Interfering resonances and bound states in the continuum

H. Friedrich and D. Wintgen
Physik Department, Technische Universität München, D-8046 Garching, West Germany
(Received 24 June 1985)

We use Feshbach's theory of resonances to demonstrate that bound states in the continuum (BIC's) can occur due to the interference of resonances belonging to different channels. If two resonances pass each other as a function of a continuous parameter, then interference causes an avoided crossing of the resonance positions and for a given value of the continuous parameter one resonance has exactly vanishing width and hence becomes a BIC. The condition for a BIC relates the positions of the noninterfering resonances with the coupling matrix elements between the various channels. In the neighborhood of the BIC point one resonance remains anomalously narrow for a finite range of values of the separation of the noninterfering resonances. Whether or not two resonances interfere is not directly related to whether or not they overlap. All these results, including the occurrence of exactly bound states in the continuum, are not consequences of approximations inherent in Feshbach's theory but are general features of a coupled-channel Schrödinger equation with only one open channel. We illustrate the results in a simple but realistic model, where all matrix elements involved can be calculated analytically. We also discuss the case of coupled Coulombic channels where BIC's are caused by perturbations interfering with a Rydberg series of autoionizing resonances. Below the continuum threshold the analogy to a BIC is an infinitely narrow perturbation of the bound-state spectrum. Near such an infinitely narrow perturbation we may observe approximate level crossings.

I. INTRODUCTION

For a Hamiltonian consisting of the kinetic energy and a purely local potential, the solutions of the stationary Schrödinger equation generally fall into two distinct categories: Below a well-defined continuum threshold the energy eigenvalues are discrete and the corresponding eigenfunctions are square-integrable bound-state wave functions. Above the continuum threshold the eigenvalues are distributed continuously and the corresponding wave functions are, in general, not normalizable.[1] Above threshold, narrow resonances with large but finite lifetimes can occur if a slight modification of the Hamiltonian would lead to an effectively higher threshold. Examples of such long-lived resonances are metastable states trapped by a large potential barrier or quasibound states in closed channels of a system with weakly coupled channels.

In contrast to these almost bound states, there are, however, examples of square-integrable and exact solutions of the Schrödinger equation above the continuum threshold. These "bound states in the continuum" (BIC's) have infinite lifetimes and are not related to subthreshold states of an approximate Hamiltonian.

In one dimension, potentials which support a BIC can be explicitly constructed by a method due to von Neumann and Wigner,[2] as shown by Stillinger and Herrick[3] and discussed by Gazdy.[4] The potentials obtained in this way by direct construction depend sensitively on the wave function of the BIC and are characterized by weakly damped oscillations; it is difficult to envisage a real physical situation described by a one-dimensional potential exactly equivalent to such a von Neumann—Wigner construction.

Examples of BIC's are more readily found if we go beyond the one-dimensional Schrödinger equation. For a given correlated two-electron wave function, Stillinger and Herrick[3] have constructed a two-electron potential for which this wave function becomes a BIC. Again, the potential has to be explicitly constructed, but in contrast to the one-dimensional example it does not have any manifestly unrealistic properties. In this example, strong correlations in the two-electron wave function, and, hence, appreciable nonseparability of the Hamiltonian, are important for the occurrence of the BIC. Fonda and Newton[5,6] showed many years ago that BIC's can occur in a system of two coupled square-well potentials if the well depths and coupling strengths are adjusted appropriately.

The aim of the present paper is to demonstrate that the occurrence of BIC's' is a natural feature of the common physical situation that two resonances associated with different channels interfere.

In Sec. II we describe the theory of two interfering resonances as developed by Feshbach.[7,8] The basic result of this section is that, if we have the freedom of varying the separation of two (noninterfering) resonances as a function of a continuous parameter, then for a well-defined value of the continuous parameter one of the interfering resonances becomes a BIC.[9] The results of Sec. II are illustrated in a simple but realistic model in Sec. III. In Sec. IV we describe the adaptation of the theory of Sec. II to the case of coupled Coulombic channels and discuss the example of a hydrogen atom in a uniform magnetic field, where a first physically real example of BIC's caused by

the interference of resonances in different channels was reported.[10]

II. THEORY OF TWO INTERFERING RESONANCES

The theory of resonances caused by the coupling of bound states in closed channels to open channels was formulated comprehensively by Feshbach many years ago.[7,8] Although the theory of isolated resonances and of overlapping resonances is well established and has found widespread application, for example, in the field of nuclear reactions,[11] to our knowledge there exists as yet no investigation of the quantitative influence of the interference of resonances on their individual positions *and* widths. This influence can be very dramatic, but the theory is remarkably simple, at least for two interfering resonances.

Following the general procedure of Ref. 7, we start from a physical system described by channel wave functions $\Psi_i(r)$ depending on the radial coordinate r. The dynamics are determined by real local potentials $V_{ii}(r)$ and real coupling potentials $V_{ij}(r)=V_{ji}(r)$, $j\neq i$. The coupled-channel equations for the wave functions are

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+V_{ii}(r)\right]\Psi_i(r)+\sum_{j\,(\neq i)}V_{ij}(r)\Psi_j(r)=E\Psi_i(r)\ . \tag{1}$$

Let us now assume that only one channel, $i=0$, is open and that only two other (closed) channels, $i=A$ and $i=B$, are important. Let us further assume that the wave functions in the closed channels are dominated by bound-state wave functions:

$$\Psi_A(r)=A\phi_A(r),\quad \Psi_B(r)=B\phi_B(r)\ , \tag{2}$$

where ϕ_A and ϕ_B are normalized eigenfunctions of the respective uncoupled channel Hamiltonians:

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+V_{AA}\right]\phi_A=E_A\phi_A\ ,$$
$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+V_{BB}\right]\phi_B=E_B\phi_B\ . \tag{3}$$

These assumptions greatly simplify the coupled channel equations (1):

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+V_{00}(r)\right]\Psi_0(r)+AV_{0A}(r)\phi_A(r)+BV_{0B}(r)\phi_B(r)=E\Psi_0(r)\ ,$$
$$AE_A\phi_A(r)+BV_{AB}(r)\phi_B(r)+V_{0A}(r)\Psi_0(r)=EA\phi_A(r)\ , \tag{4}$$
$$BE_B\phi_B(r)+AV_{AB}(r)\phi_A(r)+V_{0B}(r)\Psi_0(r)=EB\phi_B(r)\ .$$

The approximations involved in formulating Eq. (4) are well justified in situations where just two individual resonances are important. In fact, an exact reduction of the coupled-channel equations (1) to the form (4) is possible using Feshbach's projection techniques.[12] In such an exact formulation, the coupling of the wave functions Ψ_0,ϕ_A,ϕ_B to that part of Hilbert space which no longer appears in Eq. (4) leads to smoothly energy-dependent effective potentials, which remain real if the excluded part of Hilbert space contains no open channels. If, however, the transition from Eq. (1) to Eq. (4) involves the elimination of open channels, then the effective potentials in Eq. (4) will have to account for the loss of flux into these channels and will, in general, be complex. For our present purposes it is important that the potentials in Eq. (4) are all real, and hence the following discussion implies that only channel 0 is open.

