dot{q}^2-V(q)\right]+\frac{1}{2}\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt'\,K(t-t')[q(t)-q(t')]^2 , \tag{2.4}$$

where $q(t)$ is the particles trajectory and the function $K(t-t')$ represents the coupling to the environment. This function is given in terms of $J(\omega)$ by

$$K(t-t')=\int\frac{d\omega}{2\pi}J(\omega)\{[1+N(\omega)]e^{i\omega(t-t')}+N(\omega)^{-i\omega(t-t')}\} \tag{2.5}$$

in which $N(\omega)$ is the Bose-Einstein distribution function. Here, and throughout the text, we shall set $\hbar=1$.

The Lagrangian can be simplified by analytically continuing T to imaginary times $i\theta$. Since the resulting $K(\tau)$ is periodic with period $\theta=\beta$, one can continue the imaginary time τ outside the range $\theta/2>\tau>-\theta/2$ if one uses the periodic boundary conditions

$$q(\theta+\tau)=q(\tau) .$$

This allows one to write

$$S[q(\tau)]=\int_{-\theta/2}^{\theta/2}d\tau\left[\frac{M}{2}\dot{q}^2+V(q)\right]+\frac{1}{2}\int_{-\theta/2}^{\theta/2}d\tau\int_{-\infty}^{\infty}d\tau'\,k(\tau-\tau')\times[q(\tau)-q(\tau')]^2 ,$$

where

$$k(\tau)=\int_0^\infty\frac{d\omega}{2\pi}J(\omega)\exp(-\omega|\tau|) . \tag{2.6}$$

In the following text we shall use θ as the inverse temperature, and β will be reserved for a different parametrization.

As Caldeira and Leggett[1,2] have shown, the extremal path which satisfies the equation of motion

$$M\ddot{q}=\frac{\partial V}{\partial q}+2\int_{-\infty}^{\infty}d\tau'\,k(\tau-\tau')[q(\tau)-q(\tau')] \tag{2.7}$$

exhibits a similarity to the classically damped particle if

$$J(\omega)=M\eta\omega .$$

With this choice Eq. (2.6) yields

$$k(\tau)=\frac{M\eta}{2\pi}\frac{1}{\tau^2} . \tag{2.8}$$

The decay rate is given by the imaginary part of the space diagonal Green's function. In the dilute bounce gas approximation[15,16] the Green's function is given by the extremal paths which start from the metastable minimum near $q=0$, and return there at a later time, together with the quantum fluctuations about these trajectories. In the dilute-gas approximation these trajectories give an exponential contribution to the Green's function, which only depends on a single bounce trajectory. In the next section we shall evaluate the bounce trajectory.

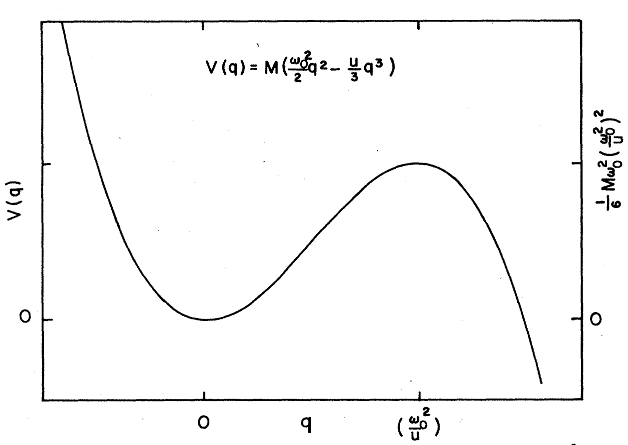

FIG. 1. The metastable potential well $V(q)=M[(\omega_0^2/2)q^2-(u/3)q^3]$. The top of the potential barrier is located at $q=\omega_0^2/u$.

III. EXACT SOLUTION FOR THE BOUNCE

We shall present an exact solution for the Euler-Lagrange equation for a bounce trajectory

$$-\frac{\partial^2 q}{\partial\tau^2}+\omega_0^2q-uq^2=\frac{\eta}{2\pi}\int_{-\infty}^{\infty}d\tau'\frac{\partial q}{\partial\tau'}\left[\frac{1}{\tau'-\tau+i\xi}+\frac{1}{\tau'-\tau-i\xi}\right] \tag{3.1a}$$

that satisfies the finite temperature periodic boundary conditions

$$q(\tfrac{1}{2}\theta)=q(-\tfrac{1}{2}\theta) . \tag{3.1b}$$

To date, the only known solutions of this equation are for the limiting case $\eta=0$ and $\eta\to\infty$. In the nondissipative case, the bounce has the form

$$q(\tau)=\frac{3}{2}\left[\frac{\omega_0^2}{u}\right]\operatorname{sech}^2\left[\frac{\omega_0\tau}{2}\right],\quad \eta=0,\ \theta\to\infty\ .$$

The other analytic solution corresponds to asymptotically large damping $\eta\to\infty$,

$$q(\tau)=\frac{\omega_0^2}{u}\frac{2\pi}{\theta\omega_0}\frac{\eta}{\omega_0}\frac{\sinh(\beta/2)\cosh(\beta/2)}{[\sinh^2(\beta/2)+\sin^2(\pi\tau/\theta)]},\quad \eta\to\infty$$

where the bounce-width parameter $\sinh^2(\beta/2)$ is found from

$$\coth\beta=\frac{\omega_0}{\eta}\frac{\omega_0\theta}{2\pi}\ .$$

For low temperatures $\theta\to\infty$, the bounce takes the form

$$q(\tau)=6\frac{\omega_0^2}{u}\left[\frac{2\pi}{\theta\omega_0}\right]^2\left[\frac{\eta\theta}{10\pi}\frac{\sinh(\beta/2)\cosh(\beta/2)}{\sinh^2(\beta/2)+\sin^2(\pi\tau/\theta)}+\frac{1}{2}\frac{\sinh^2(\beta/2)-\sin^2(\pi\tau/\theta)\cosh\beta}{[\sinh^2(\beta/2)+\sin^2(\pi\tau/\theta)]^2}\right] \tag{3.2}$$

is a solution of the bounce equation if the bounce-width parameter $\sinh^2\beta$ satisfies the simultaneous equations

$$\left[\frac{\eta\theta}{10\pi}\right]^2+6\left[\frac{\eta\theta}{10\pi}\right]\coth\beta+3\operatorname{csch}^2\beta-1-\left[\frac{\omega_0\theta}{2\pi}\right]^2=0\ ,$$

$$\left[\frac{\eta\theta}{10\pi}\right]^2\coth\beta+\left[\frac{\eta\theta}{10\pi}\right]\left[\operatorname{csch}^2\beta-\frac{1}{6}\left[\frac{\omega_0\theta}{2\pi}\right]^2\right]+\tfrac{1}{2}\coth\beta\operatorname{csch}^2\beta=0\ . \tag{3.3}$$

These coupled equations possess a solution over the entire temperature range $T_0>T>0$ in which the quantum tunneling dominates the dynamics. However, the solution only exists for a particular value of the coupling strength (η/ω_0). The crossover temperature T_0 is defined by

$$\left[\frac{\eta\theta}{10\pi}\right]=1$$

and in complete analogy with the overdamped limit the bounce trajectory becomes a constant.

