Analytic solutions of the two-state problem for a class of chirped pulses

Jakub Zakrzewski
Institute of Physics, Jagellonian University, ul. Reymonta 4, 30-059 Kraków, Poland
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Analytic solutions to the two-state-problem Bloch equations are obtained for a class of smooth chirped pulses. The asymmetric pulses introduced by Bambini and Berman as well as the chirped hyperbolic secant pulse of Hioe belong to the class of pulses discussed.

The Hamiltonian of the two-level atom driven by a smooth resonant pulse reads (in the rotating-wave approximation)

$$H=E_1|1\rangle\langle 1|+E_2|2\rangle\langle 2| +\frac{\hbar\Omega_0}{2}[F(t)\exp(i\omega_L t)|1\rangle\langle 2|+\text{H.c.}] \ , \qquad (1)$$

where ω_L is the pulse-carrier frequency and Ω_0, the Rabi frequency, is proportional to the transition dipole moment and the typical electric field strength. $F(t)$ is a complex envelope of the electric field: $F(t)=f(t)\exp[ig(t)]$, where f and g are real functions. Thus $f(t)$ describes the amplitude modulation and $g(t)$ the frequency modulation (chirp) of the pulse. We express the wave vector as

$$|\psi\rangle=\exp\frac{-iE_1t}{\hbar}\alpha_1(t)|1\rangle+\exp\frac{-iE_2t}{\hbar}\alpha_2(t)|2\rangle \ .$$

Then the Schrödinger equation reduces to

$$\dot\alpha_1=-i\Omega_0/2f(t)\exp\{i[g(t)+\Delta t]\}\alpha_2 \ , \qquad (2a)$$

$$\dot\alpha_2=-i\Omega_0/2f(t)\exp\{-i[g(t)+\Delta t]\}\alpha_1 \ , \qquad (2b)$$

where the dot indicates d/dt; $\Delta=\omega_L-\omega_0$ is the detuning, and $\omega_0=(E_2-E_1)/\hbar$. Introducing the real quantities

$$w=\alpha_2^*\alpha_2-\alpha_1^*\alpha_1 \ , \qquad (3a)$$

$$u=\alpha_2^*\alpha_1\exp\{-i[g(t)+\Delta t]\}+\text{c.c.} \ , \qquad (3b)$$

$$v=i\alpha_2^*\alpha_1\exp\{-i[g(t)+\Delta t]\}+\text{c.c.} \ , \qquad (3c)$$

Eqs. (2a) and (2b) become

$$\begin{pmatrix}\dot u\\ \dot v\\ \dot w\end{pmatrix}=\begin{pmatrix}0 & -\Delta-\dot g(t) & 0\\ \Delta+\dot g(t) & 0 & \Omega_0 f(t)\\ 0 & -\Omega_0 f(t) & 0\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix} \ , \qquad (4)$$

which are the famous Bloch equations for the optical or (originally) magnetic resonance problem. For their fundamental importance it is extremely useful to have analytic solutions to (4) [or, equivalently, (2)], as they provide deeper understanding of the underlying physics than straight numerical integration of these equations. Rosen and Zener[1] showed that for an unchirped $[\dot g(t)=0]$ pulse with envelope $f(t)=\text{sech}(t/\tau)$, where τ is the pulse duration, the solution to (2) may be expressed in terms of the hypergeometric functions.[2] Up until the 1980s the Rosen-Zener result was the only analytic solution of the problem [for smooth $F(t)$] for an arbitrary detuning Δ and Rabi frequency Ω_0. Bambini and Berman[3] were able to show that there exists a class of unchirped, asymmetric pulses for which analytic solutions of (2) may be obtained. The first solution for the chirped pulse is due to Hioe,[4] who has taken $f(t)=\text{sech}(t/\tau)$, $\dot g(t)\sim\tanh(t/\tau)$. Finally, Bambini and Lindberg[5] discussed a class of symmetric pulses, which, however, are not analytical functions of time, as they are obtained by a symmetrization procedure applied to the pulses.[3]

The aim of this Brief Report is to show that there exists an entire class of pulses described by the complex envelope $F(t)$, for which analytic solutions to (2) [and thus (4)] may be found. The chirped pulse of Hioe and the asymmetric pulses of Bambini and Berman belong to this class.