The last two equations in (4) can be simplified by taking matrix elements

$$A(E-E_A)=BM_{AB}+\langle\phi_A\,|\,V_{0A}\,|\,\Psi_0\rangle\ ,$$
$$B(E-E_B)=AM_{AB}+\langle\phi_B\,|\,V_{0B}\,|\,\Psi_0\rangle\ , \tag{5}$$

where the matrix element M_{AB} describes the direct coupling between the states ϕ_A and ϕ_B:

$$M_{AB}=\langle\phi_A\,|\,V_{AB}\,|\,\phi_B\rangle\ . \tag{6}$$

The Schrödinger equation for $\Psi_0(r)$ can be written as

$$\left[E+\frac{\hbar^2}{2M}\frac{d^2}{dr^2}-V_{00}(r)\right]\Psi_0(r)=AV_{0A}(r)\phi_A(r)+BV_{0B}(r)\phi_B(r) \tag{7}$$

and can be solved with the help of the Green's function

$$G\equiv\left[E+\frac{\hbar^2}{2M}\frac{d^2}{dr^2}-V_{00}\right]^{-1}$$

as follows:

$$\Psi_0(r)=\phi_0(r)+A\int_0^\infty G(r,r')V_{0A}(r')\phi_A(r')dr'+B\int_0^\infty G(r,r')V_{0B}(r')\phi_B(r')dr'\ , \tag{8}$$

where $\phi_0(r)$ is a solution of the uncoupled (i.e., homogeneous) equation in the open channel and obeys the following boundary conditions:

$$\phi_0(0)=0,\quad \phi_0(r\rightarrow\infty)=\sin\left[kr-\frac{l\pi}{2}+\delta_{\rm bg}\right]\ ; \tag{9}$$

here, $k=\sqrt{2ME}\,/\hbar$, and l is the orbital-angular-momentum quantum number. $\delta_{\rm bg}$ is a background phase shift due to the potential V_{00} in the open channel.

The form of Eq. (9) implies that the potentials in Eq. (4) are short ranged. The case of short-ranged potentials superimposed on a repulsive Coulomb potential can be treated by simply replacing the sine by the appropriate regular Coulomb function and (later) the cosine by the appropriate irregular Coulomb function. The interesting case of attractive Coulomb potentials is treated in Sec. IV.

Inserting the expression (8) for Ψ_0 into Eqs. (5) gives a completely determined set of two linear equations for the two unknown factors A and B:

$$A(E-E_A-G_{AA})=B(M_{AB}+G_{AB})+M_{A0}\,,$$
$$B(E-E_B-G_{BB})=A(M_{AB}+G_{AB})+M_{B0}\,. \tag{10}$$

Here, M_{A0} and M_{B0} are the matrix elements describing the direct coupling between the uncoupled open-channel wave function ϕ_0 and the states ϕ_A and ϕ_B, respectively:

$$M_{A0}=\langle\phi_A\,|\,V_{0A}\,|\,\phi_0\rangle,\quad M_{B0}=\langle\phi_B\,|\,V_{0B}\,|\,\phi_0\rangle\,. \tag{11}$$

The dynamic effect of the coupling of the channels enters through the matrix elements of the Green's function G:

$$G_{AA}=\langle\phi_A\,|\,V_{0A}GV_{0A}\,|\,\phi_A\rangle\,,$$
$$G_{BB}=\langle\phi_B\,|\,V_{0B}GV_{0B}\,|\,\phi_B\rangle\,, \tag{12}$$
$$G_{AB}=\langle\phi_A\,|\,V_{0A}GV_{0B}\,|\,\phi_B\rangle\,.$$

Equations (10) are merely a specialization to the case of two resonances of a more general set of equations for several overlapping resonances, as given in Ref. 7. For the case of two resonances, the equations are readily resolved for A and B. Using the abbreviations

$$\epsilon_A=E_A+G_{AA}(E),\quad \epsilon_B=E_B+G_{BB}(E)\,,$$
$$T_A=E-\epsilon_A,\quad T_B=E-\epsilon_B\,, \tag{13}$$

we obtain

$$A=\frac{T_BM_{A0}+(M_{AB}+G_{AB})M_{B0}}{T_AT_B-(M_{AB}+G_{AB})^2}\,,$$
$$B=\frac{T_AM_{B0}+(M_{AB}+G_{AB})M_{A0}}{T_AT_B-(M_{AB}+G_{AB})^2}\,. \tag{14}$$

For further discussion it is advantageous to take G to be the standing-wave Greens' function,

$$G(r,r')=-\frac{2M}{\hbar^2k}\phi_0(r_<)\phi_{\rm irr}(r_>)\,, \tag{15}$$

where ϕ_0 is the uncoupled regular open-channel wave function obeying the boundary conditions (9) and $\phi_{\rm irr}$ is the uncoupled open-channel wave function which is irregular at $r=0$ and behaves asymptotically as

$$\phi_{\rm irr}(r)\sim\cos\left[kr-\frac{l\pi}{2}+\delta_{\rm bg}\right]\,.$$

With this choice of G, all matrix elements in Eqs. (10) are real.

For large values of r the (coupled) open-channel wave function (8) has the form

$$\Psi_0(r)\sim\sin\left[kr-\frac{l\pi}{2}+\delta_{\rm bg}\right]+(\tan\delta)\cos\left[kr-\frac{l\pi}{2}+\delta_{\rm bg}\right]\,, \tag{16}$$

and the *additional phase shift* δ due to the coupling to the closed channels A and B is given by

$$\tan\delta=AM_{A0}+BM_{B0}\,. \tag{17}$$

From Eq. (14) we obtain

$$\tan\delta=-\frac{1}{E}\frac{T_BW_{A0}^2+T_AW_{B0}^2+2W_{A0}W_{B0}W_{AB}}{T_AT_B-W_{AB}^2}\,, \tag{18}$$

where we have redefined the coupling matrix elements

$$W_{A0}=\sqrt{k}M_{A0}=\sqrt{k}\,\langle\phi_A\,|\,V_{0A}\,|\,\phi_0\rangle\,,$$
$$W_{B0}=\sqrt{k}M_{B0}=\sqrt{k}\,\langle\phi_B\,|\,V_{0B}\,|\,\phi_0\rangle\,, \tag{19}$$
$$W_{AB}=M_{AB}+G_{AB}=\langle\phi_A\,|\,V_{AB}\,|\,\phi_B\rangle + \langle\phi_A\,|\,V_{0A}GV_{0B}\,|\,\phi_B\rangle\,;$$

all matrix elements W have the dimensions of an energy.

Equation (18) is an explicit equation for the additional asymptotic phase shift of the open-channel wave function due to the coupling of the open channel to the closed-channel wave functions ϕ_A and ϕ_B. All quantities on the right-hand side of Eq. (18) can be directly calculated from the solutions of the uncoupled equations. Note that the matrix elements containing the regular uncoupled open-channel wave function ϕ_0 and the matrix elements containing the Green's function G depend on the energy E. Since we are interested in the interaction of two resonances, we assume that the continuum in channel 0 does not contain any additional resonant structure, so the energy dependence of these matrix elements will be smooth. We now study in detail the implications of Eq. (18).

If the coupling matrix element W_{AB} is neglected, Eq. (18) becomes

$$\tan\delta=-(1/E)(W_{A0}^2/T_A+W_{B0}^2/T_B)\,. \tag{20}$$

Poles of the right-hand side of Eq. (20) occur when T_A or T_B have zeros, i.e., for

$$E=E_A+G_{AA}\quad\text{or}\quad E=E_B+G_{BB}\,. \tag{21}$$

Around these energies, the phase shift δ jumps more or less suddenly by π, i.e., we have resonances at energies

$$\epsilon_A^0=E_A+G_{AA}(\epsilon_A^0),\quad \epsilon_B^0=E_B+G_{BB}(\epsilon_B^0)\,, \tag{22}$$

which, due to the coupling of the respective closed channel A or B to the open channel, are shifted from the uncoupled eigenvalues E_A and E_B. The width Γ of a resonance is given by[1]

$$\Gamma=2\left[\frac{d\delta}{dE}\bigg|_{E=E_r}\right]^{-1}\,, \tag{23}$$

where E_r is the resonance position. Taking derivatives of Eq. (20) at $E=\epsilon_A^0$ and $E=\epsilon_B^0$ gives

$$\Gamma_A=(2/\epsilon_A^0)W_{A0}^2,\quad \Gamma_B=(2/\epsilon_B^0)W_{B0}^2\,. \tag{24}$$