The constant value, again, just corresponds to the position of the top of the potential barrier. At the crossover temperature, the coupling strength is given by

$$\frac{\eta}{\omega_0}=\frac{5}{\sqrt{6}}\ .$$

As the temperature is lowered the coupled equations still possess solutions. In Fig. 2 we show typical graphical solutions of the equations (3.3) for a sequence of decreasing temperatures. The corresponding bounce trajectories are shown in Fig. 3. As the temperature is lowered β rapidly decreases. The bounce trajectory no longer remains a constant but sharpens up. The bounce trajectories start closer to the minimum of the potential well and extend further than the maximum of the potential barrier.

of a Lorentzian of width $\tau_B=\eta/\omega_0^2$. At the crossover from thermally activation to quantum tunneling[7,8] T_0 is given by

$$\frac{\omega_0\theta_0}{2\pi}=\frac{\eta}{\omega_0},\quad \eta\to\infty\ .$$

At T_0, the bounce trajectory collapses to a constant $q(\tau)=\omega_0^2/u$ which corresponds to the top of the potential barrier. We note that the region where quantum tunneling is important becomes asymptotic small as $\eta\to\infty$. Clearly, it would be more advantageous to have an analytic solution which corresponds to a finite portion of phase space in which the particles decay predominantly by quantum-mechanical tunneling.

We find that

At zero temperature the bounce width $\tau_B=(\beta\theta/2\pi)$ is given by the unique solution of the equation

$$-\tfrac{1}{36}(\omega_0\tau_B)^6+\tfrac{5}{12}(\omega_0\tau_B)^4+\tfrac{5}{2}(\omega_0\tau_B)^2+\tfrac{25}{4}=0,\quad T=0\ .$$

The corresponding value of the coupling strength is

$$\frac{\eta}{\omega_0}=\frac{5}{\omega_0\tau_B}\left[\frac{(\omega_0\tau_B)^2-\frac{5}{2}}{\frac{(\omega_0\tau_B)^2}{6}+5}\right],\quad T=0\ .$$

This value is only about 10% different from the coupling strength at T_0. The bounce trajectory starts out at the local minimum of the potential well $[q(-\theta/2)=0]$ and has its maximum excursion from the minimum given by

$$q(0)=\frac{\omega_0^2}{u}\frac{12}{(\omega_0\tau_B)^2}\left[1+\frac{\eta}{5\omega_0}(\omega_0\tau_B)\right],\quad T\to 0\ .$$

This bounce point is intermediate between the zero damping and the infinite damping value

$$\tfrac{3}{2}\frac{\omega_0^2}{u}<q(0)<2\frac{\omega_0^2}{u},\quad T\to 0\ .$$

This has the consequence that the energy loss over the instanton trajectory is less than that associated with the strong damping limit discussed in Ref. 6, as is to be expected. In Fig. 4 we depict the temperature dependence of the extrema of the bounce trajectories. We note that at finite temperatures, the extrema of the bounce trajectory cannot, simply, be related to the energy loss during the tunneling process.

We shall utilize these bounce trajectories to calculate the decay rates. In order to calculate this we require the action corresponding to these trajectories. The action is evaluated as

