# Wave function for a system with a nonlocal potential determined from its phase shifts

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The wave function for a system with an energy-independent nonlocal potential is determined from its phase shifts to a good approximation, using a fixed-energy inversion method.

An inversion scheme[1,2] has recently been developed to determine an $l$-independent local potential from the phase shifts at fixed energy. In nuclear scattering the interaction is in general *non*local as a consequence of the Pauli principle. For such nonlocal interactions the inversion method can be employed to determine an $l$-independent equivalent local potential (ELP) from the phase shifts at a given energy.[3,4] The ELP is then in general energy dependent. Although an ELP and the corresonding nonlocal interaction are equivalent on the energy shell (phase-shift equivalence), this is not the case for off-shell processes (no wave-function equivalence). It would therefore be of considerable interest to extend the inversion scheme such as to yield, besides the ELP, also the "nonlocal" wave function. This is the subject of the present article.

An energy-independent nonlocal potential $V_N(\mathbf{r},\mathbf{r}')$ associated with a nonlocal wave function $\psi(\mathbf{r})$ satisfying

$$(-\hbar^2\nabla^2/2\mu)\psi(\mathbf{r})+\int V_N(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'=E\psi(\mathbf{r}) , \quad (1)$$

is associated with an equivalent local potential $V_L(E,\mathbf{r})$ with a local wave function $\phi(\mathbf{r})$ satisfying

$$[T+V_L(E,r)]\phi(\mathbf{r})=E\phi(\mathbf{r}) , \quad (2)$$

where

$$\psi(\mathbf{r})=f(r)\phi(\mathbf{r}), \text{ with } f(r)\to 1 \text{ as } r\to\infty . \quad (3)$$

If, in particular, we assume a linear energy dependence for the ELP[5]

$$V_L(E,r)=V_0(r)+EW_0(r) , \quad (4)$$

then the energy dependence can be eliminated by transforming (2) to

$$[f^{-1}(T+V_0+EW_0)f^{-1}]\psi=Ef^{-2}\psi , \quad (5)$$

using (3) and setting

$$f^2(r)=1-W_0(r) . \quad (6)$$

With the effective mass

$$\mu^*(r)=[1-W_0(r)]\mu \quad (7)$$

(5) reduces to a wave equation with a velocity-dependent potential,

$$\{[\mu^*(r)]^{-1/2}(-\hbar\nabla^2/2)[\mu^*(r)]^{-1/2}+[\mu/\mu^*(r)]V_0(r)\}\psi(\mathbf{r})=E\psi(\mathbf{r}) , \quad (8)$$

which represents a particular form of (1).

In this case we can thus determine the velocity-dependent potential, and the associated nonlocal wave function, exactly from our knowledge of the ELP, which is in turn obtained by inversion of the phase shifts. However, it is not enough to determine $V_L(E,r)$ of (4) at a single energy. We also have to calculate $\partial V_L(E,r)/\partial E=W_0(r)$, which requires a knowledge of $V_L(E,r)$ in a neighborhood of energies around the fixed $E$.

The simple model (4) does not generally represent nuclear scattering, but it has proven to be approximately applicable over limited energy ranges. Moreover, it has recently been shown by Horiuchi[6,7] (see also, de Forest[5] and Rook[8]) that in the Wentzel-Kramers-Brillouin (WKB) approximation to energy-independent nonlocal interactions, the damping factor $f(r)$ of (3) is given by

$$f^2(r)=1-\partial V_L(E,r)/\partial E . \quad (9)$$

Relation (9) is valid for any form of energy dependence of $V_L(E,r)$ in the WKB approximation, and it is exact for the linear case (4), cf. Eq. (6). This indicates that the relation (9) is likely to be a good approximation in general. Therefore we propose to extend the inversion scheme to nonlocal energy-independent interactions, by assuming that the "inverted" nonlocal partial wave function $\tilde{u}_l(E,r)$ is given, in terms of the local wave function $v_l(r)$, by

$$\tilde{u}_l(E,r)=[1-\partial V_L(E,r)/\partial E]^{1/2}v_l(E,r) , \quad (10)$$

where $v_l(E,r)$ is calculated from the equivalent local potential $V_L(E,r)$, as determined by inversion of the phase shifts.

An important check on the $\tilde{u}_l(E,r)$ is to what extent they fulfill the "orthogonality condition" for the exact $u_l(E,r)$

$$\int_0^\infty [u_l(E_l,r)u_l(E_2,r)-v_l(E_1,r)v_l(E_2,r)]dr = \int_0^\infty \frac{V_L(E_2,r)-V_L(E_1,r)}{E_2-E_1}v_l(E_1,r)v_l(E_2,r)dr , \quad (11)$$

which in the limit $E_2\to E_1=E$, reduces to

$$\int_0^\infty [u_l^2(E,r)-v_l^2(E,r)]dr=-\int_0^\infty \frac{\partial V_L(E,r)}{\partial E}v_l^2(E,r)dr . \quad (12)$$

[These relations follow from the application of Green's theorem to the nonlocal and equivalent local Schrödinger equations and the condition that $u_l(r)\to v_l(r)$ for $r\to\infty$.]

