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Construction of solvable Hill equations

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By extending the work of Casperson [Phys. Rev. A **30**, 2749 (1984); **31**, 2743(E) (1985)], a procedure to construct an analytically-solvable second-order linear differential equation with continuous periodic coefficients is found and presented here. An illuminating example with three-parameter coefficients is given which might be applicable to the problem of an electron in two-atom semiconductor materials.

I. INTRODUCTION

Many physical systems that involve periodic variations in time and space can be described, in their reduced form, by the Hill equation,

$$\frac{d^2y}{dx^2}+f(x)y=0\ , \tag{1}$$

where $f(x)$ is a periodic function of the independent variable x. The simplest form of $f(x)$ that is continuous in all orders of derivatives is the sinusoidal variation. The result is the Mathieu equation,

$$\frac{d^2y}{dx^2}+[a-2q\cos(2x)]y=0\ . \tag{2}$$

Equation (2) has been studied extensively, and the solutions are expressed in special functions.[1] In general, those functions are difficult to work with, not only analytically, but numerically. In most cases, approximate solutions of Eq. (1) are possible using perturbation or stationary-phase method if the periodic terms in $f(x)$ are small or fast varying.[2]

Recently, a class of solvable Hill equations with two-parameter continuous periodic functions was reported.[3] (A few applications of the Hill equation can be found in the literatures cited therein.) The periodic coefficient and the solution given by Casperson[3] are

$$f(x)=\frac{F}{[1+G\cos(2x)]^4}+\frac{4G\cos(2x)}{1+G\cos(2x)}\ , \tag{3}$$

$$y(x)=[1+G\cos(2x)]\cos[P(x)+a]\ , \tag{4}$$

where

$$P(x)=\frac{F^{1/2}}{2(1-G^2)}\left[\frac{G\sin(2x)}{1+G\cos(2x)}-\frac{2}{(1-G^2)^{1/2}}\times\tan^{-1}\left(\frac{(1-G^2)^{1/2}\tan x}{1+G}\right)\right]. \tag{5}$$

In Eqs. (3)–(5), a is an arbitrary constant; F and G are the independent parameters. This is the first Hill equation with a smooth coefficient $f(x)$ that can be solved exactly in terms of elementary functions. It is not a special case of the Mathieu equation and neither of the reverse. However, both equations are reduced to the same form in the limit of small q and G. The coefficient in Eq. (3) was pointed out to be applicable to the problem of an electron in a one-dimensional periodic potential with single or double wells.

It is then irresistible to raise the following questions: Is this the only type of solvable Hill equation? If not, what other forms of coefficient could be? How can we find them? How many independent parameters in the coefficient are allowed, etc? The purpose of this paper is to complete the classification of solvable Hill equations and to answer the questions mentioned above. It is concluded in this paper that one can construct as many solvable Hill equations as one can imagine. The constructing procedure is simple and straightforward.

II. THEORY

In general, the solution $y(x)$ is obtained by solving Eq. (1) for a given $f(x)$. Reversely, $f(x)$ can also be found if $y(x)$ is known,

$$f(x)=-\frac{1}{y(x)}\frac{d^2y(x)}{dx^2}\ . \tag{6}$$

It is required that $y(x)$ not be zero, such that $f(x)$ is meaningful, over an extensive region. Equation (6) implies that the forms of $f(x)$ can be as versatile as one can give for the periodic wave function, $y(x)$. For example, if $y(x)$ is chosen as $[1+G\cos(2x)]$, the coefficient becomes $4G\cos(2x)/[1+G\cos(2x)]$ which is identical to the second term in Eq. (3).

It is interesting to study if a class of solvable Hill equations can be evolved from a single given $y(x)$. These equations should describe the systems with similar wave functions. Assume a periodic function $g(x)$ is to be added to the coefficient $f(x)$ in Eq. (1). The equation becomes

$$\frac{d^2y(x)}{dx^2}+[g(x)+f(x)]y(x)=0\ . \tag{7}$$

The solution of Eq. (7) is assumed to be

$$y(x)=y_0(x)\exp[iy_1(x)]\ , \tag{8}$$

where $y_0(x)$ is the original solution satisfying Eq. (1) and $y_1(x)$ is the modification function to be solved. Both $y_0(x)$ and $y_1(x)$ are real functions. By substituting Eq. (8) into Eq. (7) and neglecting the common exponential factor,