This is, of course, the well-known formula for isolated Breit-Wigner resonances.[7]

Interference of the two resonances affects their positions and widths. Poles of the right-hand side of Eq. (18), i.e., resonances, occur when the denominator

$$D(E)=(E-\epsilon_A)(E-\epsilon_B)-W_{AB}^2 \tag{25}$$

has zeros; these zeros are

$$E_{1/2}=(\epsilon_A+\epsilon_B)/2\pm[(\epsilon_A-\epsilon_B)^2/4+W_{AB}^2]^{1/2} . \quad (26)$$

Considering that W_{AB}, ϵ_A, and ϵ_B depend (smoothly) on the energy E, Eq. (26) actually represents two transcendental equations for the new positions E_1 and E_2 of the interfering resonances. If the separation of the noninterfering resonance positions is much larger than $|W_{AB}|$, then interference has little effect on the resonance positions. For small values of $\epsilon_A-\epsilon_B$, inteference pushes the resonances apart, so that their separation is at least $2|W_{AB}|$. Thus interference leads to an avoided crossing of the resonance positions. The effect of the interference of the resonances on their widths is quite dramatic.

If we write

$$\tan\delta=-N(E)/D(E) , \quad (27)$$

with

$$N(E)=(T_B W_{A0}^2+T_A W_{B0}^2+2W_{A0}W_{B0}W_{AB})/E , \quad (28)$$

then the derivative of the phase shift at resonance, i.e.,for $D=0$, is simply

$$\left.\frac{d\delta}{dE}\right|_{D=0}=\frac{D'}{N} . \quad (29)$$

The widths Γ_1,Γ_2 of the resonances are given via Eq. (23):

$$\Gamma_1=\frac{1}{E_1}\left[W_{A0}^2+W_{B0}^2+\frac{(W_{A0}^2-W_{B0}^2)(\epsilon_A-\epsilon_B)/2+2W_{A0}W_{B0}W_{AB}}{[(\epsilon_A-\epsilon_B)^2/4+W_{AB}^2]^{1/2}}\right] ,$$
$$\Gamma_2=\frac{1}{E_2}\left[W_{A0}^2+W_{B0}^2-\frac{(W_{A0}^2-W_{B0}^2)(\epsilon_A-\epsilon_B)/2+2W_{A0}W_{B0}W_{AB}}{[(\epsilon_A-\epsilon_B)^2/4+W_{AB}^2]^{1/2}}\right] . \quad (30)$$

We have written the equations separately for Γ_1 and Γ_2 in order to remind the reader that the smoothly energy-dependent matrix elements W and energies ϵ_A,ϵ_B are to be taken at $E=E_1$ in the formula for Γ_1 and at $E=E_2$ in the formula for Γ_2.

In deriving Eqs. (24) and (30) the energy dependences of the various matrix elements have been neglected. If all energy dependences are taken into account exactly, the right-hand sides of Eqs. (24) have to be divided by $1-G'_{AA}$ and $1-G'_{BB}$, respectively, and the right-hand sides of Eqs. (30) have to be divided by

$$1-\frac{d}{dE}\left\{\frac{\epsilon_A+\epsilon_B}{2}\pm\left[\left[\frac{\epsilon_A-\epsilon_B}{2}\right]^2+W_{AB}^2\right]^{1/2}\right\} .$$

Because all energy dependences are smooth, these divisors are close to unity.

Equation (30) shows that the sum of the widths of the two interfering resonances is roughly equal to the sum of the noninterfering widths (24), because the quotients in the large parentheses in Eq. (30) approximately cancel. This result is well known from the theory of overlapping resonances.[7] Less well known is the fact that the distribution of the total width over the two resonances is very uneven and depends sensitively on the separation of the noninterfering resonances and on the coupling matrix elements W. The width of one of the resonances can, in fact, vanish exactly. From Eq. (29) and from the fact that $D(E)$, defined by Eq. (25), obviously has no multiple zeros, we see that an infinite derivative of $\delta(E)$ corresponding to vanishing width of a resonance occurs when a zero of $N(E)$ coincides with a zero of $D(E)$. This happens if

$$T_A=-\frac{W_{A0}}{W_{B0}}W_{AB} \text{ and } T_B=-\frac{W_{B0}}{W_{A0}}W_{AB} , \quad (31)$$

or, in other words,

$$E=\epsilon_A-\frac{W_{A0}}{W_{B0}}W_{AB}=\epsilon_B-\frac{W_{B0}}{W_{A0}}W_{AB} . \quad (32)$$

If Eq. (32) is fulfilled, both numerator and denominator in Eq. (18) [Eq. (27)] vanish simultaneously and the asymptotic phase shift becomes indeterminate because we have a bound state in the continuum.

Imagine a situation in which the separation of the uncoupled energy eigenvalues E_A,E_B depends on a continuous parameter such as the strength of an external field. At this point it is important to remember that ϕ_A and ϕ_B are states in different channels and that the uncoupled energies E_A,E_B are determined independently by their respective channel Hamiltonians. Thus the energies E_A and E_B may pass each other as the continuous parameter is varied. Then there is a well-defined value of the continuous parameter for which the BIC condition (32) is fulfilled.

If, for example, the energy dependence of ϵ_A, ϵ_B, W_{AB}, and the quotient W_{A0}/W_{B0} is negligible, then the BIC condition becomes

$$\epsilon_A^0-\epsilon_B^0=W_{AB}\left[\frac{W_{A0}}{W_{B0}}-\frac{W_{B0}}{W_{A0}}\right] , \quad (33a)$$

and the energy of the BIC is given by either of Eqs. (32). In the general case with all energy dependences included, the BIC condition is

$$E_A-E_B=G_{BB}-G_{AA}+W_{AB}\left[\frac{W_{A0}}{W_{B0}}-\frac{W_{B0}}{W_{A0}}\right] , \quad (33b)$$

with the matrix elements on the right-hand side taken at one of the resonance energies E_1 or E_2, depending on the relative sign of the coupling matrix elements (see below). The resonance energies depend on E_A and E_B via Eqs.

(13) and (26), and so Eq. (33b) involves a self-consistency condition. However, we can always fulfill (33b) if we have the freedom of varying $E_A - E_B$. Thus the occurrence of bound states in the continuum is a general feature of situations in which two resonances pass each other as a function of a continuous parameter.

Which of the two interfering resonances becomes a BIC when Eq. (32) is fulfilled depends on the sign of the product $W_{A0}W_{AB}W_{B0}$. Using Eqs. (32) and (33b) in Eqs. (26) and (30), we see that the resonance at E_2 has vanishing width Γ_2 if $W_{A0}W_{AB}W_{B0}>0$, and that the resonance at E_1 has vanishing width Γ_1 if $W_{A0}W_{AB}W_{B0}<0$. In other words, if the product $W_{A0}W_{AB}W_{B0}$ is positive, it is always the lower of the two interfering resonances which becomes a BIC; if the product is negative the upper resonance becomes a BIC.