The wave functions $\tilde{u}_l(E,r)$ of (10) satisfy (11) to second

 

TABLE I. N-$\alpha$ phase shifts at five energies (expressed in units of MeV). At 25.0 MeV we also give the phase shifts corresponding to the parametrized $S$ function used in the fitting procedure of the inversion scheme, see Ref. 2.

| $l$ | 22.5 | 23.75 | 25.0 | 25.0 (fit) | 26.25 | 27.5 |
|---|---|---|---|---|---|---|
| 0 | 88.54 | 86.77 | 85.08 | 85.08 | 83.47 | 81.92 |
| 1 | 60.15 | 59.71 | 59.23 | 59.23 | 58.72 | 58.20 |
| 2 | 14.06 | 15.04 | 15.97 | 15.97 | 16.85 | 17.69 |
| 3 | 2.43 | 2.75 | 3.07 | 3.08 | 3.41 | 3.75 |
| 4 | 0.38 | 0.45 | 0.53 | 0.53 | 0.62 | 0.71 |

order in $E_2-E_1$ and (12) exactly.

If the nonlocal interaction is itself energy dependent ("intrinsic" energy dependence), our inversion method is not applicable. However, there are many cases in nuclear physics where the intrinsic energy dependence can be neglected, e.g., in the renormalized version of the resonating group method for the scattering of two clusters at lower energies, where the spurious linear-energy dependence has been transformed away to obtain wave functions with the correct probability interpretation.[7] For nucleons it has been found that the intrinsic energy dependence is only important below 100 MeV.[9]

To test the proposed method for the determination of nonlocal wave functions from the phase shifts, we consider the nucleon–alpha-particle (N-$\alpha$) interaction of Lassaut and Vinh Mau,[10,11] derived in an antisymmetrized folding model for a variety of effective nucleon-nucleon forces. This interaction is real and energy independent below $E_{\text{c.m.}}=20$ MeV, where the lowest inelastic channel opens in the N-$\alpha$ system.

We consider the N-$\alpha$ potential generated by the Serber force. Its parameters are given in Table I of Ref. 10. Our choice is motivated by the fact in this case our inversion method achieves the required high degree of accuracy, while the WKB approximation of Perey and Buck[12] (Horiuchi[5]) is inaccurate. This makes it a more stringent test for our proposed method, which combines an exact quantal inversion (for the ELP) with an approximate (WKB) relation for the damping factor.

We calculate the phase shifts $\delta_l$ of the nonlocal N-$\alpha$ potential exactly, without the customary Perey-Buck approximation of replacing $R=|\mathbf{r}+\mathbf{r}'|/2$ by $R=(r+r')/2$, at five laboratory energies $E_N=22.5$ to 27.5 MeV, in steps of $\Delta E=1.25$ MeV. This step size proved to be optimal to reconcile the conflicting requirements of the accuracy of the inversion and the numerical differentiation of $V_L(E,r)$. The phase shifts $\delta_l$ to be inverted are given in Table I ( we ignore the weak inelasticity between 25 and 27.5 MeV, which is irrelevant for our purposes). For the inversion we employed the mixed rational-nonrational method of Ref. 2. The quality of our fit is shown in Table I for the energy $E_N=25.0$ MeV.

To check the accuracy of our inversion we reconstructed the direct potential[10] $V_D(r)$ from its phase shifts at $E_N=25$ MeV. The result is shown in Fig. 1. It is seen that the reconstruction is quite accurate for $r\gtrsim 0.4$ fm, but less so for smaller distances. The effect of the small discrepancy for $r\lesssim 0.4$ fm is of course magnified in the numerical differentiation of $V_L(E,r)$, which becomes unreliable in that region. It is seen from Fig. 1 that the ELP of Perey and Buck[12] is quite inaccurate for N-$\alpha$ scattering. In Figs. 2 and 3 we show the exact nonlocal wave function $u_l(r)$ in comparison to the equivalent local function $v_l(r)$ and the "inverted" nonlocal function $\tilde{u}_l(r)$ at $E_N=25$ MeV for $l=0$ and 1, respectively. These results confirm that $\tilde{u}_l(r)\approx u_l(r)$ to a high degree of accuracy, particularly for the $p$ waves. The accuracy for $s$ waves, though less impressive, is still quite adequate. Because of the inaccuracy of $V_L(E,r)$ for $r\lesssim 0.4$ fm we have not plotted $\tilde{u}_0(r)$ for $r<0.2$ fm. If we compare these wave functions with $v_l(r)$ it is clear that the inclusion of the damping factor $f(r)$ is essential.