$\exp[iy_1(x)]$, we have

$$\left[\frac{d^2y_0(x)}{dx^2}-y_0(x)\left(\frac{dy_1(x)}{dx}\right)^2+y_0(x)g(x)+y_0(x)f(x)\right]+i\left(y_0(x)\frac{d^2y_1(x)}{dx^2}+2\frac{dy_0(x)}{dx}\frac{dy_1(x)}{dx}\right)=0\ . \quad (9)$$

Since y_0 and y_1 are real, the real and imaginary part of Eq. (9) should be identical to zero, separately. Using Eq. (1), we have

$$y_0(x)\frac{d^2y_1(x)}{dx^2}+2\frac{dy_0(x)}{dx}\frac{dy_1(x)}{dx}=0\ , \quad (10)$$

$$\left(\frac{dy_1(x)}{dx}\right)^2=g(x)\ . \quad (11)$$

Equations (10) and (11) can be solved to obtain

$$y_1(x)=\pm F^{1/2}\int\frac{1}{y_0^2(x)}dx\ , \quad (12)$$

and

$$g(x)=\frac{F}{y_0^4(x)}\ , \quad (13)$$

where F is an arbitrary constant. Choosing a starting function as

$$y_0(x)=1+G\cos(2x)\ , \quad (14)$$

we find

$$y_1(x)=\pm F^{1/2}\int\frac{1}{[1+G\cos(2x)]^2}dx=\pm P(x)\ , \quad (15)$$

$$g(x)=\frac{F}{[1+G\cos(2x)]^4}\ , \quad (16)$$

which is exactly the equation and solution reported in Ref. 3. The new parameter F controls the magnitude of the new superposing periodic function $g(x)$.

III. SUMMARY

The general constructing procedure of analytically solvable Hill equations can now be summarized in a sequence of steps:

(A) Choose a suitable periodic wave function $y_0(x)$.

(B) Calculate the coefficient $f(x)$ following Eq. (6). If additional independent terms in the coefficient are desired, continue the following steps.

(C) Use Eqs. (12) and (13) to obtain the new coefficient $g(x)$ and the modification function $y_1(x)$.

(D) The new solvable Hill equation and its solutions are constructed according to Eqs. (7) and (8).

(E) Steps (C) and (D) can be repeated, if analytically possible, to obtain new coefficients and solutions with more independent parameters.

In contrast to the ordinary equation-solving technique, the constructing steps (A) and (B) lead to finding the equation by knowing the wave function in the first place. Because of the second-order derivative in Eq. (6), the form of the coefficient is usually more complicated than the wave function itself. However, if the construction proceeds to the steps (C) and (D), the new coefficient is obtained algebraically and the new solution is found through an indefinite integral. The form of the coefficient might be much simpler than the wave function, which is observed in most physical systems. Since step (C) involves an integration in Eq. (12), it is important to choose a function form such that the squared inverse of the function is analytically integrable.[4] This is the major difficulty found in repeating the steps (C) and (D).

IV. EXAMPLES

Except for the result given in Ref. 3, the procedure is demonstrated in the following two examples using a similar but slightly different starting function. The second example becomes the first solvable Hill equation with a three-parameter continuous coefficient.

(A) Choose the starting function as

$$y_0(x)=[1+G\cos(2x)]^{1/2}\ . \quad (17)$$

Following the procedure, we obtain the equation and the solution,

$$\frac{d^2y(x)}{dx^2}+\left(\frac{F+G^2\sin^2(2x)}{[1+G\cos(2x)]^2}+\frac{2G\cos(2x)}{1+G\cos(2x)}\right)y(x)=0\ , \quad (18)$$

$$y(x)=[1+G\cos(2x)]^{1/2}\cos[Q(x)+a]\ , \quad (19)$$

where

$$Q(x)=\int\frac{1}{1+G\cos(2x)}dx=\frac{F^{1/2}}{(1-G^2)^{1/2}}\tan^{-1}\left(\frac{(1-G^2)^{1/2}\tan x}{1+G}\right)\ . \quad (20)$$

The coefficient appearing in Eq. (18) exhibits second-order singularities when G is larger than one. In comparison to $f(x)$ in Eq. (3), which has fourth-order singularities, it represents a smoother potential variation.