It is interesting and important to see what happens if the BIC condition is almost fulfilled. Let E_n be the energy of the resonance which would be a BIC if Eq. (32) were exactly fulfilled (i.e., $E_n=E_1$ if $W_{A0}W_{AB}W_{B0}<0$, $E_n=E_2$ if $W_{A0}W_{AB}W_{B0}>0$), and let us assume that, at $E=E_n$, the BIC condition (33b) is almost fulfilled:

$$\epsilon_A-\epsilon_B=W_{AB}\left[\frac{W_{A0}}{W_{B0}}-\frac{W_{B0}}{W_{A0}}\right]+\epsilon\ . \tag{34}$$

Then, from (30), the width of the corresponding resonance is, to leading order in ϵ,

$$\Gamma_n=(2/E_n)\,|\,W_{A0}W_{B0}\,|\,\frac{\epsilon^2}{W_{AB}^2}\left|\frac{W_{A0}}{W_{B0}}+\frac{W_{B0}}{W_{A0}}\right|^{-3}\ . \tag{35}$$

Equation (35) shows that even if the BIC condition (33b) is not fulfilled exactly, anomalously small widths of one of the interfering resonances persist over a finite range of deviations ϵ in Eq. (34). In Eq. (35), $(2/E_n)\,|\,W_{A0}W_{B0}\,|$ is roughly equal to the geometric mean of the widths of the noninterfering resonances. If the matrix elements W_{A0} and W_{B0} are comparable in magnitude, then the range of values of E_A-E_B, over which the width of the narrow resonance is much smaller than the noninterfering widths, is of the order of $|\,W_{AB}\,|$. If the magnitudes of W_{A0} and W_{B0} are very different, then the "narrowing factor"

$$(\epsilon^2/W_{AB}^2)\,|\,W_{A0}/W_{B0}+W_{B0}/W_{A0}\,|^{-3}$$

in Eq. (35) will be anomalously small for a much larger range of values of ϵ.

Note that the formula (26) describing the avoided crossing of the two resonances does not contain the matrix elements W_{A0}, W_{B0} which determine the widths of the noninterfering resonances according to Eq. (24). Also, the BIC condition (32), (33b), and the narrowing factor in Eq. (35) depend only on the ratio of W_{A0} and W_{B0} and not on their absolute magnitudes. Thus the most important effects of the interference of the two resonances do not depend on the absolute widths of the (noninterfering) resonances.

If the widths of the noninterfering resonances are comparable, i.e., if the magnitudes of W_{A0} and W_{B0} are not very different, then according to Eq. (33) the BIC point occurs for relatively small separations of the noninterfering resonance positions, i.e., near the avoided crossing of the positions of the interfering resonances. However, if the widths of the noninterfering resonances are very different, the BIC point may occur far from the avoided crossing of the resonance positions.

If the widths of the noninterfering resonances are both small in comparison with the coupling matrix element $|\,W_{AB}\,|$, then the widths of the interfering resonances will always be smaller than their separation, which is at least $2\,|\,W_{AB}\,|$. Thus the resonances never overlap, even though the effects of their intereference are just as dramatic as if their original (noninterfering) widths were much larger than $|\,W_{AB}\,|$. Whether or not two resonances interfere strongly does not depend on their overlapping. We may have strong interference effects in a situation where the widths of the resonances are much smaller than their separation; conversely, if, e.g., $|\,W_{A0}\,|$ and $|\,W_{B0}\,|$ are large but $|\,W_{AB}\,|$ is small, we may have very little interference between two resonances which overlap strongly.

III. A SIMPLE MODEL

In order to illustrate the effects discussed in Sec. II, this section presents a model calculation for a simple but realistic system of interfering resonances.

The system consists of three coupled s-wave channels with the following diagonal potentials:

$$V_{00}(r)=0\ ,$$

$$V_{AA}(r)=I_A-\frac{\hbar^2}{2M}\frac{\alpha_2^2-\alpha_1^2}{e^{(\alpha_2-\alpha_1)r}-1},\quad 0<\alpha_1<\alpha_2\ , \tag{36}$$

$$V_{BB}(r)=I_B-\frac{\hbar^2}{2M}\frac{\beta_2^2-\beta_1^2}{e^{(\beta_2-\beta_1)r}-1},\quad 0<\beta_1<\beta_2$$

and with the coupling potentials

$$V_{0A}(r)=v_{0A}e^{-\alpha r},\quad V_{0B}(r)=v_{0B}e^{-\beta r}\ ,$$
$$V_{AB}(r)=v_{AB}e^{-\gamma r}\ , \tag{37}$$

with constant parameters v_{0A}, v_{0B}, and v_{AB}.

The potentials V_{AA} and V_{BB} are proportional to $1/r$ near the origin and approach the respective inelastic thresholds I_A and I_B exponentially for large r. The uncoupled ground-state wave functions ϕ_A and ϕ_B in these potentials are simple differences of exponentials:

$$\phi_A(r)=N_A(e^{-\alpha_1 r}-e^{-\alpha_2 r}),\quad \phi_B(r)=N_B(e^{-\beta_1 r}-e^{-\beta_2 r})\ , \tag{38}$$

with normalization constants N_A, N_B given, e.g., by

$$N_A=[2\alpha_1\alpha_2(\alpha_1+\alpha_2)]^{1/2}/(\alpha_2-\alpha_1)\ .$$

The uncoupled energy eigenvalues E_A and E_B of the ground states (38) are

$$E_A=I_A-\frac{\hbar^2}{2M}\alpha_1^2,\quad E_B=I_B-\frac{\hbar^2}{2M}\beta_1^2\ . \tag{39}$$

The regular uncoupled open-channel wave function ϕ_0 is

$$\phi_0(r)=\sin(kr),\quad \hbar^2k^2=2ME\ , \tag{40}$$

and the open-channel Green's function is simply the free s-wave Green's function

$$G(r,r')=\begin{cases}-(2M/\hbar^2k)\sin(kr)\cos(kr'), & r<r'\\ -(2M/\hbar^2k)\cos(kr)\sin(kr'), & r>r'\ .\end{cases} \tag{41}$$

We have studied the three-channel problem defined by the above potentials by two different methods:

(i) Direct numerical integration of the coupled-channel equations (1).

(ii) Evaluation of the analytic formulas based on Eq. (18), which was derived from Eq. (4) and hence involves the approximation that each closed channel is dominated by the single bound state ϕ_A or ϕ_B.

Comparing the results of the "exact" solution (i) with the one-state-per-closed-channel approximation (ii) gives an estimate of the importance of the influence of the other states in the channels A and B.

With the above choice of the potentials [Eqs. (36) and (37)], all matrix elements appearing in the analytic formulas of method (ii) can be evaluated analytically: The matrix element $W_{A0}=\sqrt{k}\,\langle\phi_A\,|\,V_{0A}\,|\,\phi_0\rangle$, which describes the direct coupling of the state in channel A with the open channel, is

$$W_{A0}=N_Av_{0A}k^{3/2}(\alpha_2-\alpha_1)(\alpha_1+\alpha_2+2\alpha)[k^2+(\alpha+\alpha_1)^2]^{-1}\times[k^2+(\alpha+\alpha_2)^2]^{-1}\ . \tag{42}$$

W_{B0} is given by an analogous expression. The energy-independent matrix element $M_{AB}=\langle\phi_A\,|\,V_{AB}\,|\,\phi_B\rangle$, which describes the direct coupling between the states in channel A and B, is

$$M_{AB}=N_AN_Bv_{AB}[(\alpha_1+\beta_1+\gamma)^{-1}+(\alpha_2+\beta_2+\gamma)^{-1}-(\alpha_1+\beta_2+\gamma)^{-1}-(\alpha_2+\beta_1+\gamma)^{-1}]\ . \tag{43}$$

The matrix elements involving the Green's function (41) are most conveniently expressed in terms of the (energy-dependent) function

$$F(x,y)=(xy-k^2)[(k^2+x^2)(k^2+y^2)(x+y)]^{-1}\ . \tag{44}$$

With this definition of F, the energy shift

$$G_{AA}=\langle\phi_A\,|\,V_{0A}GV_{0A}\,|\,\phi_A\rangle$$

in the position ϵ_A of the (noninterfering) channel-A resonance is

$$G_{AA}=-\frac{2M}{\hbar^2}N_A^2v_{0A}^2[F(\alpha+\alpha_1,\alpha+\alpha_1)+F(\alpha+\alpha_2,\alpha+\alpha_2)-2F(\alpha+\alpha_1,\alpha+\alpha_2)]\ , \tag{45}$$

and there is an analogous expression for G_{BB}. The matrix element

$$G_{AB}=\langle\phi_A\,|\,V_{0A}GV_{0B}\,|\,\phi_B\rangle\ ,$$

which describes the indirect coupling via channel zero of the states in channels A and B, is

$$G_{AB}=-\frac{2M}{\hbar^2}N_AN_Bv_{0A}v_{0B}\times[F(\alpha+\alpha_1,\beta+\beta_1)+F(\alpha+\alpha_2,\beta+\beta_2)-F(\alpha+\alpha_1,\beta+\beta_2)-F(\alpha+\alpha_2,\beta+\beta_1)]\ . \tag{46}$$

For explicit calculations we have taken $\hbar^2/2M$ to be unity. This means that k, α, β, etc. are all in units of an (arbitrary) inverse length and all energies are in units of the square of this inverse length. The radial coordinate r will always be given in units of the corresponding length.