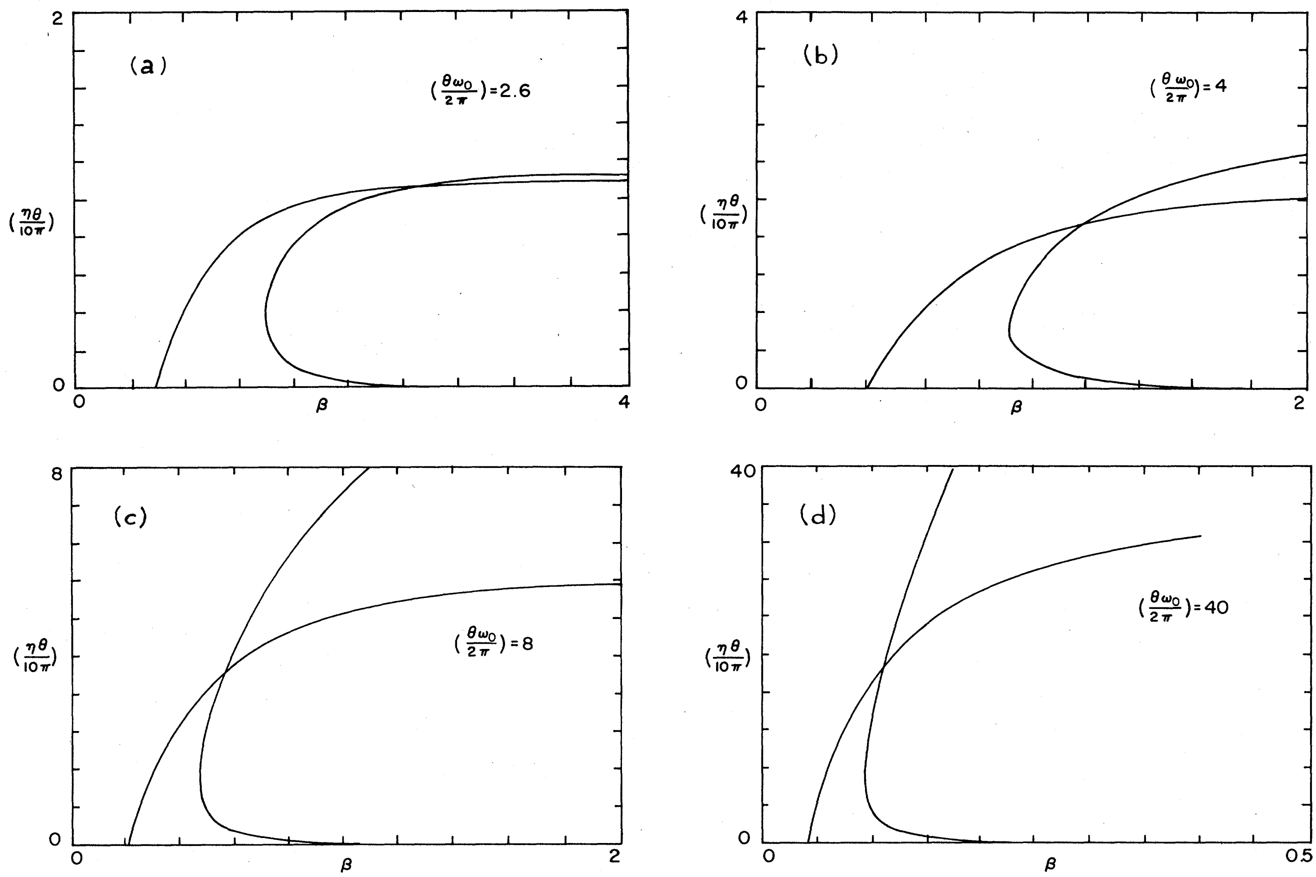

FIG. 2. The graphical solutions of the two coupled quadratic equations (3.3) for a sequence of decreasing temperatures. The intersection of the two curves defines $(\eta\theta/10\pi)$ and β.

$$S_B = 2\pi M \left[\frac{\omega_0^2}{u}\right]^2 \frac{\eta}{5}\left[1+15\left[\frac{\eta}{5\omega_0}\right]^2\left[\frac{2\pi}{\omega_0\theta}\right]^2 \operatorname{csch}^2\beta + 33\left[\frac{\eta}{5\omega_0}\right]^1\left[\frac{2\pi}{\omega_0\theta}\right]^3 \coth\beta\operatorname{csch}^2\beta \right.$$
$$+\frac{21}{2}\left[\frac{\eta}{5\omega_0}\right]^0\left[\frac{2\pi}{\omega_0\theta}\right]^4 \operatorname{csch}^2\beta(3\operatorname{csch}^2\beta+2)$$
$$\left. +3\left[\frac{\eta}{5\omega_0}\right]^{-1}\left[\frac{2\pi}{\omega_0\theta}\right]^5 \coth\beta\operatorname{csch}^2\beta(3\operatorname{csch}^2\beta+1)\right] . \tag{3.4}$$

The bounce action takes the value

$$S_B(T_0) = 2\pi M\left[\frac{\omega_0^2}{u}\right]^2 \frac{\eta}{5}, \quad T=T_0 ,$$

at the crossover temperature, and it increases at lower temperatures to the zero-temperature value

$$S_B(0) = 2\pi M\left[\frac{\omega_0^2}{u}\right]^2 \frac{\eta}{5}(1.42), \quad T\to 0 .$$

The zero-temperature value is intermediate between the undamped and overdamped values of the bounce action. We also note the temperature variation of the bounce action suggests that the decay rate will be lowest at zero temperature, as has been previously suggested.[22,23] The total decay rate is, of course, given by the product of the prefactor as well as the exponential term. We shall calculate the prefactor in the next section.

IV. QUANTUM FLUCTUATIONS

The decay rate, evaluated in the bounce approximation, is given by the ratio of the contributions to the Green's function from the paths close to the bounce trajectory relative to the paths which always remain close to the metastable minima. Since these contributions are dominated by the extrema of the action, the decay rate is dominated by the exponential of the bounce action relative to the action of the path $q(\tau)=0$. The prefactor can be written as the ratio of the small fluctuations about these extremal paths. Thus the decay rate may be written as

$$\Gamma = [2(\mathrm{Im} Z_B)/(Z_0\theta)]\exp(-S_B) . \tag{4.1}$$

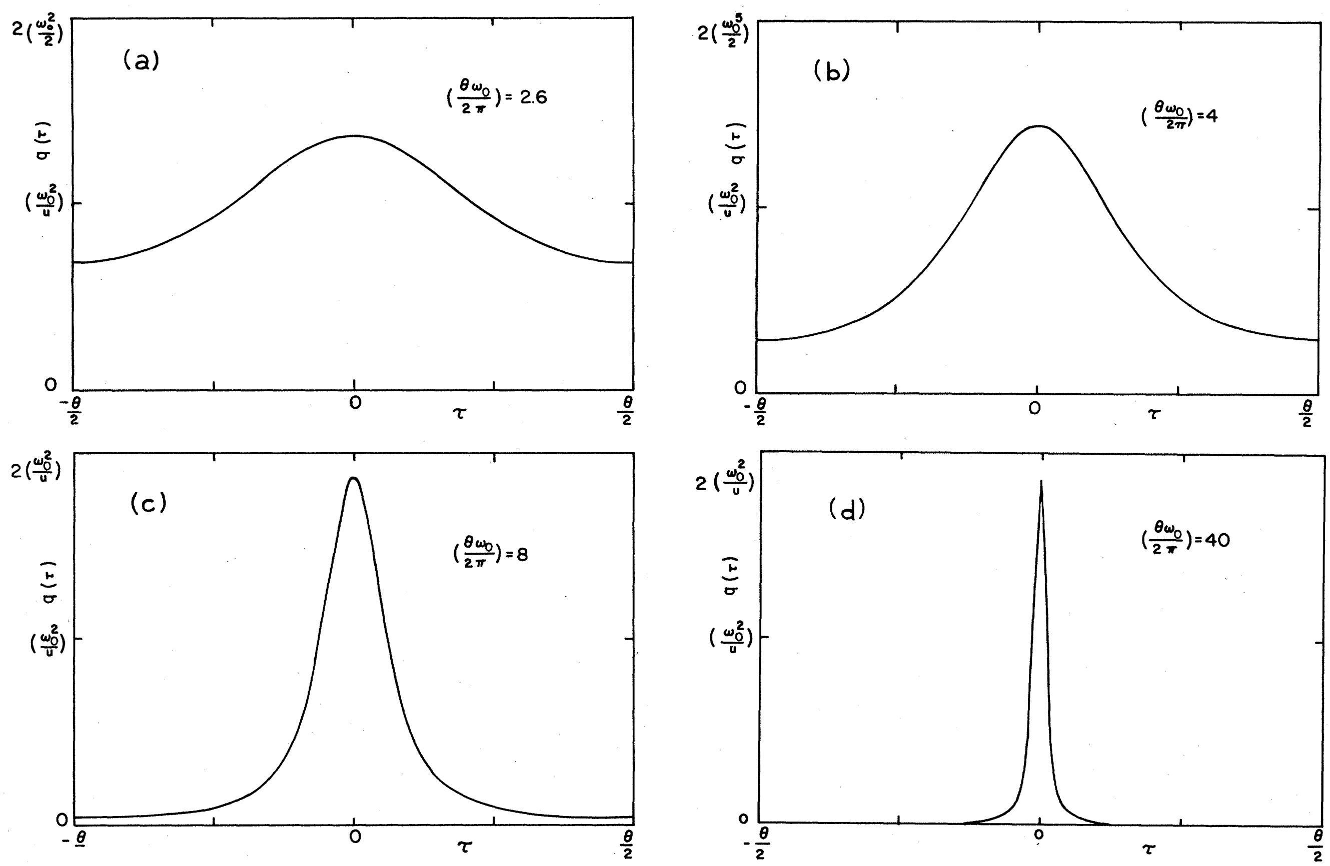

FIG. 3. The bounce trajectories corresponding to the same sequence of decreasing temperatures depicted in Fig. 2.

