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Dynamic scaling and the surface structure of Eden clusters

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The evolution of the surface of two-dimensional Eden deposits grown in a strip of width L is related to the dynamics of a set of "normal modes" of wave number q. Monte Carlo simulations show striking similarities with critical phenomena. The amplitude squared of the modes relax in the long-time limit ($t\rightarrow\infty$) to a value $S(q)\sim q^{-2}$, and the relaxation towards the steady state is dominated by a relaxation time scaling as $\tau(q)\sim q^{-z}$, with $z=1.55\pm0.15$. This implies that the surface width has the scaling form $\xi(t,L)\sim L^{1/2}G(t/L^z)$ with $G(x)\rightarrow G(\infty)\neq0$ as $x\rightarrow\infty$ and $G(x)\sim x^{1/(2z)}$ for $x\rightarrow 0$.

The development of structure in growth and aggregation models has been a subject of intense study in recent years.[1] Despite this activity no satisfactory analytic theory has emerged, and it is therefore still necessary to study the simpler growth models. One such model is the Eden model.[2] Its simplicity lies in the fact that clusters grown according to the Eden algorithm are compact,[3] i.e., their fractal dimension (D) is equal to the Euclidean dimension (d) of the space in which growth takes place. The nontrivial aspect of the Eden process is the irregularity of the surface of the clusters. As we have shown by simulating the Eden process in $d=2$ and 3 dimensions,[4-6] the width (ξ) of the active zone, i.e., the width of the surface zone of the clusters grows according to the relation $\xi\sim N^{\bar{\nu}}\sim R^{d\bar{\nu}}$, where N is the number of particles in the cluster and R is the mean radius. Since $\bar{\nu}$ was found to be not equal to $1/d$, our result implies the existence of a second relevant length scale (ξ) besides the mean cluster radius $R\sim N^{1/d}$. Clearly, a theory of the Eden process should be aimed at explaining the appearance of this second length scale and at calculating its exponent $\bar{\nu}$. In this paper we make a small step towards such a theory by demonstrating the following two points.

(a) There is a close analogy between the Eden process and relaxation in a thermodynamic system at its critical point. Namely, appropriately defined surface perturbations of wave number q have characteristic growth and decay rates scaling as q^z with $z=1.55\pm0.15$ in $d=2$.

(b) The second length scale ξ is intimately related to the critical dynamics of the process, and $\bar{\nu}$ can be expressed in terms of the dynamical critical exponent z. For example, in $d=2$ one finds $\bar{\nu}=(4z)^{-1}$.

In order to establish the above points, we shall work with version C of the Eden model on the square lattice. In this version introduced by Jullien and Botet[7] new particles are added to the cluster according to the following two-step algorithm: (i) An occupied surface site of the cluster is chosen with the probability $1/n_s$, where n_s is the total number of such sites. (ii) The new particle is then added equiprobably to one of the adjacent empty sites.

It will be convenient to study this growth process in a strip geometry,[7,8] i.e., to restrict the growth to a strip of width L (in units of the lattice spacing) and to use periodic boundary conditions in the direction perpendicular to the strip. As an initial condition we shall use a state (substrate) in which all the sites are occupied up to a given height h_0, which may be chosen to be the reference level $h_0=0$. (The height is also measured in units of the lattice spacing along the strip).

The strip geometry provides a convenient separation of the control parameters. The width of the strip L and the average height of the deposit $\bar{h}$, or, in appropriate units, the time of the growth $t\sim\bar{h}$ can be varied independently. This is to be contrasted with the "circular" geometry usually considered,[6,9,10] where the cluster grows from a seed particle and a single parameter N controls both the "height," i.e., the mean radius $R\sim N^{1/2}$, and the "strip width" which is equivalent to the circumference at the mean radius, $2\pi R\sim N^{1/2}$. One expects that curvature effects are negligible for $N\rightarrow\infty$ and, consequently, the scaling properties of Eden clusters in "circular" geometry can be obtained from those in strip geometry provided $t\sim\bar{h}\sim L\sim N^{1/2}$ is chosen.

The advantage of separating the control parameters is the freedom to consider limits which are simple but inaccessible in circular geometry. In particular, the $t\rightarrow\infty$ limit is important, since in this limit the growth process becomes stationary and the surface properties of the moving front become time independent.[8] One might hope that this stationary state can be more readily treated by analytical methods and that one might then connect to the $t\sim\bar{h}\sim L\sim N^{1/2}$ limit by studying relaxation towards the stationary state.