The potential parameters we used are summarized in Table I. The relative positions of the thresholds of channels A and B (and hence of the uncoupled energy eigenvalues E_A and E_B) are considered variable. Figure 1 shows the uncoupled channel potentials V_{AA} and V_{BB} (solid curves), together with the uncoupled ground-state wave functions ϕ_A and ϕ_B (dashed curves). With our choice of potential parameters the uncoupled channel potentials V_{AA} and V_{BB} each support only one bound state. Calculations were performed with different signs of the coupling strength v_{0B} in order to illustrate the dependence of the interference effects on the relative signs of the coupling matrix elements.

We now keep the mean position $\frac{1}{2}(E_A+E_B)$ of the uncoupled energy eigenvalues fixed (at the value 7.0) and vary their separation in order to study the interference of the two resonances. The positions and widths of the resonances are plotted as functions of E_A-E_B in Fig. 2. The lower two parts of Fig. 2 correspond to different signs of v_{0B} and illustrate the strong effect of the relative signs of the coupling matrix elements on the widths of the interfering resonances. Changing the sign of v_{0B} also affects the positions of the interfering resonances because it changes the sign of G_{AB} but not of M_{AB} in the formula for the coupling matrix element W_{AB} [see Eq. (19)]. However, this effect is negligible in the present case, be-

TABLE I. Parameters defining the potentials (36) and (37) of the simple three-channel model (the range parameters α_1, α_2, α, β_1, etc., are all in units of an arbitrary inverse length and the coupling strengths v_{0A}, v_{0B}, and v_{AB} are in units of the square of this inverse length).

α_1	α_2	α	β_1	β_2	β	γ	v_{0A}	v_{0B}	v_{AB}
1.5	2.5	1.3	2.0	3.0	1.5	1.4	−1.0	±1.0	−1.0

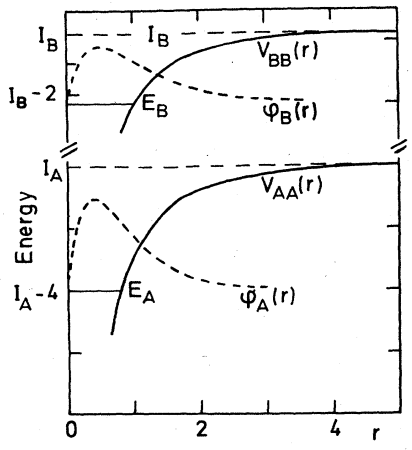

FIG. 1. Diagonal channel potentials $V_{AA}(r)$ and $V_{BB}(r)$ (solid lines) as defined by Eq. (36) with the parameters of Table I. The uncoupled energy eigenvalues E_A and E_B are shown relative to the respective channel thresholds I_A and I_B, and the corresponding uncoupled ground-state wave functions $\phi_A(r)$ and $\phi_B(r)$ are shown as dashed lines.

cause $|G_{AB}|$ is much smaller than $|M_{AB}|$. Throughout the present calculations the matrix elements (12) involving the Green's function are small in comparison with the coupling matrix elements W.

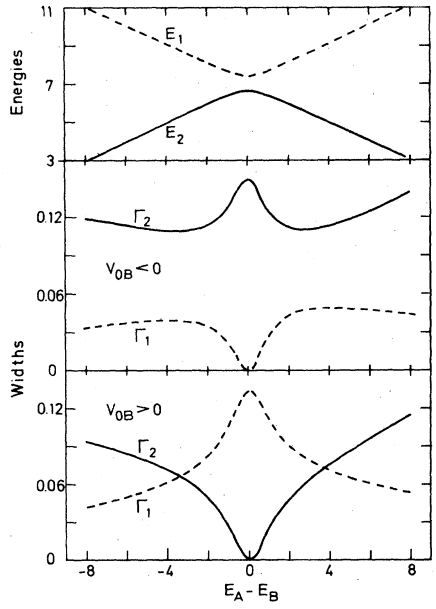

FIG. 2. Top part of the figure shows the positions E_1 and E_2 of the interfering resonances as functions of the separation E_A-E_B of the uncoupled energies of the closed-channel states. The energies obtained with the two different signs of the coupling strength v_{0B} agree to within the thickness of the lines. The bottom part of the figure shows the widths Γ_1 and Γ_2 of the intefering resonances for the two different choices of the sign of v_{0B}.

As shown in the top part of Fig. 2, the interference of the resonances causes an avoided crossing of the resonance positions. In the lower part of the figure we see that, for a given value of E_A-E_B, which lies close to zero in the present case, one of the resonance widths vanishes giving a BIC, while the other increases in order to keep the sum of the widths roughly constant. The case $v_{0B}<0$ corresponds to $W_{A0}W_{AB}W_{B0}<0$, and hence the upper resonance becomes a BIC, while $v_{0B}>0$ corresponds to $W_{A0}W_{AB}W_{B0}>0$ and the lower resonance becomes a BIC.

Figure 3 shows the asymptotic phase shift $\delta(E)$ as a function of energy for two fixed values of the separation E_A-E_B of the uncoupled energy eigenvalues. For $E_A-E_B=-1.5$ we see two isolated resonances of comparable width and, although there is some interference, its effects are not dramatic. For $E_A-E_B=0$ we are close to the avoided crossing and also, in the present example, to the BIC point. Here interference effects keep the actual resonances apart and lead to a concentration of almost all of the total width in the upper resonance; the lower resonance is anomalously narrow. Note that although interference effects are dramatic in the right half of Fig. 3, the resonances are still "isolated," because the noninterfering widths are smaller than $|W_{AB}|$ in the present example. This illustrates the point made in Sec. II that interfering resonances need not neceesarily be overlapping resonances.

On the scale of Fig. 3 the difference between the exact phase shifts obtained by numerical solution of the coupled-channel equations and the results obtained from the analytic formula (18) is smaller than the thickness of the lines. In order to illustrate the difference of the two methods, Fig. 4 shows—on an expanded scale—the width of the narrow resonance in the vicinity of the point of vanishing width (for the case $v_{0B}>0$). The dashed curve shows the widths derived from the analytic formulas in Sec. II and the solid curve shows the exact results obtained by numerical integration of the coupled-channel equations (1). A comparison of the two curves in Fig. 4 shows that the analytic formulas based on the one-state-per-closed-channel approximation indeed reproduce the results of the exact calculation very well. Both curves

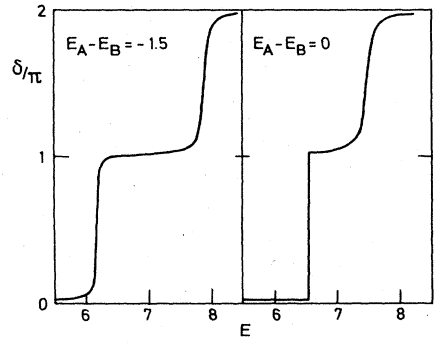

FIG. 3. Asymptotic phase shift $\delta(E)$ of the open-channel wave function for two different values of the separation E_A-E_B of the uncoupled energies of the closed-channel states (here, $v_{0B}>0$).