Let us combine (2a) and (2b) into a single, second-order differential equation for the lower-level amplitude $\alpha_1(t)$:

$$\ddot\alpha_1+[-i(\Delta+\dot g)-\dot f/f]\dot\alpha_1+f^2(\Omega_0^2/4)\alpha_1=0 \qquad (5)$$

(a similar equation can be written for α_2). We introduce the new variable $z=z(t)$ defined by

$$t=\tau/2\ln[z/(1-z)^{1+\lambda}], \quad \lambda>-1 \qquad (6)$$

which is a one-to-one mapping of the entire time axis on the [0,1] interval.[3] In (6) τ is the characteristic time scale in the problem. We make also the ansatz[3]

$$f(t)=2\frac{z^{1/2}(1-z)^{1/2}}{\lambda z+1} \ , \qquad (7)$$

in order to recover the results of Bambini and Berman in the limiting case $\dot g(t)=0$. The choice (7) implies

$$S=\Omega_0\int_{-\infty}^{+\infty}dt\,f(t)=\Omega_0\tau\pi \ , \qquad (8)$$

where S has the meaning of the unchirped-pulse area. In terms of the new variable z, Eq. (5) becomes

$$z(1-z)\alpha_1''+\left(\frac{1-i\Delta\tau}{2}-\frac{i\Delta\tau\lambda z}{2}-z-ig'z(1-z)\right)\alpha_1' +\tau^2(\Omega_0^2/4)\alpha_1=0 \ , \qquad (9)$$

where the prime denotes d/dz. We will be able to write down the general solution for the amplitude $\alpha_1(t)$ immediately if Eq. (9) has the form of the hypergeometric equation[2]

$$z(1-z)\alpha_1''+[c-(a+b+1)z]\alpha_1'-ab\alpha_1=0 \ . \qquad (10)$$

This requires the g' to be of the form

$$g'=\frac{\psi z+\phi}{z(1-z)} \ , \qquad (11)$$

where, since g' should be real, the same holds for ψ and ϕ.

In terms of the time variable, Eq. (11) yields

$$\dot{g}=\frac{2}{\tau}\left(\frac{\psi z+\phi}{\lambda z+1}\right) . \tag{11a}$$

The quantity $\omega_L+\dot{g}$ has the meaning of the instantaneous pulse frequency [compare (2) or (4)]; thus, $\dot{g}$ should vanish for maximum of $f(t)$ (as ω_L is supposed to be the carrier frequency). This requirement allows us to eliminate one of the parameters ψ, ϕ. For amplitude modulation (7) the maximum of $f(t)$ corresponds[3] to $z_{\max}=1/(2+\lambda)$. Therefore, (11a) may be expressed as

$$\dot{g}=\beta\frac{(2+\lambda)z-1}{\lambda z+1} , \tag{12}$$

where $\beta=2\psi(2+\lambda)\tau^{-1}$ is the amplitude of the chirp in the frequency units. As t goes from $-\infty$ to $+\infty$ (z changes from 0 to 1), $\dot{g}$ changes from $-\beta$ to β. For $\lambda=0$, Eqs. (7) and (12) become

$$f(t)=\operatorname{sech}(t/\tau) ,$$
$$\dot{g}(t)=\beta\tanh(t/\tau) , \tag{13}$$

reproducing the pulse of Hioe. Obviously, for $\beta=0$ we get $\dot{g}=0$, and the unchirped pulses of Bambini and Berman are recovered.