(B) In this example, the starting function is

$$y_0(x)=[1+G\cos(2x)]^{1/2}[1+H\cos(2x)]^{1/2}\ , \quad (21)$$

which has two independent parameters, G and H. The equation and the solution are found following the procedure

$$\frac{d^2y(x)}{dx^2}\left[\frac{F}{[1+G\cos(2x)]^2[1+H\cos(2x)]^2}+\frac{2G\cos(2x)}{1+G\cos(2x)}+\frac{2H\cos(2x)}{1+H\cos(2x)}+\left(\frac{G}{1+G\cos(2x)}-\frac{H}{1+H\cos(2x)}\right)^2\sin^2(2x)\right]y(x)=0\ , \quad (22)$$

$$y(x)=[1+G\cos(2x)]^{1/2}[1+H\cos(2x)]^{1/2}\cos[R(x)+a]\ , \quad (23)$$

$$R(x)=\frac{1}{G-H}\{G[Q(x)]_{G=G}-H[Q(x)]_{G=H}\}\ . \quad (24)$$

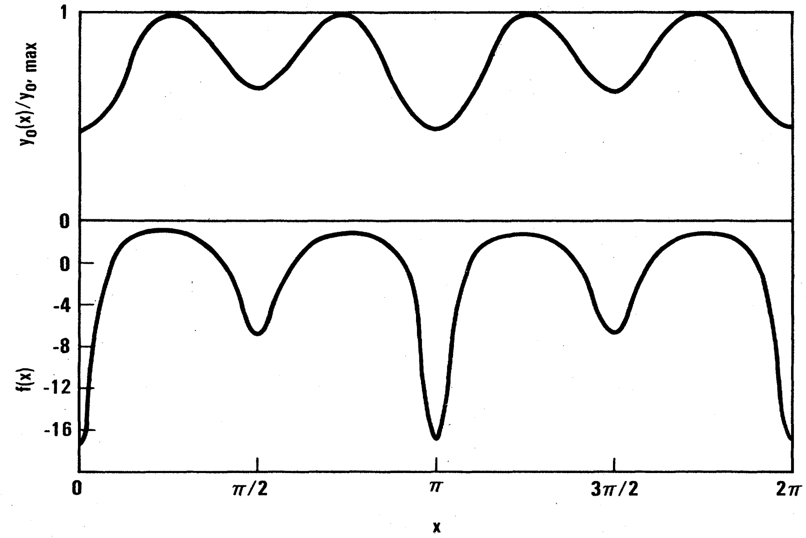

FIG. 1. Coefficient $f(x)$ and the corresponding wave function $y(x)$ for a three-parameter Hill equation with $F=0$, $G=0.8$, and $H=-0.9$ in Eq. (22).

Comparing the starting functions in Eqs. (14), (17), and (21), we found that the previous two examples are the special cases of the last one, with H equal to G and 0, respectively. When H is zero, it is easy to see that Eqs. (22)–(24) are reduced to Eqs. (18)–(20). When H is equal to G, the reduction of first two equations is obvious except for the function $R(x)$. The reduced form of $R(x)$ should be

$$R(x) \underset{H=G}{\longrightarrow} \frac{d[GQ(x)]}{dG} = Q(x) + G\frac{dQ(x)}{dG} . \qquad (25)$$

Using the integral form for $Q(x)$ in Eq. (20), we found that

$$R(x) \underset{H=G}{\longrightarrow} \int \frac{1}{[1+G\cos(2x)]^2}\,dx , \qquad (26)$$

which is identical to the form shown in Eq. (15).

The coefficient in Eq. (22) has three independent parameters and represents variations with two different amplitudes, if the absolute values of G and H are not equal. For example, we can choose $F=0$, $G=0.8$, and $H=-0.9$. The normalized solution and the coefficient in this case are shown in Fig. 1. It is interesting that $f(x)$ is similar to a periodic one-dimensional potential with alternating different interaction strengths. This might be applied to the problem of an electron in two-atom crystals, such as GaAs or other III-V and II-VI compounds.

V. CONCLUSION

A general procedure has been established for the construction of analytically solvable Hill equations. The result is applied to obtain a solvable equation with three-parameter continuous periodic coefficients. The equation can be applied to the problem of an electron in some two-atom compounds. Other choices of the starting function might also lead to useful types of Hill equations.

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