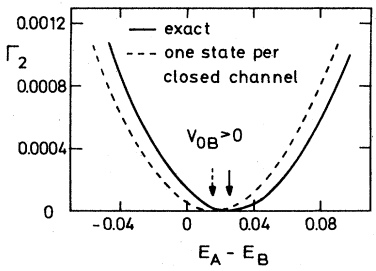

FIG. 4. Width of the narrow resonance in the vicinity of the BIC point. Here, $v_{0B}>0$, so the lower resonances is the narrow one. The dashed curve shows the width obtained from the analytic formulas derived in Sec. II and the solid line shows the exact results obtained by numerical integration of the coupled-channel equations. In each case the BIC point is marked by an arrow.

show a point of vanishing width (in the numerical calculation the width at this point was less than 10^{-8}) and the same quadratic dependence of the narrow width on the continuous parameter E_A-E_B in the vicinity of this BIC point [compare Eq. (35)]. The essential difference between the two calculations is a small shift of the BIC point which lies at $E_A-E_B=0.025$ in the exact calculation and at $E_A-E_B=0.016$ from the analytic formulas of Sec. II.

The deviation of the dashed curve in Fig. 4 from the solid curve reflects the effect of neglecting the excited (continuum) states in the closed channels in the present three-channel model. Incorporating the effects of these states into the reduced version (4) of the coupled-channel equations (1) leads to a smooth energy dependence of the potentials in Eq. (4), as would the consideration of the effects of other closed channels. All the results derived in Sec. II are valid in the presence of such a (smooth) energy dependence of the potentials in Eq. (4). It is important to realize that the results derived in Sec. II, including the occurrence of exactly bound states in the continuum, are not restricted to the one-state-per-closed-channel approximation or to a three-channel system, but are quite general features of interfering resonances in a multichannel system with only one open channel.

IV. COULOMBIC CHANNELS

An interesting variation of the situation described in Sec. II occurs when the diagonal channel potentials $V_{ii}(r)$ behave asymptotically (large r) as

$$V_{ii}(r)\sim -e^2/r \ . \tag{47}$$

In this case a resonance from one channel may interfere simultaneously with a whole Rydberg series of resonances in another channel. The fact that interference in coupled Coulombic channels can and does lead to bound states in the continuum was recently shown in Ref. 10. In the absence of channel coupling the eigenstates of the Hamiltonian form uncoupled Rydberg series with energy eigenvalues $E_{n,i}$ in each channel i converging to the channel threshold I_i:

$$E_{n,i}=I_i-\mathscr{R}/(n-\mu_{n,i})^2 \ , \tag{48}$$

where $\mathscr{R}=Me^4/2\hbar^2$ is the Rydberg energy and $\mu_{n,i}$ form, for each i, a convergent series of weakly n-dependent quantum defects which are due to the short-ranged deviations of the channel potentials from a pure Coulomb potential. In a given channel the quantum defects μ_n of the bound states below threshold are intimately connected with the asymptotic phase shifts of the continuum wave functions above threshold, as was first pointed out by Seaton.[13]

In the presence of channel coupling, ordinary bound states exist only below the lowest channel threshold I_1. States lying above I_1 become resonances and can decay by autoionization. In the case of comparatively weak coupling, the states associated with higher channels form Rydberg series of autoionizing resonances converging to the respective threshold. If a state associated with a higher channel happens to lie below the lowest threshold, it causes a pseudoresonant perturbation in the quantum defects of the Rydberg series of bound states.[14]

Coupled Coulombic channels are comprehensively described by "multichannel quantum-defect theory" (MQDT), which has been developed by Seaton[15] and by Fano and collaborators.[16] Here we use a representation recently introduced by Giusti-Suzor and Fano,[17] where the basic equation of MQDT reads

$$\det|\tan[\pi(\nu_i+\mu_i)]+R_{ij}|=0 \ . \tag{49}$$

The elements R_{ij} define a real symmetric coupling matrix with no diagonal elements. The μ_i describe the diagonal effects of the short-ranged deviations from the case of uncoupled pure Coulomb potentials. If all important channels are included, the μ_i and R_{ij} depend only weakly on energy. In closed channels i, the quantity ν_i in Eq. (49) stands for the effective quantum number and is related to the energy E by

$$E-I_i=-\mathscr{R}/\nu_i^2 \ . \tag{50}$$

In the absence of channel coupling, Eq. (49) yields uncoupled Rydberg series with energy eigenvalues in the ith channel determined by

$$\tan[\pi(\nu_i+\mu_i)]\equiv T_i(E)=0 \ . \tag{51}$$

If channel 1 is open, then $\pi\mu_1$ represents a smooth background phase shift of the wave function and $\delta=-\pi\nu_1-\pi\mu_1$ is the additional phase shift due to channel-coupling effects.

For three coupled Coulombic channels, channel 1 open and channels 2 and 3 closed, Eq. (49) becomes

$$\tan\delta=-\frac{R_{12}^2T_3+R_{31}^2T_2-2R_{12}R_{23}R_{31}}{T_2T_3-R_{23}^2} \ , \tag{52}$$

with T_2 and T_3 as defined in Eq. (51). Equation (52) bears a remarkable resemblance to (18) describing the open-channel phase shift due to interfering resonances. A superficial difference between Eqs. (52) and (18) is that all quantities in Eq. (52) are dimensionless. The essential difference is that the zeros of $T_A(E)$ and $T_B(E)$ in Eq. (18) define the positions of just two noninterfering reso-

nances [see Eq. (13)], whereas the zeros of $T_2(E)$ and $T_3(E)$ in Eq. (52) define whole Rydberg series of (noninterfering) resonances. The Coulombic generalization of Eq. (24) for the widths of the noninterfering resonances is, e.g., in channel 2,

$$\Gamma_n/\mathscr{R}=\frac{4R_{12}^2}{\pi}\frac{1}{\nu_2^3}\ . \tag{53}$$

Note that the autoionization widths decrease with the inverse cube of the effective quantum number as we approach the series limit, i.e., at the same rate as the separation of successive resonances.

As in the case of only two resonances described in Sec. II, interference of the resonances associated with two different Rydberg series has a dramatic effect on the widths of the resonances. Infinite derivatives of the phase shift corresponding to exactly bound states in the continuum occur when both numerator and denominator of the quotient on the right-hand side of Eq. (52) vanish simultaneously. In analogy to Eq. (31), the BIC condition is now

$$T_2(E)=R_{23}R_{12}/R_{31},\quad T_3(E)=R_{23}R_{31}/R_{12}\ . \tag{54}$$

Each of Eqs. (54) taken separately defines a Rydberg series of energies which, due to channel coupling, are somewhat shifted from the positions of the noninterfering resonances as defined by Eq. (51). Each branch of the tangent function defines one (shifted) resonance position. The BIC condition (54) is fulfilled whenever two shifted resonance positions belonging to different channels coincide. If we have a continuous parameter to vary the separations of the thresholds I_2 and I_3 of the closed channels, then the resonances of the upper channel 3 wander through the Rydberg series of autoionizing resonances associated with channel 2, and a BIC will occur whenever one resonance from channel 3 passes one of the resonances associated with channel 2.

An interesting situation occurs when a resonance associated with channel 3 lies at or just below the second threshold I_2. Since the width of this resonance is finite and the separations (and widths) of resonances converging to I_2 become smaller and smaller near the series limit, this one resonance of channel 3 will simultaneously interfere with a large, in fact, infinite, number of resonances associated with channel 2. Now the BIC condition (54) will, in general, be fulfilled for at most one pair of resonances, but approximate fulfillment of the BIC condition may lead to anomalously narrow autoionization widths for a whole series of resonances converging to I_2.