The wave functions $\tilde{u}_l(r)$ satisfy (8) with $V_0(r)$ replaced by $V_L(E,r)-E\,\partial V_L(E,r)/\partial E$. Both the effective mass and the potential are now (weakly) energy dependent. It has previously been shown that any nonlocal wave equation can be exactly transformed to such a velocity-dependent form.[13] In this case $V_0(r)$ must be replaced by $V_L(E,r)-E[1-f^2(E,r)]$ in (8).

We have demonstrated that our extension of the inversion method to energy-independent nonlocal interactions is

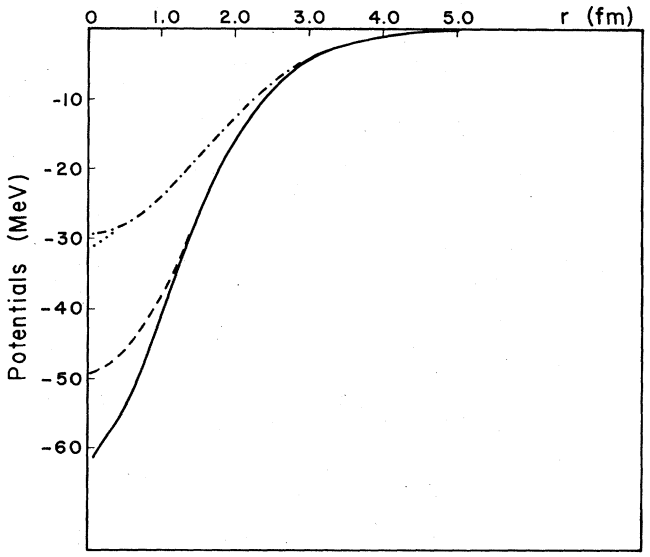


FIG. 1. The approximate ELP of Perey and Buck (– – –) compared to the exact ELP obtained by inversion (——) for the N-$\alpha$ potential of Lassaut and Vinh Mau, see Ref. 10, at $E_N=25$ MeV. The direct potential (– · – · –) and its reconstruction by inversion (· · · · ·) are also shown.

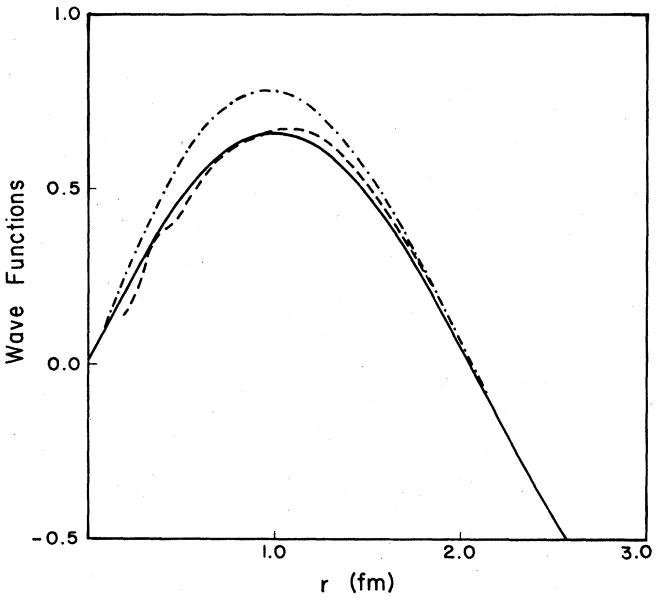


FIG. 2. The equivalent local $s$ wave function $v_0(r)$ ($- \cdot - \cdot$) and the "inverted" nonlocal wave function $\tilde{u}_0(r)$ ($- - -$) compared to the exact nonlocal wave function $u_0(r)$ (——) at $E_N = 25$ MeV.

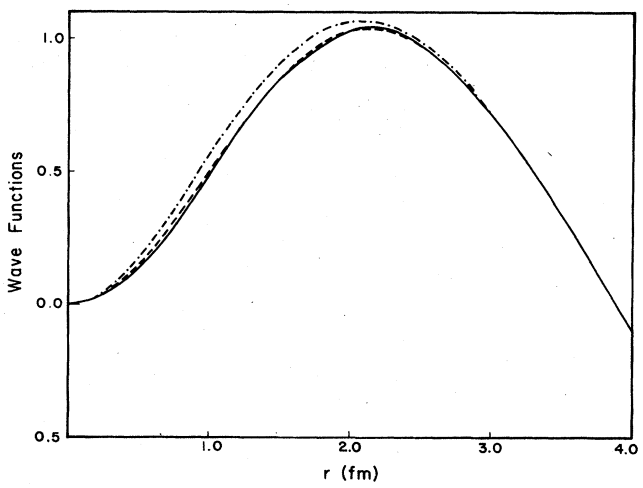


FIG. 3. The same as in Fig. 2 but for $l = 1$.

practical in the context of nuclear physics. The method should also be applied in other fields like atomic and molecular physics.

For nonlocal interactions with an intrinsic energy dependence the phase shifts do not contain enough information. Properties of the potential itself must be considered in this case. We refer to attempts which have been made to disentangle the intrinsic energy dependence from the nonlocality by means of a dispersion relation.[14]

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