For τ fixed, the duration of the pulse T defined as the full width at half maximum of the amplitude modulation $f(t)$ strongly depends[3] on λ; a similar λ dependence exists for the time interval over which $\dot{g}$ changes significantly. To compare the pulses with equal durations one must, therefore, for each λ value, choose τ appropriately. This adjustment has been applied to plot the amplitude $f(t)$ and the frequency $\dot{g}(t)$ modulations for different values of λ in Figs. 1 and 2. We have chosen Ω_0 equal to 1 (in some frequency units) for $\lambda=0$, and for better comparison of curves changed Ω_0 with λ to assure equal areas S (8) under the amplitude-modulation curves ($\Omega_0=1/\tau$). Note that the chirp functions $\dot{g}$ have rather similar time dependences, whereas the shape of $f(t)$ changes appreciably with changing λ.

With $\dot{g}$ of the form (12), Eq. (9) becomes the hypergeometric equation with parameters a,b,c given by

$$a,b=(i\gamma_1\pm\gamma_2)/\pi, \quad c=\tfrac{1}{2}+i\gamma_3/\pi , \tag{14}$$

where for future purposes we introduced the short notation

$$\gamma_1=\frac{\pi\tau}{4}[\lambda\Delta+(2+\lambda)\beta] , \tag{14a}$$

$$\gamma_2=\frac{\pi\tau}{2}\{\Omega_0^2-[\lambda\Delta+(2+\lambda)\beta]^2/4\}^{1/2} , \tag{14b}$$

$$\gamma_3=\pi\tau(\beta-\Delta)/2 . \tag{14c}$$

Thus, the general solution for the probability amplitude α_1 reads[2]

$$\alpha_1=A_1\,{}_2F_1(a,b;c;z)+B_1z^{1-c}\,{}_2F_1(a-c+1,b-c+1;2-c;z) , \tag{15}$$

where ${}_2F_1(x)$ denotes the hypergeometric function. The second amplitude of interest, α_2, may be obtained with the help of Eq. (2b) by differentiation of (15). We can also make use of (3) to obtain the components of the Bloch vector. This way, analytic expressions for the atom transient response to the chirped pulse defined by Eqs. (6), (7), and (12) may be found.

We discuss in detail only the long-time properties of the solution. Assuming the initial conditions

$$\alpha_1(t=-\infty)=\alpha_1(z=0)=1 ,$$
$$\alpha_2(t=-\infty)=\alpha_2(z=0)=0 , \tag{16}$$

i.e., the atom which was in the lower state in the remote

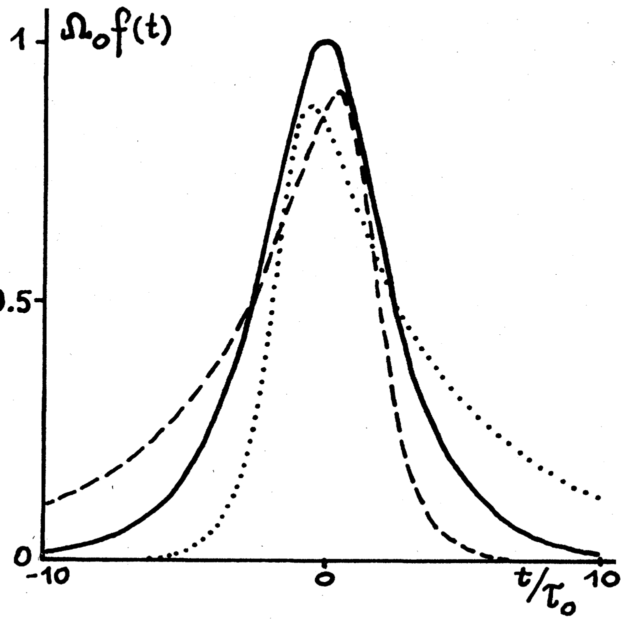

FIG. 1. Amplitude modulations $\Omega_0 f(t)$ as a function of time in units of τ_0, the value of τ for $\lambda=0$. Solid, dashed, and dotted lines correspond to $\lambda=0$, $\lambda=-0.8$, and $\lambda=5$, respectively. For further explanation, see text.