Approximate fulfillment of the BIC condition (54) can be expressed as

$$T_2(E)=R_{23}R_{12}/R_{31},\quad T_3(E)=R_{23}R_{31}/R_{12}+\lambda\ , \tag{55}$$

where λ is small compared with unity. The expression analogous to Eq. (35) for the width Γ_n of the narrow resonance in the vicinity of a BIC point is then

$$\frac{\Gamma_n}{\mathscr{R}}=\frac{4}{\pi\nu_2^3}\lambda^2R_{12}^2\left[R_{23}^4+\frac{R_{23}^2R_{31}^2}{R_{12}^2}+\left[\frac{\nu_3}{\nu_2}\right]^3R_{23}^2\left[\frac{R_{23}^2R_{31}^2}{R_{12}^2}+1\right]\right]^{-1} \tag{56}$$

and depends quadratically on λ. Near the second threshold, i.e., for $\nu_2\rightarrow\infty$, we obtain

$$\frac{\Gamma_n}{\mathscr{R}}=\frac{4R_{12}^2}{\pi\nu_2^3}\frac{\lambda^2}{R_{23}^4+R_{31}^2R_{23}^2/R_{12}^2}\ , \tag{57}$$

i.e., the width is reduced by a factor

$$\lambda^2(R_{23}^4+R_{31}^2R_{23}^2/R_{12}^2)^{-1}$$

compared with the widths expected for an unperturbed Rydberg series of resonances. The range of values of λ over which the widths of the perturbed series of resonances is anomalously narrow is at least of the order of R_{23}^2. This translates into a finite-energy range via the derivative $dE/d\lambda$ defined by

$$\tan[\pi(\nu_3+\mu_3)]=R_{23}R_{31}/R_{12}+\lambda\ , \tag{58}$$

but the magnitude of the energy range depends strongly on the value of the effective quantum number ν_3 at the second threshold I_2. Close enough to threshold, however, interference due to one perturber from channel 3 can always lead to a Rydberg series of anomalously narrow autoionizing resonances.

One physically real situation, where the perturbation of a Rydberg series of autoionizing resonances leads to BIC's, is, as recently shown in Ref. 10, the hydrogen atom in a uniform magnetic field. For a large range of field strengths B, this system is accurately described by the one-electron Hamiltonian[18]

$$H=\frac{p^2}{2m_e}-\frac{e^2}{r}+\omega l_z+\frac{m_e}{2}\omega^2(x^2+y^2)\ , \tag{59}$$

where $\omega=eB/2m_ec$ is half the cyclotron frequency. The azimuthal quantum number m and parity π are good quantum numbers; in each m^π subspace we have a system of coupled channels defined by the Landau quantum number N for the motion of the electron perpendicular to the direction of the field, while the motion parallel to the field is governed by Coulombic potentials. For field strengths near or greater than

$$B_0=m_e^2e^2c/\hbar^3\simeq 2.35\times10^9\ \text{G}\ ,$$

the energy between the thresholds of successive Landau channels is greater than the Rydberg energy and we obtain energetically well-separated Rydberg series of bound states ($N=0$) and autoionizing resonances ($N>0$). As the field strength is reduced, the various Rydberg series begin to overlap and interfere.[19]

Figure 5 shows the behavior of the energies and widths of the lowest eight Landau excited states in the $m^\pi=2^+$ subspace at field strengths at which this series of $N=1$ resonances is being perturbed by the lowest state of the $N=2$ series. The top half of the figure shows the energies

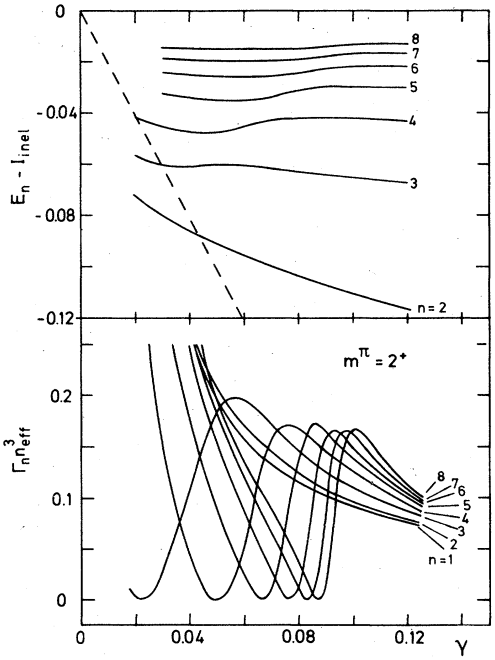

FIG. 5. Energies and widths (in Rydberg units) of the lowest eight Landau excited states in the $m^\pi = 2^+$ subpsace as functions of the magnetic-field-strength parameter $\gamma = B/B_0$. The energies (top half) are shown relative to the inelastic threshold which lies $2\gamma\mathscr{R}$ above the ionization threshold (dashed line). The widths (bottom half) are multiplied by the cube of the effective quantum number n_{eff} defined by Eq. (60). The results in this figure were obtained by solving ten coupled Landau channel equations.

of the resonances relative to the first Landau threshold (the inelastic threshold) I_{inel}, which lies $2\hbar\omega = 2(B/B_0)\mathscr{R}$ above the ionization threshold. Below the ionization threshold (dashed line), the energies correspond to the positions of pseudoresonant perturbations of the Rydberg series of bound states. The bottom half of the figure shows the widths multiplied by the cube of the respective effective quantum numbers n_{eff} defined by

$$E_n - I_{\text{inel}} = -\mathscr{R}/n_{\text{eff}}^2 . \tag{60}$$

In an unperturbed Rydberg series of autoionizing resonances, $\Gamma_n n_{\text{eff}}^3$ is essentially independent of n [see Eq. (53)].

Figure 5 clearly shows how, as the perturber wanders downward in the Rydberg series of autoionizing resonances with decreasing field strength, one resonance after another is slightly pushed down in energy and becomes a BIC for a given value of the magnetic field strength. The effect that a perturber close to the inelastic threshold causes several anomalously narrow widths simultaneously can also be seen in Fig. 5, e.g., at $B = 0.08B_0$ the widths of the $n=6$, 7, and 8 states are very much smaller than the widths expected from the $\Gamma_n n_{\text{eff}}^3$ scaling law.

The effect described here, namely the suppression of autoionization due to perturbation of a Rydberg series of autoionizing resonances, has recently been observed experimentally by Neukammer *et al.*[20] in the $[5d_{3/2}nd_{3/2}]_{J=0}$ series in barium, which is perturbed around $n=30$ by the $[5d_{5/2}13d_{5/2}]_{J=0}$ state. Although there is no continuous parameter in this system, the density of states is already so high near $n=30$ that the BIC condition (33b) is almost fulfilled for some states. For the $n=26$ state, Neukammer *et al.* observed a suppression of the autoionization rate by almost three powers of 10.

Finally, we discuss what happens when the interference of two perturbers occurs below the ionization threshold. In this case we are not dealing with the interference of two autoionizing resonances but of two pseudoresonant perturbations. Just as the interference above threshold leads to a BIC, the interference below threshold leads to an infinitely narrow perturbation of the Rydberg series of bound states. When a narrow perturbation wanders through a Rydberg series of bound states, it leads to approximate level crossings. In the limit of small widths of Γ of the pertuber, the magnitude of the anticrossings it causes, i.e., of the minimal separations of adjacent energy levels, is proportional to $\Gamma^{1/2}$ (see Ref. 14).

Thus a point of vanishing width Γ corresponds to a point of vanishing anticrossing in the bound-state spectrum. However, the point of vanishing anticrossing need not coincide with the point of closest approach of two adjacent levels, so a vanishing width Γ for one value of a continuous parameter will not, in general, lead to the exact degeneracy of two bound states. On the other hand, anomalously narrow widths of the pertruber persist over a finite range of values of the parameter, and this leads to anomalously small anticrossings, i.e., approximate level crossings, in the neighborhood of the point of vanishing anticrossing.