At this moment, however, we are still limited to studying the process by Monte Carlo simulations. The first Monte Carlo results are due to Jullien and Botet,[8] who studied both the relaxational and the stationary-state properties of the surface thickness of the deposit. In our work we have undertaken a more detailed characterization of the surface by decomposing it into Fourier modes and investigating the static and dynamic properties of these modes. Working with the Fourier modes makes the analogies with dynamic critical phenomena transparent, and, since the surface thickness can be expressed through these modes, the derivation of the formula $\bar{\nu}=(4z)^{-1}$ becomes straightforward. Furthermore, we, of course, recover all of the results of Jullien and Botet.[7,8]

Turning to the details of the calculations, note that in order to define the Fourier modes we need a single-valued function for the surface height. Although version C of the Eden algorithm suppresses to some extent the formation of overhangs and holes, the surface may consist of more than one occupied site above a particular point of the substrate. In order to obtain a single-valued representation of the surface height, we first construct a local average by defining a mean surface height $h(i,t)$ for substrate point i at time t

through the equation

$$h(i,t)=\frac{1}{n_s(i,t)}\sum_{j=1}^{n_s(i,t)} h_j(i,t) \quad , \tag{1}$$

where $h_j(i,t)$ is the height of the jth surface site in column i, $n_s(i,t)$ is the number of surface sites in column i at time t, and t is measured in number of particles deposited per substrate site. The width of the surface is then given by

$$\xi^2(L,t)=\frac{1}{L}\sum_{i=1}^{L}[h(i,t)-\bar{h}(t)]^2 \quad , \tag{2}$$

where $\bar{h}(t)=\sum_i h(i,t)/L$ is the average height. This definition is a bit different from the usual definition[7,8] of surface width $\sigma(L,t)$:

$$\sigma^2(L,t)=\frac{1}{n_s}\sum_{i,j}[h_j(i,t)-\tilde{h}(t)]^2 \quad , \tag{3}$$

where n_s is the total number of surface sites and $\tilde{h}(t)=\sum_{ij}h_j(i,t)/n_s$. As L becomes large, however, we find that the scaling properties of ξ and σ are identical.

Given a single-valued representation of the cluster boundary, one can carry out a Fourier analysis of the surface by defining

$$\hat{h}(q,t)=\frac{1}{\sqrt{L}}\sum_{j=1}^{L}[h(j,t)-\bar{h}(t)]e^{iqj} \quad , \tag{4}$$

with $q=\pm 2k\pi/L$, $k=1,2,\ldots,L-1$. In terms of $\hat{h}$ we can write

$$\xi^2(L,t)=\frac{1}{L}\sum_q\langle|\hat{h}(q,t)|^2\rangle=\frac{1}{L}\sum_q S(q,t) \quad , \tag{5}$$

where the angular brackets denote averaging over different clusters. In Fig. 1 Monte Carlo results are displayed for the function $q^2S(q,t)$ for various values of L in the limit $t\to\infty$. One can see that this function is practically independent of L and approaches a finite limit as $q\to 0$. This immediately establishes that in the stationary state we have

$$\xi^2(L,\infty)\sim\int_{2\pi/L}^{\pi}dq/q^2\sim L \quad , \tag{6}$$

a result also observed in the simulations of Jullien and Botet.[7,8] The facts that the amplitude of the Fourier modes scale as $\langle|\hat{h}(q,\infty)|^2\rangle\sim q^{-2}$ and that the dependence on the finite size (L) appears only through the lower cutoff in the possible values of q are similar to those found in various roughening models,[11] and for Ising interfaces.[12] The similarity, however, stops at the static aspects of the phenomena. As we shall see below, the dynamics of the surface in the Eden model is distinct from the dynamics of the interface in the dynamical generalizations of the above models.

In order to study the relaxational properties of the Eden surface, we first consider the time correlations in the stationary state, i.e., we "measure" the correlation function

$$\phi(q,\tau)=\lim_{t\to\infty}\frac{1}{S(q,t)}\langle\hat{h}(q,t+\tau)\hat{h}(-q,t)\rangle \quad , \tag{7}$$

for various values of q and L. In Fig. 2, $\phi(q,\tau)$ is displayed as a function of the scaled time $\bar{\tau}=q^z\tau$, and one can see that the simulation points fall on two universal curves provided $z=1.55$. (The range of z which yields satisfactory collapse of the data is $z=1.55\pm0.15$.) The upper curve describes the decay of the smallest q mode, while the lower one represents the relaxation of modes with $q>2\pi/L$. The exponent z, however, is the same for all modes. Thus the finite-size effects for dynamics are more complicated than for statics. Apart from the cutoff in the possible values of q, the smallest $q=2\pi/L$ mode is singled out; it decays with a smaller time constant. As the size of the substrate is doubled, $L'=2L$, the $q=2\pi/L$ mode becomes part of the