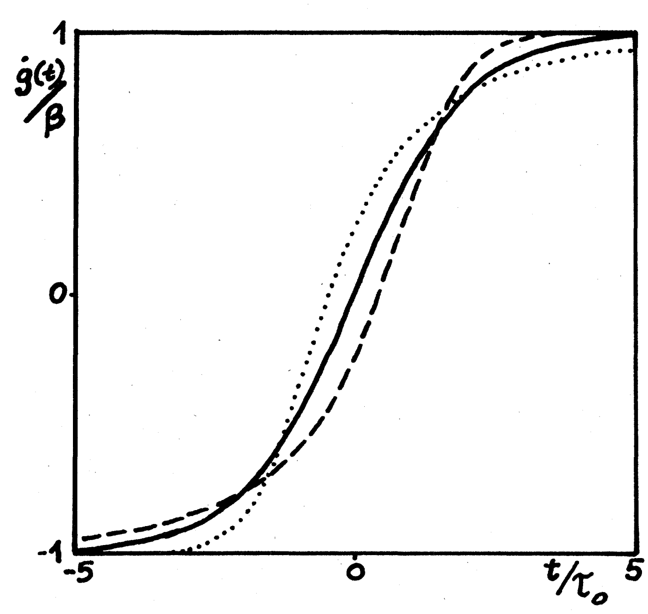

FIG. 2. Frequency modulations (chirp) $\dot{g}(t)/\beta$ for the same values of parameters as in Fig. 1.

past, we look at the probability for the atom to remain in its initial state: $P_1=\lim_{t\to\infty}|\alpha_1(t)|^2$. Alternatively, we could discuss the transition probability

$$P_2=\lim_{t\to\infty}|\alpha_2(t)|^2=1-P_1$$

or the long-time inversion $w(t=+\infty)=P_2-P_1$. With the initial conditions (16), the solution (15) becomes

$$\alpha_1(t)={}_2F_1(a,b;c;z) \quad . \tag{17}$$

The known properties of the hypergeometric function[2]

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b)>0 \tag{18a}$$

and the Euler gamma function[2]

$$\Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z)=\pi\sec(\pi z) \tag{18b}$$

allow us to express P_1 in terms of elementary functions:

$$P_1=\mathrm{sech}\gamma_3\,\mathrm{sech}(\gamma_3-2\gamma_1)\times[\cosh^2(\gamma_1-\gamma_3)\cos^2\gamma_2+\sinh(\gamma_1-\gamma_3)\sin^2\gamma_2] \quad . \tag{19}$$

To get the insight into the nature of the solution (19), let us discuss the special cases.

(i) $\lambda=\beta=0$. We obtain the Rosen-Zener[1] solution for the hyperbolic secant pulse. Then

$$\gamma_1=0,\quad \gamma_2=\pi\tau\Omega_0/2,\quad \gamma_3=-\pi\tau\Delta/2 \tag{20}$$

and

$$P_1=1-\mathrm{sech}^2(\pi\tau\Delta/2)\sin^2(\pi\tau\Omega_0/2) \quad , \tag{21}$$

and the behavior of P_1 is governed by the pulse area $S=2\gamma_2$ (8). The atom always returns to its initial state for $S=2\pi n$ regardless of the detuning. The notion of the pulse area plays an important role in the theory of light propagation through the resonant media. The famous self-induced transparency solution of McCall and Hahn[6] corresponds to $n=1$ (pulse with area $S=2\pi$). On the other hand, for $S=(2n-1)\pi$ the transition probability P_2 is maximal: $P_2=\mathrm{sech}^2(\pi\tau\Delta/2)$ and for $\Delta=0$ all atoms end up in the upper state.