The factor Z_0 can be approximated by quadratic fluctuations about $q=0$. This factor is most easily evaluated if one expresses the paths as

$$q(\tau)=0+\sum_{n=-\infty}^{\infty} C_n q_n(\tau)\,, \tag{4.2}$$

where the $q_n(\tau)$ are eigenfunctions of the second functional derivative of the action

$$-\frac{\partial^2 q_n}{\partial\tau^2}+\omega_0^2 q_n-\frac{\eta}{2\pi}\int_{-\infty}^{\infty} d\tau' \frac{\partial q_n}{\partial\tau'} \times\left[\frac{1}{\tau'-\tau+i\zeta}+\frac{1}{\tau'-\tau-i\zeta}\right] =\frac{\Lambda_n^0}{M}q_n\,, \tag{4.3}$$

where $q_n(\frac{1}{2}\theta)=q_n(-\frac{1}{2}\theta)$.

We note that there are, currently, two different forms of boundary conditions being used in the literature. The other form is that $q_n(\frac{1}{2}\theta)=q_n(-\frac{1}{2}\theta)=0$. We choose periodic boundary conditions since in that case the calculated rate smoothly continues onto the classical rate at the crossover temperature T_0, as discussed in the Appendix. The other choice would lead to a breakdown of the instanton method before T_0 is reached.

Due to the orthonormality properties of the eigenfunction, we may write

$$Z_0=\prod_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dC_n}{(2\pi)^{1/2}}\exp\left[-\frac{\Lambda_n^0}{2}C_n^2\right] =\left[\prod_{n=-\infty}^{\infty}\Lambda_n^0\right]^{-1/2}. \tag{4.4}$$

This is simply related to the partition function of the damped harmonic oscillator, and can be written as

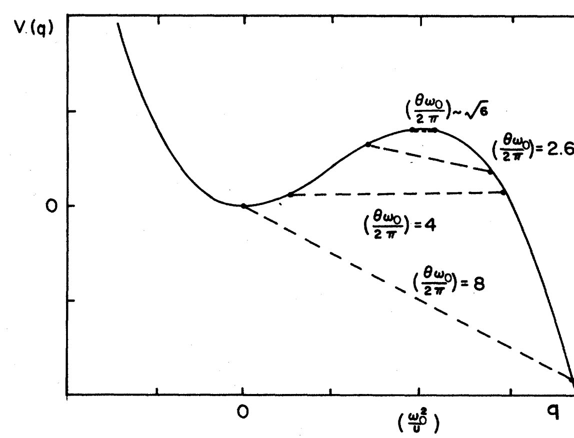

FIG. 4. The variation of the extrema of the bounce trajectories as the temperature is decreased. At T_0, the crossover temperature, the bounce trajectory remains at the top of the potential barrier. As T is decreased the bounce exists over a finite spatial region. The spatial extent saturates at $T=0$.

$$Z_0=\frac{1}{(M\omega_0^2)^{1/2}}\prod_{n=1}^{\infty}\left[\frac{\frac{1}{M}\left[\frac{\theta}{2\pi}\right]^2}{n^2+n\left[\frac{\eta\theta}{2\pi}\right]+\left[\frac{\omega_0\theta}{2\pi}\right]^2}\right],$$

$$Z_0=\frac{1}{(M\omega_0^2)^{1/2}}\prod_{n=1}^{\infty}\left[\frac{\frac{1}{M}\left[\frac{\theta}{2\pi}\right]^2\exp\left[-\left[\frac{\eta\theta}{2\pi}\right]/n\right]}{n^2}\right]\exp\left[\left[\frac{\eta\theta}{2\pi}\right]\gamma\right]$$
$$\times\left|\Gamma\left\{1+\left[\frac{\eta\theta}{4\pi}\right]+i\left[\left[\frac{\omega_0\theta}{2\pi}\right]^2-\left[\frac{\eta\theta}{4\pi}\right]^2\right]^{1/2}\right\}\right|^2 \quad (4.5)$$

in which γ is Euler's constant.

The calculation of Z_B would follow much the same lines, if it were not for the fact that the second functional derivative of the action possesses zero eigenvalues

$$-\frac{\partial^2 q}{\partial\tau^2}n+\omega_0^2 q_n-2uq_B(\tau)q_n-\frac{\eta}{2\pi}\int_{-\infty}^{\infty}d\tau'\frac{\partial q}{\partial\tau'}\left[\frac{1}{\tau'-\tau+i\zeta}+\frac{1}{\tau'-\tau-i\zeta}\right]=\frac{\Lambda_n}{M}q_n\,, \quad (4.6)$$

where $q_n(\frac{1}{2}\theta)=q_n(-\frac{1}{2}\theta)$.

The zero eigenvalue, which is always present, is a Goldstone-like mode that restores the continuous-time translational invariance broken by the choice of the bounce phase at $\tau=0$ to be zero. The eigenfunction corresponds to phase functions about the bounce, and is thus proportional to $\dot{q}_B(\tau)$. The method of handling such zero eigenvalues is well known[24] and we shall closely follow the logic of Larkin and Ovchinnikov, which we shall repeat below.