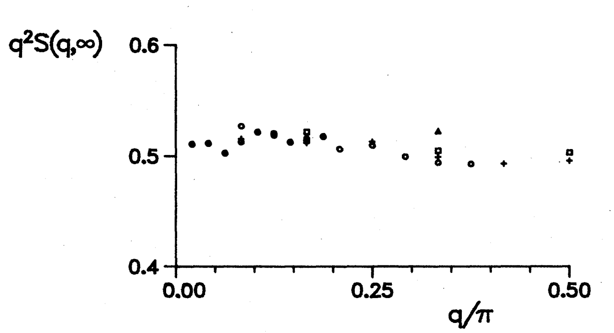

FIG. 1. The equal-time correlation function $S(q,\infty)$ [Eq. (5)] multiplied by $(q/\pi)^2$ for strips of width $L=6$, 12, 24, 48, and 96. Clusters were first grown to a depth (time) $40L$ before calculation of $S(q,\infty)$ and $\phi(q,\tau)$ [Eq. (7)] began. The total number of samples used was 30000 for $L=6$, 12, and 24, 15000 for $L=48$, and 7500 for $L=96$. For $L=48$ and 96, $S(q,\infty)$ and $\phi(q,\tau)$ were calculated for the first nine points, $q=2\pi j/L$, $j=1,\ldots,9$ in the Brillouin zone.

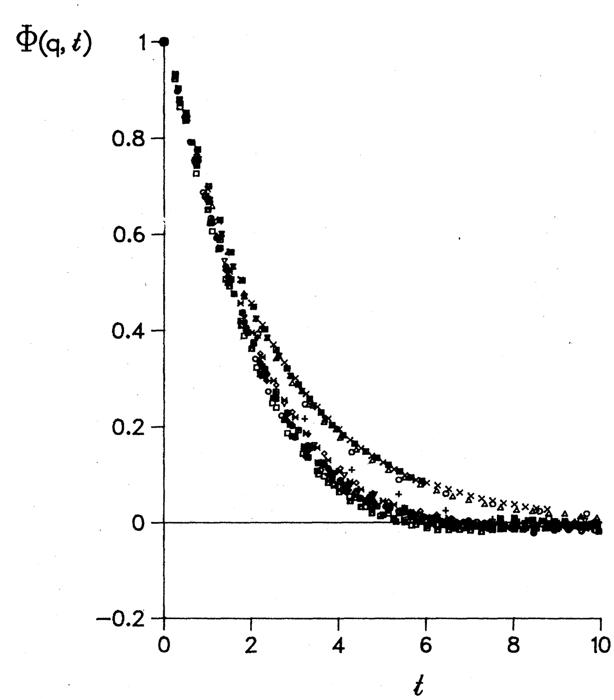

FIG. 2. The relaxation function $\phi(q,\tau)$ [Eq. (7)] in the steady-state regime plotted as a function of the scaled time $t=\tau q^z$ with $z=1.55$ for $L=6$, 12, 24, 48, and 96. The same equilibration and averaging procedure as used to obtain Fig. 1 was employed. The upper cluster of points represents $\phi(q,\tau)$ for $q=2\pi/L$ for all L. The lower set of data is $\phi(q,\tau)$ for $q=\pi/3$ for $L=12$, $q=\pi/6$ and $\pi/4$ for $L=24$, $q=\pi/12$, $\pi/8$, and $\pi/6$ for $L=48$, and $q=\pi/24$, $\pi/16$, $\pi/12$, $5\pi/48$, $\pi/8$, $7\pi/48$, and $\pi/6$ for $L=96$.

emerging continuum (lower curve) and decays with the time constant characterizing the rest of the Brillouin zone. Since the upper curve represents only two points in the Brillouin zone, it does not play any role in the calculation of macroscopic averages such as the width of the surface zone.

The observed scaling $\phi(q,\tau) \simeq f(q^z\tau)$ of the relaxation function is a characteristic feature of the dynamics of systems at a critical point. This observation establishes our first point (a) about the critical dynamics of the Eden model. The result $z \approx 1.55$ shows that the Eden dynamics belongs to a different universality class than the dynamics introduced into the roughening models where $z \geq 2$ is found.[11,13]

In order to make connection with the "circular" geometry, we need the limit $t \sim L$. Thus we must consider the far-from-stationary-state dynamics of the surface. For this purpose we studied the initial growth of correlations, i.e., the correlation function

$$\psi(q,t) = \frac{S(q,\infty) - S(q,t)}{S(q,\infty)} \quad . \tag{8}$$

In Fig. 3, $\phi(q,t)$ is plotted for various values of L and q as a function of $\tilde{t} = q^z t$. We see again the collapse of data for the same range of the dynamical critical exponent $z = 1.55 \pm 0.15$. In this case, however, the lowest q modes do not form a separate branch of the scaling function. The reason for this may be insufficient resolution in our data, or it may be that the finite-size effects are simpler for the initial growth of the amplitude of the modes.