(ii) *Case $\lambda=0$, the Hioe*[4] *chirped pulse.* In that case,

$$\gamma_1=\pi\tau\beta/2,\quad \gamma_2=\pi\tau(\Omega_0^2-\beta^2)^{1/2}/2 \tag{22}$$

and for $\Delta=0$, one obtains

$$P_1=\mathrm{sech}(\pi\tau\beta/2)\cos^2[\pi\tau(\Omega_0^2-\beta^2)^{1/2}/2] \quad ,$$

which is maximal for

$$\tilde{S}=\pi\tau(\Omega_0^2-\beta^2)^{1/2}=2\gamma_2=2\pi n \tag{23}$$

and $P_1=0$ for $\tilde{S}=(2n-1)\pi$. Thus, for the chirped pulse of Hioe the behavior of P_1 is governed by $\tilde{S}$ rather than by S. Hioe named $\tilde{S}$ the "effective area" of the chirped pulse. This notion allows for discussion of the chirped-pulse (13) solution, with the help of concepts borrowed from the propagation theory.

(iii) *Case $\beta=0$, the asymmetric pulses of Bambini and Berman.*[3] For $\beta=0$ one obtains (14):

$$\gamma_1=\pi\tau\lambda\Delta/4,\quad \gamma_2=\pi\tau(\Omega_0^2-\lambda^2\Delta^2/4)^{1/2}/2 \quad ,\quad \gamma_3=-\pi\tau\Delta/2 \quad . \tag{24}$$

P_1 may be expressed as

$$P_1=\mathrm{sech}\frac{\pi\tau\Delta}{2}\,\mathrm{sech}\frac{\pi\Delta(\lambda+1)}{2}\times\left[\sinh^2\frac{\pi\tau\Delta(\lambda+2)}{4}+\cos^2\frac{\pi\tau(\Omega_0^2-\lambda^2\Delta^2/4)^{1/2}}{2}\right] \quad , \tag{25}$$

and, only for $\Delta=0$, P_1 is governed by the area S (8). For $\Delta\neq 0$, P_i oscillates[7] between

$$P_{\max}=\mathrm{sech}\frac{\pi\tau\Delta}{2}\,\mathrm{sech}\frac{\pi\tau\Delta(1+\lambda)}{2}\cosh^2\frac{\pi\tau\Delta(\lambda+2)}{4}$$

and

$$P_{\min}=\mathrm{sech}\frac{\pi\tau\Delta}{2}\,\mathrm{sech}\frac{\pi\tau\Delta(1+\lambda)}{2}\sinh^2\frac{\pi\tau\Delta(\lambda+2)}{4} \quad .$$

Oscillations of P_1 are determined by

$$\gamma_2=\tfrac{1}{2}[S^2-(\pi\tau\lambda\Delta/4)^2]^{1/2} \quad , \tag{26}$$

which depends not only on the pulse area S but also on the asymmetry parameter λ as well as on the detuning Δ.

(iv) $\lambda,\beta\neq 0$. In the most general $\lambda,\beta\neq 0$ case the oscillatory behavior of P_1 (19) is governed by

$$\gamma_2=\frac{\pi\tau}{2}[\Omega_0^2-(\lambda\Delta+(2+\lambda)\beta)^2/4]^{1/2} \tag{27}$$

and depends on all pulse parameters Ω_0, β, λ and on the detuning Δ. Note, however, that for the chirp amplitude $\beta=-\lambda\Delta/(2+\lambda)$, P_1 (19) reduces to

$$P_1=1-\sin^2(\pi\tau\Omega_0/2)\,\mathrm{sech}^2\left[\frac{\pi\tau\Delta(1+\lambda)}{2(2+\lambda)}\right] \quad , \tag{28}$$

and oscillations of P_1 are determined by the area of the corresponding *unchirped* pulse. In that case, for arbitrary pulse shape the atom returns to its ground state whenever this area equals $2\pi n$. In other words, for a given chirped asymmetric pulse (λ,β fixed) there exists a detuning such that the tendency of the chirped pulse to invert the atomic population[8] is compensated by the asymmetric pulse shape.