The zero eigenvalue can be replaced by the normalization of the corresponding eigenfunction by the following trick. First, we introduce a functional that describes the fluctuations about the bounce

$$D(\tau'\,|\,q(\tau))=\int_{-\theta/2}^{\theta/2}d\tau[q(\tau)-q_B(\tau-\tau')]^2 \quad (4.7)$$

and define a time τ_0 through the relation

$$\left.\frac{\partial D}{\partial\tau'}\right|_{\tau'=\tau_0}=0\,. \quad (4.8)$$

The factor Z_B can be simply written as

$$Z_B=\exp(S_B)\int_{-\theta/2}^{\theta/2}d\tau'\int Dq(\tau)\exp\{-S[q,(\tau)]\}\times\delta(\tau'-\tau_0[q(\tau)])\,. \quad (4.9)$$

The zeros of the Dirac delta function are defined through (4.8). A simple substitution yields

$$Z_B=\exp(S_B)\int_{-\theta/2}^{\theta/2}d\tau'\int Dq(\tau)\exp(-S[q(\tau)])\times\delta\left[\frac{\partial D}{\partial\tau'}\right]\left|\frac{\partial^2 D}{\partial(\tau')^2}\right|. \quad (4.10)$$

We shall use the expansion of $q(\tau)$ about the bounce in terms of the eigenfunctions of (4.6)

$$q(\tau)=q_B(\tau)+\sum_n C_n q_n(\tau)\,.$$

As $\dot{q}_B(\tau)$ is proportional to the $n=-1$ eigenfunction of (4.6) we find

$$\frac{\partial D(\tau'\,|\,q(\tau))}{\partial\tau'}=2C_{-1}\left[\int_{-\theta/2}^{\theta/2}\left[\frac{\partial q_B}{\partial\tau}\right]^2 d\tau\right]^{1/2}, \quad (4.11)$$

where we have used the orthonormality properties of the $q_n(\tau)$. Likewise the Jacobian can be written in terms of the zero-mode normalization

$$\frac{\partial^2 D}{\partial(\tau')^2}=2\int_{-\theta/2}^{\theta/2}d\tau\left[\frac{\partial q_B}{\partial\tau}\right]^2. \quad (4.12)$$

Thus Z_B can be written as

$$Z_B=\left[\int_{-\theta/2}^{\theta/2}d\tau\left[\frac{\partial q_B}{\partial\tau}\right]^2\right]^{1/2}\theta\prod_n\left[\left[\int_{-\infty}^{\infty}\frac{dC_n}{(2\pi)^{1/2}}\right]\exp\left[-\frac{\Lambda_n C_n^2}{2}\right]\right]\delta(C_{-1})\,. \quad (4.13)$$

This takes care of the Goldstone mode. The negative eigenvalue must be analytically continued to obtain a finite result. This analytic continuation introduces a factor of $\frac{1}{2}$ and turns Z_B into an imaginary quantity. This derivation is presented to emphasize that the correct normalization of the Goldstone mode involves the quantity

$$N=\int_{-\theta/2}^{\theta/2}d\tau\left[\frac{\partial q_B}{\partial\tau}\right]^2$$

and not S_B/M as has often been assumed. We evaluate this quantity as

$$N=2\pi\omega_0\left[\frac{\omega_0^2}{u}\right]^2\left[\frac{2\pi}{\omega_0\theta}\right]^5(18\,\mathrm{csch}^2\beta)\left[\left[\frac{\eta\theta}{10\pi}\right]^2\coth\beta+\left[\frac{\eta\theta}{10\pi}\right](2+3\,\mathrm{csch}^2\beta)+\coth\beta(1+3\,\mathrm{csch}^2\beta)\right]. \tag{4.14}$$

The eigenvalues of (4.6) can be found exactly, at the crossover temperature T_0. They are simply expressed as

$$\Lambda_n=M\left[\frac{2\pi}{\theta}\right]^2\left[|n|^2+|n|\left[\frac{\eta\theta}{2\pi}\right]+\left[\frac{\omega_0\theta}{2\pi}\right]^2-\frac{12}{5}\left[\frac{\eta\theta}{2\pi}\right]\right]. \tag{4.15}$$

We see that in addition to the negative eigenvalue Λ_0, there are two zero eigenvalues $\Lambda_1=\Lambda_{-1}=0$. The new zero mode corresponds to amplitude fluctuations about the bounce solution.[7,8] Following this earlier work we eliminate both zero modes by considering the effect that the cubic interaction has. This yields

$$Z_B=\frac{i}{2|\Lambda_0|^{1/2}}\prod_{\substack{n\\ n\neq 0,\pm1}}\left[\frac{1}{\Lambda_n}\right]^{1/2}\int\int\frac{dC_1dC_{-1}}{2\pi}\exp\left[-\frac{\Lambda_1}{2}(C_1^2+C_{-1}^2)+\frac{M^2u^2}{2\theta}\left[\frac{1}{\Lambda_0}+\frac{1}{2\Lambda_2}\right](C_1^2+C_{-1}^2)^2\right]. \tag{4.16}$$

Since $1/\Lambda_0+1/2\Lambda_2<0$, the integral converges and Z_B is found as

$$Z_B=\frac{i}{2|\Lambda_0|^{1/2}}\prod_{n=2}^{\infty}\left[\frac{1}{\Lambda_n}\right]\left[\frac{1}{2Mu}\right]\left[\frac{\pi\theta|\Lambda_0|\Lambda_2}{2\Lambda_2+\Lambda_0}\right]^{1/2}\mathrm{erfc}\left[\frac{\theta^{1/2}|\Lambda_0|^{1/2}\Lambda_1\Lambda_2^{1/2}}{2Mu(2\Lambda_2+\Lambda_0)^{1/2}}\right]\exp\left[\frac{\theta|\Lambda_0|\Lambda_1^2\Lambda_2}{4M^2u^2(2\Lambda_2+\Lambda_0)}\right]\quad\text{at } T=T_0, \tag{4.17}$$

where $\mathrm{erfc}(x)=1-\mathrm{erf}(x)$. This yields the tunneling rate Γ, as

$$\Gamma=35\omega_0\left[\frac{4\sqrt{6}}{10}\frac{V_{\max}}{\omega_0}\right]^{1/2}\exp\left[-2\pi\sqrt{6}\frac{V_{\max}}{\omega_0}\right],\quad T=T_0$$

in which $V_{\max}$ is the potential barrier height

$$V_{\max}=\frac{M}{6}\omega_0^2\left[\frac{\omega_0^2}{u}\right]^2.$$

We note that the exponential term in the rate matches smoothly onto the Arrehenius factor. The authors of Ref. 8 use an expression similar to Eq. (4.17) for temperatures $T<T_0$. They use a negative value for the argument of the error function, which would imply that $\Lambda_1=\Lambda_{-1}$ and that both are negative. The discrepancy arises from neglecting some nonlinear terms of order $(T-T_0)$ in the action, while retaining other terms of this order. Such an approximation is not consistent for $T_0>T$. In the bounce method, if one includes the fluctuations about the bounce consistently to order (T_0-T), one finds the degeneracy between Λ_1 and Λ_{-1} is broken, and that $\Lambda_1>0$ while $\Lambda_{-1}\equiv 0$. At $T=T_0$, these differences vanish and all the expressions become identical.