Figure 3 implies that the $S(q,t)$ obeys dynamic scaling

$$S(q,t) \sim \frac{1}{q^2}[1 - g(q^z t)] \quad . \tag{9}$$

The scaling function $g(x)$ is exponentially small for $x \to \infty$, while $1 - g(x) \sim x$ for $x \to 0$. The consequence of this functional form for the width of the surface is immediate, since

$$\xi^2(L,t) \sim \int_{2\pi/L}^{\pi} \frac{dq}{q^2}[1 - g(q^z t)] \quad , \tag{10}$$

and letting $q = x/L$, we find a scaling form as $L \to \infty$:

$$\xi(L,t) \sim L^{1/2} G(t/L^z) \quad , \tag{11}$$

with $G(x) \to G(\infty) \neq 0$ for $x \to \infty$ and $G(x) \sim x^{1/(2z)}$ for $x \to 0$. The scaling form (11) has already been observed in the Monte Carlo simulations of Jullien and Botet, and their estimate of the dynamic critical exponent $z = 1.7 \pm 0.3$ is in agreement with ours; $z = 1.55 \pm 0.15$. What is new in our results is that the small-x limit of $G(x)$ is derived from Eq. (10), while their simulations were not sufficient to obtain $G(x) \sim x^{1/(2z)}$; they conjectured this behavior.

The small-x behavior of $G(x)$ is important in deriving the $t \sim L$ limit of the surface width. Indeed, we have

$$\xi(L,L) \sim L^{1/2} G(L^{1-z}) \sim L^{1/(2z)} \quad , \tag{12}$$

and, since $L \sim N^{1/2}$, the surface width in the circular geometry is given by $\xi \sim N^{1/(4z)} \sim N^{\bar{\nu}}$, and thus we have arrived at the second (b) point of our paper, namely, we have shown that the surface exponent can be expressed through the dynamic exponent z in the following form:

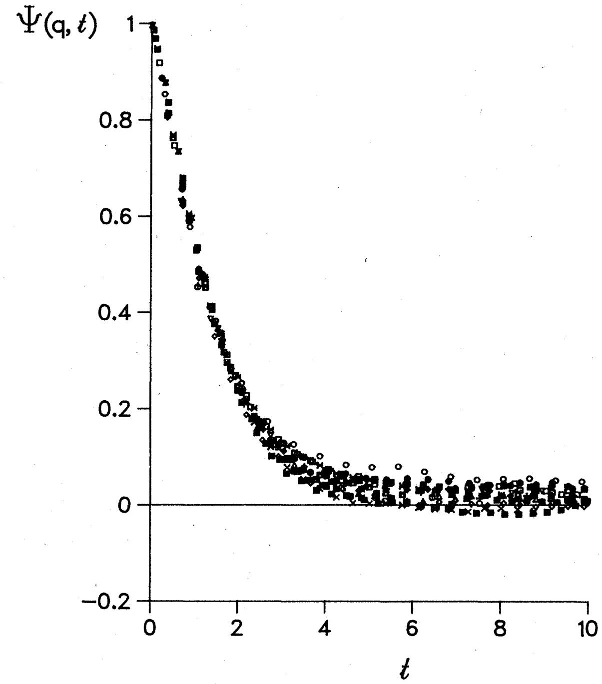

FIG. 3. Relaxation function $\phi(q,\tau)$ in the far-from-stationary-state regime [Eq. (8)] plotted as a function of the scaled time $t = q^z\tau$ with $z = 1.55$ for $L = 6$, 12, 24, 48, and 96. Clusters were started from the initial configuration and grown to a depth of $3L$. Each point on the graph is obtained from an average of over 20 000 different clusters. Data for the same set of q's as in Fig. 2 are plotted.

$$\bar{\nu} = (4z)^{-1} \quad . \tag{13}$$

The exponent $\bar{\nu}$ is not accurately known. Its effective value in subsequent decades,[6] is $\bar{\nu} \approx 0.18$ ($10^2 < N < 10^3$), $\bar{\nu} = 0.05$ ($10^3 < N < 10^4$), and $\bar{\nu} \simeq 0.21$ ($10^4 < N < 10^5$). Thus even clusters of size 10^5 in circular geometry are not large enough to test this formula. Clearly, finite-size effects in circular geometry are more important than in strip geometry. Whether this is an effect of curvature or perhaps due to a distortion of circular clusters into a diamond shape at large sizes because of the fourfold symmetry of the square lattice remains to be understood.

The importance of Eq. (13) is the connection which it establishes between dynamics and the surface properties of Eden clusters. We believe that this connection is quite general and that the dynamic scaling (9) which we have found for the Eden model will apply to a large number of other growth processes. This connection also demonstrates how the description of growth processes involves at least two length scales.

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