In the present example illustrated in Fig. 5, the point of vanishing width of the $n=3$ state actually occurs for an energy below the ionization threshold and thus corresponds not to a BIC but to a point of vanishing anticrossing leading to approximate level crossings in the bound-state spectrum. In 1980 Zimmermann *et al.*[21] discovered approximate level crossings in the spectrum of the hydrogen atom in comparatively weak (laboratory) magnetic fields. These approximate level crossings can be understood as a consequence of an approximate separability of the Hamiltonian at low field strengths.[21–24] We now present an alternative mechanism for generating approximate level crossings in the region of somewhat stronger magnetic fields. The mechanism is the interference of two perturbers and is not related to any exact or approximate symmetry.

V. SUMMARY AND CONCLUSION

The subject of this paper has been the interference of resonances associated with different channels of a coupled-channel Schrödinger equation. When two resonances interfere, their interference affects their positions and widths. If we have the freedom of varying the separation of the noninterfering resonance positions as a function of a continuous parameter, the interference effects lead to an avoided crossing of resonance positions.

As shown by Eq. (26), the minimal separation of the resonance positions is of the order of the coupling matrix element $|W_{AB}|$. Also, the sum of the widths of the interfering resonances is roughly equal to the sum of the widths of the noninterfering resonances.

These results are quite well known. Less well known is the fact that the distribution of the width among two interfering resonances is extremely uneven. For a well-defined separation of the noninterfering resonance positions, one of the resonances loses all width and becomes a BIC. If the BIC condition is not fulfilled exactly, anomalously narrow widths of one of the two resonances are observed for a finite range of separations; this range is at least of the order of $|W_{AB}|$ and may be much larger if the widths of the noninterfering resonances are very different [see Eq. (35)]. Which of the two interfering resonances becomes a BIC at the BIC point and an anomalously narrow resonance in the vicinity of the BIC point depends on the relative signs of the matrix elements coupling the various channels.

The essential features of the interference effects do not depend on the absolute magnitudes of the widths of the noninterfering resonances but only on their ratio. If the widths of the noninterfering resonances are of the same order of magnitude, then the BIC point lies near the avoided crossing of the positions of the interfering resonances; if the noninterfering widths are very different, the BIC point may lie far from the avoided crossing.

Whether or not two resonances interfere is not directly related to whether or not they overlap. "Isolated" resonances, i.e., resonances whose separation is larger than their widths, can show dramatic interference effects, while the interference between two strongly overlapping resonances may be negligible.

All the results mentioned above are derived by applying Feshbach's theory of resonances to a system of two bound states in closed channels coupled to an open channel. All results remain valid if the effective potentials contain an energy dependence to account for the coupling to other states in the closed channels and to other closed channels. The results derived in Sec. II and illustrated in Sec. III, including the occurrence of exactly bound states in the continuum, are quite general features of interfering resonances in a multichannel system with only one open channel.

If the bound states in the closed channels are coupled not to an open channel but to a more or less dense spectrum of bound states, as may exist below the continuum threshold of the "open channel," then we are no longer talking about resonances, but rather about "pseudoresonant perturbations" of the bound-state spectrum. Instead of leading to a BIC, the interference of two such perturbations leads to a perturbation of vanishing width which corresponds to a "point of vanishing anticrossing" in the bound-state spectrum. Near this point, approximate level crossings of the bound-state energies may occur. How small an anticrossing actually is depends on how close it lies to the point of vanishing anticrossing.

In a system of Coulombic channels, the coupling of a Rydberg series of states in a closed channel to an open channel leads to a Rydberg series of autoionizing resonances. Interference of a resonance from a third channel with any member of this series can lead to a BIC. Near the series limit of the Rydberg series of autoionizing resonances, i.e., near the inelastic threshold, the BIC points lie very close and interference due to a single perturber can affect several resonances simultaneously and may even lead to a Rydberg series of anomalously narrow resonances. Autoionization rates which are strongly suppressed by such interference have recently been observed in laboratory experiments on barium. The general features of a perturbed series of autoionizing resonances are nicely illustrated in the example of a hydrogen atom in a uniform magnetic field.

The occurrence of BIC's and of anomalously narrow resonances due to the interference of resonances in different channels is a very general effect that may be important in many areas of physics. The fact that the effects of interference on the resonance widths are so dramatic and depend on the relative signs of the coupling matrix elements suggests that the identification of such interference effects could play an important role in atomic, molecular, or nuclear spectroscopy, or wherever the interaction between resonances might be important.

ACKNOWLEDGMENT

We are grateful to the Deutsche Forschungsgemeinschaft for support and encouragement.

[1]G. Baym, *Lectures on Quantum Mechanics* (Benjamin, New York, 1969).
[2]J. von Neumann and E. Wigner, Phys. Z. **30**, 465 (1929).
[3]F. H. Stillinger and D. R. Herrick, Phys. Rev. A **11**, 446 (1975).
[4]B. Gazdy, Phys. Lett. **61A**, 89 (1977).
[5]L. Fonda and R. G. Newton, Ann. Phys. (N.Y.) **10**, 490 (1960).
[6]R. G. Newton, *Scattering Theory of Waves and Particles,* 2nd ed. (Springer, Berlin, 1982), pp. 548ff.
[7]H. Feshbach, Ann. Phys. (N.Y.) **5**, 357 (1958).
[8]N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon, Oxford, 1965), pp. 357ff.
[9]A brief account of this result has been given in H. Friedrich and D. Wintgen, Verhandl. Deutsch. Phys. Ges. **20**, 433 (1985).
[10]H. Friedrich and D. Wintgen, Phys. Rev. A **31**, 3964 (1985).
[11]C. Mahaux, in *Resonances—Models and Phenomena,* Vol. 211 of *Lecture Notes in Physics,* edited by S. Albeverio, L. S. Ferreira, and L. Streit (Springer, Heidelberg, 1984), p. 139.
[12]H. Feshbach, Ann. Phys. (N.Y.) **19**, 287 (1962).
[13]M. J. Seaton, Mon. Not. Astron. Soc. **118**, 504 (1958).
[14]H. Friedrich and D. Wintgen, Phys. Rev. A **31**, 1920 (1985).
[15]M. J. Seaton, Rep. Prog. Phys. **46**, 167 (1983), and references therein.
[16]C. Greene, U. Fano, and G. Strinati, Phys. Rev. A **19**, 1485 (1977); C. Greene, A. R. P. Rau, and U. Fano, *ibid.* **26**, 2441 (1982).
[17]A. Giusti-Suzor and U. Fano, J. Phys. B **17**, 215 (1984).
[18]C. W. Clark, K. T. Lu, and A. T. Starace, in *Progress in Atomic Spectroscopy,* edited by H. Beyer and K. Kleinpoppen

(Plenum, New York, 1984), Pt. C, p. 1; J. C. Gay, *ibid.*, p. 177.

[19]H. Friedrich and M. Chu, Phys. Rev. A **28**, 1423 (1983); M.-C. Chu and H. Friedrich, *ibid.* **29**, 675 (1984).

[20]J. Neukammer, H. Rinneberg, G. Jönsson, W. E. Cooke, H. Hieronymus, A. König, K. Vietzke, and H. Spinger-Bolk, Phys. Rev. Lett. (to be published).

[21]M. L. Zimmermann, M. M. Kash, and D. Kleppner, Phys. Rev. Lett. **45**, 1092 (1980).

[22]D. R. Herrick, Phys. Rev. A **26**, 323 (1982).

[23]D. Delande and J. C. Gay, J. Phys. B **17**, L335 (1984).

[24]D. Wintgen and H. Friedrich, J. Phys. B (to be published).