Below T_0, the amplitude mode acquires a finite eigenvalue Λ_1. Although the eigenvalues given in (4.15) are not exact, they still represent a reasonable approximation to the solutions of (4.6) near $T\simeq T_0$. The corrections to the eigenvalues given in (4.15) can be estimated from perturbation theory as

$$\delta\Lambda_n\simeq\frac{2M\left\{12\left[1+\left[\frac{\eta\theta}{10\pi}\right]\right]e^{-\beta}\right\}^2}{\left\{\left[2n+\left[\frac{\eta\theta}{2\pi}\right]\right]^2-1\right\}},\quad |n|\geq 1. \tag{4.18}$$

This tends to increase the eigenvalues Λ_n. We note that at T_0 the eigenvalues corresponding to q_n and q_{-n} are degenerate. The coupling to this order $(T-T_0)$, lifts the degeneracy between q_1 and q_{-1} at temperatures below T_0.

At zero temperature, the expression for Z_B can be found with the aid of scattering theory,[25] since the bounce extends over a finite time $\sim\tau_B=\beta\theta/2\pi$ while the boundaries $(-\frac{1}{2}\theta,\frac{1}{2}\theta)$ are removed to infinity.

We define the Green's function for the full scattering problem through the equation

$$\left[-\frac{\partial^2}{\partial\tau^2}+\omega_0^2-\frac{\Lambda}{M}-2uq_B(\tau)\right]G(\tau;\tau')+\frac{\eta}{2\pi}\int_{-\infty}^{\infty}d\tau''\frac{\partial G}{\partial\tau''}(\tau'';\tau')\left[\frac{1}{\tau-\tau''+i\epsilon}+\frac{1}{\tau-\tau''-i\epsilon}\right]=\delta(\tau-\tau'). \tag{4.19}$$

The unperturbed Green's function $G^0(\tau-\tau')$ is defined by

$$\left[-\frac{\partial^2}{\partial\tau^2}+\omega_0^2-\frac{\Lambda}{M}\right]G^0(\tau-\tau')+\frac{\eta}{2\pi}\int_{-\infty}^{\infty}d\tau''\frac{\partial G^0}{\partial\tau''}(\tau''-\tau')\left[\frac{1}{\tau-\tau''+i\epsilon}+\frac{1}{\tau-\tau''-i\epsilon}\right]=\delta(\tau-\tau'). \tag{4.20}$$

This last equation has a simple solution. Henceforth, we shall use the $\Lambda/M\to\Lambda$ substitution of Λ. If $\Lambda\geq\omega_0^2$, $G^0(\tau)$ has a continuous spectrum. We find

$$\mathrm{Re}G^0(\tau)=\frac{1}{2\pi}\frac{1}{\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2}}\mathrm{Re}\left[\left[\mathrm{Si}(\omega_\beta\tau)-\frac{\pi}{2}\right]\sin(\omega_\beta\tau)-\left[\mathrm{Si}(\omega_\alpha\tau)+\frac{\pi}{2}\right]\sin(\omega_\alpha\tau)+\mathrm{Ci}(\omega_\beta\tau)\cos(\omega_\beta\tau)-\mathrm{Ci}(\omega_\alpha\tau)\cos(\omega_\alpha\tau)\right],$$

where

$$\omega_\alpha=-\frac{\eta}{2}+\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2}$$

and

$$\omega_\beta=+\frac{\eta}{2}+\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2}.$$

The imaginary part is given by

$$\mathrm{Im}G^0(\tau)=\frac{1}{2}\frac{\cos(\omega_\alpha\tau)}{\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2}}. \tag{4.21}$$

In the range where $\omega_0^2\geq\Lambda\geq\omega_0^2-(\eta/2)^2$ the imaginary part of $G^0(\tau)$ is zero, we find

$$\mathrm{Re}G^0(\tau)=\frac{1}{2\pi}\frac{1}{\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2}}\mathrm{Re}\left[\left[\mathrm{Si}(\omega_\beta\tau)-\frac{\pi}{2}\right]\sin(\omega_\beta\tau)-\left[\mathrm{Si}(\omega_\alpha\tau)-\frac{\pi}{2}\right]\sin(\omega_\alpha\tau)+\mathrm{Ci}(\omega_\beta\tau)\cos(\omega_\beta\tau)-\mathrm{Ci}(\omega_\alpha\tau)\cos(\omega_\alpha\tau)\right]$$

and

$$\mathrm{Im}G^0(\tau)=0. \tag{4.22}$$

Here, we have used the notation

$$\omega_\alpha=\frac{\eta}{2}-\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2},$$

$$\omega_\beta=\frac{\eta}{2}+\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2}.$$

The last regime, $\omega_0^2-(\eta/2)^2>\Lambda$ gives results similar to (4.22) if we analytically continue ω_α and ω_β to complex values. Thus we find that the unperturbed density of eigenvalues, given by $1/\pi\ \mathrm{Im}G(\tau=0)$ forms a continuum with $\Lambda>\omega_0^2$. This continuum starts at the same point as the nondissipative density, merely because of our inclusion of the counter term (2.1) in the Lagrangian. However, the dissipation has the effect of suppressing the square-root singularity of the one-dimensional density of states to a finite value of $1/\pi\eta$. The dissipation shifts the spectral density to higher eigenvalues, as can be seen by inspecting the finite temperature version of $\mathrm{Im}G^0(\tau=0)$. The suppression of the singularity in the one-dimensional density of states allows us to use perturbation theory advantageously.

The full Green's function of the scattering problem $G(\tau;\tau')$ can be written in terms of the unperturbed Green's function, through the Lippmann-Schwinger equation

$$G(\tau;\tau')=G^0(\tau-\tau')+\int d\tau''\,G^0(\tau-\tau'')2uq_B(\tau'')G(\tau'';\tau'). \tag{4.23}$$

Upon Fourier transforming, we obtain the equation

$$G(\omega_1;\omega_2)=G^0(\omega_1)2\pi\delta(\omega_1-\omega_2)+G^0(\omega_1)\int\frac{d\omega}{2\pi}2uq_B(\omega)G(\omega_1-\omega;\omega_2). \tag{4.24}$$

Thus we may express $G(\omega_1;\omega_2)$ in terms of the T matrix $T(\omega_1;\omega_2)$ as

$$G(\omega_1;\omega_2)=G^0(\omega_1)2\pi\delta(\omega_1-\omega_2)+G^0(\omega_1)T(\omega_1;\omega_2)G^0(\omega_2)$$

and $T(\omega_1;\omega_2)$ is given by the usual expression

$$T(\omega_1;\omega_2)=2uq_B(\omega_1-\omega_2)+\int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}2uq_B(\omega)G(\omega_1-\omega;\omega_2+\omega')2uq_B(\omega')\ . \tag{4.25}$$