The second interesting special case occurs when $\Delta=-\beta$. Then the instantaneous detuning $\Delta+\dot{g}$ vanishes at $t\to\infty$, i.e., the pulse instantaneous frequency $\omega_L+\dot{g}$ comes closer and closer to the atomic frequency ω_0. In that case

$$\gamma_1=\pi\tau\beta/2,\quad \gamma_2=\pi\tau(\Omega_0^2-\beta^2)^{1/2}/2,\quad \gamma_3=\pi\tau\beta \tag{29}$$

and the long-time population P_1 becomes completely independent of the shape of amplitude modulation $f(t)$, i.e., λ:

$$P_1=\mathrm{sech}(\pi\tau\beta)\left[\sinh^2\frac{\pi\tau\beta}{2}+\cos^2\frac{\pi\tau(\Omega_0^2-\beta^2)^{1/2}}{2}\right] \quad . \tag{30}$$

The oscillations of P_1 are governed now by $\pi\tau(\Omega_0^2-\beta^2)^{1/2}$, i.e., by the "effective area" $\tilde{S}$ (23) introduced by Hioe for the symmetric pulse (13) only. P_1 oscillates between

$$P_{\max}=\mathrm{sech}(\pi\tau\beta)\cosh^2(\pi\tau\beta/2)$$

and

$$P_{\min}=\mathrm{sech}(\pi\tau\beta)\sinh^2(\pi\tau\beta/2) \quad ,$$

being maximal for $\tilde{S}=2\pi n$ and minimal for $S=(2n-1)\pi$.

Therefore, the notion of the "effective area" may be useful also for chirped asymmetric pulses.[9] It is easily understood if we compare this result with the resonant, unchirped-pulse case $\Delta,\beta=0$. Then for arbitrary pulse shape the atomic evolution (and thus the long-time behavior also) is entirely governed by the pulse area.[8] For $\beta=-\Delta$, the chirped pulse is at long times practically resonant; thus long-time probability P_1 should not depend on the pulse shape but rather on the pulse "effective area."

We have so far discussed the "undamped" two-level system. Robiscoe[10] has shown that any nonresonant two-level problem soluble without damping can be easily treated so as to include damping caused by the escape out of the system, e.g., spontaneous decay of the upper level towards additional uncoupled levels. In such cases the important information about atomic evolution can be obtained from the energy spectra of spontaneously emitted photons. These spectra exhibit an interesting feature; they are multipeaked, with the number of peaks determined by the pulse area. For details see Lewenstein, Zakrzewski, and Rzażewski,[11] who discuss only the hyperbolic secant pulse, but the analytic solutions may be obtained for the entire class of pulses discussed here.[12] The damped Bloch equations[8] which may describe, e.g., spontaneous decay within the two-level system, are more difficult to handle, but some exact solutions have also been obtained.[13,14]

Finally, let us point out that the specific form of the time-variable transformation (6) is the only transformation which allows us to express Eq. (5) in the form of a hypergeometric equation [with $f(t)$ and $g(t)$ given by (7) and (12)], provided that we require the transformation to be a one-to-one mapping of the time axis on [0,1] interval. The proof may be carried out, e.g., in a manner similar to that proposed by Bambini and Berman[3] for the unchirped pulses.

To summarize, we have found the entire class of pulses having both amplitude and frequency modulations for which the analytic solution of the undamped two-state problem may be obtained (15). Analytical time pulses introduced before[3,4] belong to the class of pulses discussed here; thus in a sense we generalize the previous treatments of the problem. It turns out that for every unchirped asymmetric pulse[3] there exists its generalization in the form of a chirped pulse for which an analytic solution of the problem may be given. We found that the notion of "effective area" introduced by Hioe[4] for the symmetric pulse (13) is useful also for the discussed, more general class of pulses.

Since this paper was submitted I have learned that essentially the same results have been obtained independently and a bit earlier by F. Hioe and C. Carrol.[15]

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