The prefactor of the decay rate can be expressed in terms of the density of eigenvalues through the relation

$$\prod_{\substack{n=-\infty\\(n\neq 1,0,-1)}}^{\infty}\Lambda_n=\exp\left[\sum_{\substack{n=-\infty\\(n\neq 1,0,-1)}}^{\infty}\ln\Lambda_n\right]$$

which for the continuum portion of the spectrum, can be written as

$$\prod_{n=-\infty}^{\infty}\Lambda_n=\exp\left[\int_{\omega_0^2}^{\infty}d\Lambda\frac{1}{\pi}\mathrm{Im}G(\tau=0)\ln\Lambda\right]=\exp\left[\int\frac{d\Lambda}{\pi}\mathrm{Im}G^0(\tau=0)\ln\Lambda\right]$$
$$\times\exp\left[\int\frac{d\Lambda}{\pi}\mathrm{Im}\left[\int\frac{\partial G^0(\omega)}{\partial\Lambda}T(\omega,\omega)\frac{d\omega}{2\pi}\right]\ln\Lambda\right]\ . \tag{4.26}$$

The first term can be related to the product of the eigenvalues

$$\prod_{n=-\infty}^{\infty}\Lambda_n^0$$

in the absence of the bounce, and thus they cancel in the expression for the decay rate. After considerable manipulation, we can rewrite the last factor in the form used by Chang and Chakravarty[25]

$$\exp\left[+\int_{\omega_0^2}^{\infty}d\Lambda\,\frac{1}{\pi}\frac{\partial\delta(\Lambda)}{\partial\Lambda}\ln\Lambda\right] \tag{4.27}$$

in which $\delta(\Lambda)$ is the phase shift. Levinson's theorem takes on the form

$$3=-\frac{1}{\pi}\int d\Lambda\frac{\partial\delta(\Lambda)}{\partial\Lambda}$$

and can be used to demonstrate that (4.27) has the dimensions associated with Λ^{-3}. This is expected since the ratios of the determinants, including $n=0,\pm1$, are dimensionless. Thus (4.27) should be of the same dimensions as $(\Lambda_{-1}\Lambda_0\Lambda_1)^{-1}$. An expansion of the T matrix yields the result

$$\delta(\Lambda)=\delta^{(1)}(\Lambda)+\delta^{(2)}(\Lambda)+\cdots\ ,$$

where

$$\delta^{(1)}(\Lambda)=\frac{12\pi}{5}\frac{\eta}{\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2}}$$

and

$$\delta^{(2)}(\Lambda)=\frac{(12\pi)^2}{\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2}\left[2\frac{\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2}}{\tau_B}-\left[\left[\omega_\alpha-\omega_\beta+\frac{\eta}{5}\right]^2e^{-2\tau_B(\omega_\alpha-\omega_\beta)}\mathrm{Ei}[2\tau_B(\omega_\alpha-\omega_\beta)]\right.\right.$$
$$\left.+\left[\omega_\beta-\omega_\alpha+\frac{\eta}{5}\right]^2e^{-2\tau_B(\omega_\beta-\omega_\alpha)}E_1(2\tau_B(\omega_\alpha-\omega_\beta))\right]$$
$$+\left[\left[\omega_\alpha-\omega_\beta+\frac{\eta}{5}\right]^2e^{+2\tau_B\omega_\beta}\mathrm{Ei}(-2\tau_B\omega_\beta)-\left[\frac{4\eta}{5}\right]^2e^{-2\tau_B\omega_\beta}E_1(-2\tau_B\omega_\beta)\right.$$
$$\left.\left.-\left[2\omega_\alpha+\frac{\eta}{5}\right]^2e^{-2\tau_B\omega_\alpha}\mathrm{Ei}(2\tau_B\omega_\alpha)+\left[\frac{\eta}{5}\right]^2e^{+2\tau_B\omega_\alpha}E_1(2\tau_B\omega_\alpha)\right]e^{-2\tau_B\omega_\alpha}\right],\quad\Lambda\geq\omega_0^2$$

etc. In these expressions we have used

$$\left.\begin{matrix}\omega_\alpha\\ \omega_\beta\end{matrix}\right\}=-\frac{\eta}{2}\pm\left[\Lambda-\omega_0^2+\left[\frac{\eta}{2}\right]^2\right]^{1/2}$$

and

$$\tau_B=\frac{\beta\theta}{2\pi} \qquad (4.28)$$

as before.

As previously noted by Chang and Chakravarty,[25] the series rapidly converges for large Λ or η. We find that, in the lowest-order (Born) approximation, the continuum contributes a factor of

$$\exp\left[\frac{+\frac{48}{5}\left[\frac{\eta}{2\omega_0}\right]}{\left[\left[\frac{\eta}{2\omega_0}\right]^2-1\right]^{1/2}}\times\coth^{-1}\left[\frac{\left[\frac{\eta}{2\omega_0}\right]}{\left[\left[\frac{\eta}{2\omega_0}\right]^2-1\right]^{1/2}}\right]\right] \qquad (4.29)$$

to the expression (4.27) similar to the expression derived in Ref. 2. The accuracy of this expression can be checked indirectly, by comparison with the numerical work of Chang and Chakravarty.[25] From this comparison, we estimate the error in (4.29) to be of order 2.

Even though the perturbation series for the phase shift $\delta(\Lambda)$ is rapidly varying for eigenvalues Λ close to the edge of the continuum ω_0^2, the corresponding series for the prefactor expression

$$(\omega_0^2)^{-3}\exp\left[-\int_{\omega_0^2}^{\infty}d\Lambda\frac{\delta(\Lambda)}{\pi\Lambda}\right] \qquad (4.30)$$

is well behaved. The integral is dominated by the region of large $\Lambda \gg \omega_0^2$. In this region the first Born approximation dominates the phase shift and thus (4.29) is the largest contribution to the prefactor. For values of Λ close to ω_0^2, we may perform a partial summation of an infinite subseries for $\delta(\Lambda)$ to obtain the approximation

$$\delta(\Lambda)=\delta^{(+)}(\Lambda)+\delta^{(-)}(\Lambda)\ ,$$

where

$$\tan\delta^{(\pm)}(\Lambda)=\left[\frac{u}{\omega_\alpha-\omega_\beta}\right][q_B(0)\pm q_B(-2\omega_\alpha)] + \left[\frac{u}{\omega_\alpha-\omega_\beta}\right]^2\int\frac{d\omega\,\theta(\omega)}{\pi}[q_B(\omega-\omega_\alpha)\pm q_B(\omega+\omega_\alpha)]^2\left[\frac{1}{\omega-\omega_\alpha}-\frac{1}{\omega+\omega_\beta}\right]+\cdots\ . \qquad (4.31)$$

We believe that this represents a reasonable approximation since $\delta(\Lambda)$ does closely approach the correct value (as defined by Levinson's theorem) at the edge of continuum. The integration over Λ is performed numerically to find the expression for the tunneling rate as

$$\Gamma\cong 88\omega_0\left[\frac{5.18}{2\pi}\frac{V_{\max}}{\omega_0}\right]^{1/2}\exp\left[-8\pi\frac{V_{\max}}{\omega_0}\right]$$

when $T=0$.

This prefactor is slightly higher than that given in Chang and Chakravarty,[25] but is less then the first Born-approximation-estimate.

V. CONCLUSIONS

We have addressed the problem of quantum tunneling out of a metastable state, in the presence of coupling to a heat bath. We have used the dilute bounce gas approximation. The main result of this work is that we have found an exact solution of the Euler-Lagrange equations for a specific value of the ratio (η/ω_0) which characterizes the strength of the heat-bath coupling. The instanton trajectory exists for the entire temperature range $T_0 \geq T \geq 0$. At T_0 the instanton trajectory tends to a constant value $q_B(\tau)=\omega_0^2/u$, which corresponds to the maximum of the potential barrier. This has been previously identified[7,8] as the temperature at which the crossover from quantum tunneling to thermal activation occurs. We have obtained the exponential factor in the tunneling rate for this entire range of temperatures. The prefactor is given by the quantum fluctuations about the extrema of the action. We have obtained analytic expression for the prefactor at $T=T_0$ at $T=0$ assuming periodic boundary conditions for the fluctuations about the extremal trajectories. We note that the change in the prefactor between $T=T_0$ and $T=0$ indicates a sizable temperature variation of the prefactor.

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APPENDIX

The choice of boundary conditions $q_n(\frac{1}{2}\theta)=q_n(-\frac{1}{2}\theta)$ can be rationalized as follows. The small quantum fluctuations about the trajectory $q(\tau)=0$, should resemble the quantum fluctuations around the minima of a simple harmonic oscillator. Using the periodic boundary conditions we have

$$Z_0 = \left[\prod_{n=-\infty}^{\infty} \Lambda_n^0 \right]^{-1/2} . \tag{A1}$$

From this we find this result

$$Z_0 \propto \left| \Gamma \left\{ 1 + \left[\frac{\eta\theta}{4\pi} \right] + i \left[\left[\frac{\omega_0\theta}{2\pi} \right]^2 - \left[\frac{\eta\theta}{4\pi} \right] \right]^{1/2} \right\} \right|^2 \tag{A2}$$

which expresses Z_0 as being proportional to the partition function of the damped harmonic oscillator.[19] The limit $\eta \rightarrow 0$, yields the familiar expression

$$Z_0 \propto [\sinh(\beta\omega_0/2)]^{-1} . \tag{A3}$$

Thus the periodic boundary conditions yields the proportionality to the partition function as expected. The use of the other boundary condition, $q_n(\frac{1}{2}\theta) = q_n(-\frac{1}{2}\theta) = 0$, yields

$$Z_0 = \left[\prod_{n=0}^{\infty} \Lambda_n^0 \right]^{-1/2}$$

which results in the proportionality to the square root of the partition function. Thus, based on this physical consideration we choose the periodic boundary condition for the fluctuations about $q(\tau)=0$.

We insist on using the boundary conditions for the fluctuations about any other path [i.e., $q(\tau)=q_B(\tau)$] since the corresponding measures for the path integrals will cancel in a trivial manner.

[1]A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981).

[2]A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983); **153**, 445(E) (1984).

[3]A. J. Bray and M. A. Moore, Phys. Rev. Lett. **49**, 1546 (1982).

[4]S. Chakravarty, Phys. Rev. Lett. **49**, 681 (1982).

[5]A. Schmid, J. Low Temp. Phys. **49**, 609 (1982).

[6]U. Weiss, P. Riseborough, P. Hänggi, and H. Grabert, Phys. Lett. **104A**, 10 (1985); **104A**, 492(E) (1985).

[7]H. Grabert and U. Weiss, Phys. Rev. Lett. **53**, 1787 (1984).

[8]A. J. Larkin and Y. Ovchinnikov, Zh. Eksp. Teor. Fiz. **86**, 719 (1984) [Sov. Phys.—JETP **59**, 420 (1984)].

[9]T. Holstein, Ann. Phys. (N.Y.) **8**, 325 (1959); **8**, 343 (1959).

[10]C. P. Flynn and A. M. Stoneham, Phys. Rev. B **1**, 3966 (1970).

[11]P. S. Riseborough, Phys. Status Solidi B **117**, 381 (1983).

[12]J. P. Sethna, Phys. Rev. B **25**, 5050 (1982); **24**, 698 (1981).

[13]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[14]L. S. Schulman, *Techniques and Applications of Path Integrals* (Wiley, New York, 1981).

[15]J. S. Langer, Ann. Phys. **41**, 108 (1967).

[16]C. G. Callan and S. Coleman, Phys. Rev. D **16**, 1762 (1977).

[17]P. Ullersma, Physica (Utrecht) **32**, 27 (1966).

[18]P. S. Riseborough, P. Hanggi, and U. Weiss, Phys. Rev. A **31**, 471 (1985).

[19]H. Grabert, U. Weiss, and P. Talkner, Z. Phys. B **55**, 87 (1984); F. Haake and R. Reibold, Acta Phys. Austriaca **56**, 37 (1984).

[20]A. Widom and T. D. Clarke, Phys. Rev. Lett. **48**, 63 (1982).

[21]R. P. Feynman, *Statistical Mechanics* (Benjamin, New York, 1972).

[22]H. Grabert, U. Weiss, and P. Hanggi, Phys. Rev. Lett. **52**, 2193 (1984); H. Grabert and U. Weiss, Z. Phys. B **56**, 171 (1984).

[23]A. I. Larkin and Y. Ovchinnikov, Zh. Eksp. Teor. Fiz. **37**, 322 (1983) [Sov. Phys.—JETP **37**, 382 (1983)].

[24]J. Zittartz and J. S. Langer, Phys. Rev. **148**, 741 (1966).

[25]L. D. Chang and S. Chakravarty, Phys. Rev. B **29**, 130 (1983); **30**, 1566(E) (1984).