# Spectral density for a nonlinear Fokker-Planck model: Monte Carlo and analytical studies

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A Monte Carlo method is used to generate trajectories for a nonlinear stochastic model. A set of initial conditions distributed according to the equilibrium distribution function of the system is considered. Then, after averaging, the equilibrium time correlation function can be computed. The numerical results are compared with those obtained by means of the van Kampen expansion method and also by the local equilibrium approximation. The latter is shown to give a good description of the behavior over a wide range of values of the parameters.

## I. INTRODUCTION

In the last few years, a great deal of work has been devoted to the study of the coupling between nonlinearities and fluctuations in physical systems. Despite this, there are few results firmly established, both from an experimental and a theoretical point of view. Even computer experiments which have provided a sufficient and accurate simulation of equilibrium situations are plagued with difficulties when trying to describe nonequilibrium states. Perhaps the main reason for it is the lack of a kind of ergodic hypothesis.

In this paper we study a very simple system described by a nonlinear Fokker-Planck (FP) equation or by the corresponding equivalent Langevin equation. The quantity we will be interested in is the equilibrium time correlation function of the relevant variable. It is a well-known fact that this quantity relaxes exponentially at a constant rate for a linear system. The question arises on whether the exponential character of the decay remains valid in the nonlinear case and, if so, how the nonlinearity would affect the rate of relaxation.[1]

Several theoretical schemes have been proposed to answer the questions posed above. In particular, we will consider here the van Kampen expansion method[1] and the local equilibrium (LE) approximation.[2] In order to have numerical data for comparative purposes, we have carried out a stochastic simulation of the system. The idea is to use a Monte Carlo procedure to simulate the stochastic evolution of the system starting from a set of initial conditions distributed according to the equilibrium probability.

The plan of the paper is as follows. In the remainder of this section we describe our model system and introduce the notation for the different distribution functions. In Sec. II the numerical method is presented. As a test of its validity we consider the simplest case, namely, a linear model. The data fit the exact analytical solution quite well. In Sec. III the van Kampen expansion procedure is applied to our model and the equilibrium time correlation function to first order in the diffusion constant is computed. The LE approximation is considered in Sec. IV where it is shown that it reduces to van Kampen's expansion in the limit of the small diffusion constant. The several results are compared and discussed in the final section, both for the equilibrium time correlation function and its Fourier transform, the spectral density.

Let us consider a system characterized by the dynamical variable $x$, whose evolution is given by the first-order Langevin equation:

$$\frac{dx}{dt} = -U'(x) + F(t) , \tag{1}$$

where $U'(x)$ is the derivative of

$$U(x) = \frac{x^2}{2} + b\frac{x^4}{4} \tag{2}$$

and $F(t)$ represents a Gaussian, white-noise process with zero mean, i.e.,

$$\langle F(t) \rangle_{\text{noise}} = 0, \quad \langle F(t)F(s) \rangle_{\text{noise}} = 2\alpha\delta(t-s) . \tag{3}$$

The $\langle\ \rangle_{\text{noise}}$ denote an average over the realizations of the noise and $\alpha$ is the diffusion constant. Equation (1) may describe, for instance, the evolution of the position of an overdamped nonlinear Brownian oscillator.

The equivalent FP description of the system is given by an equation for the conditional probability density $P(x,t \mid x',t')$

$$\frac{\partial}{\partial t}P(x,t \mid x',t') = \frac{\partial}{\partial x}[U'(x)P(x,t \mid x',t')] + \alpha\frac{\partial^2}{\partial x^2}P(x,t \mid x',t') \tag{4}$$

and the initial condition

$$P(x,t' \mid x',t') = \delta(x-x') . \tag{5}$$

Since we are dealing with a Markov process, the knowledge of $P(x,t \mid x',t')$ allows us to construct all the multitime probability distributions for a given initial distribution function $P(x,0)$. In particular, we have

 

$$P(x,t)=\int dx' P(x,t\,|\,x',0)P(x',0) \tag{6}$$

for the one-time distribution, and

$$P(x,t;x',t')=P(x,t\,|\,x',t')P(x',t') \tag{7}$$

for the two-time joint distribution.

In this paper we will concentrate on the calculation of the two-time correlation function, defined as

$$\begin{aligned} C(\tau;t)&=\langle x(t+\tau)[x-\langle x(t)\rangle]\rangle \\ &\quad+\int dx\,dx'x\,\Delta x'(t)P(x,t+\tau;x',t) \\ &=\int dx'\Delta x'(t)\langle x(t+\tau)\,|\,[x',t]\rangle P(x',t)\,, \end{aligned} \tag{8}$$

where $\Delta x(t)=x-\langle x(t)\rangle$ and where we have introduced the conditional average value

$$\langle x(t+\tau)\,|\,[x',t]\rangle=\int dx\,x\,P(x,t+\tau\,|\,x',t)\,. \tag{9}$$

When the system is at equilibrium, it is

$$C(\tau)=\int dx'x'\langle x(\tau)\,|\,[x']\rangle P_{\rm eq}(x') \tag{10}$$

with

$$P_{\rm eq}(x')=A\exp\left[-\frac{1}{\alpha}U(x')\right], \tag{11}$$

$A$ being the normalization factor. A related quantity of interest is the spectral density defined as the cosine Fourier transform of the time correlation function $C(\tau)$,

$$S(\omega)=\frac{2}{\pi}\int_0^\infty d\tau\,C(\tau)\cos(\omega\tau)\,. \tag{12}$$

## II. MONTE CARLO CALCULATION OF THE EQUILIBRIUM TIME CORRELATION FUNCTION

The numerical method we will follow to evaluate the equilibrium time correlation function $C(\tau)$ is based upon Eq. (10). According to it, the calculation can be carried out in two steps. Firstly, one evaluates the conditional average value $\langle x(\tau)\,|\,[x']\rangle$ as a function of $x'$. Afterwards, the equilibrium average of $x'\langle x(\tau)\,|\,[x']\rangle$ has to be performed.

To calculate the conditional average we make use of Eq. (1). We have integrated it numerically, by means of a Runge-Kutta algorithm, with a time interval $h$. The random noise has been generated by the expression

$$F(t)=\sqrt{2\alpha h}\,\gamma(t)\,, \tag{13}$$

where the value of $\gamma$ at each time is obtained through the Box-Mueller formula

$$\gamma=(-2\ln\eta_1)^{1/2}\cos(2\pi\eta_2) \tag{14}$$

with $\eta_1$ and $\eta_2$ being two independent random numbers uniformly distributed between 0 and 1.

Starting from a given initial value $x'$, we generate a sufficiently large number, $M$, of stochastic trajectories, $x_j(t;x')$. The conditional average value is then given by

$$\langle x(t)\,|\,[x']\rangle=\frac{1}{M}\sum_{j=1}^{M}x_j(t;x')\,. \tag{15}$$

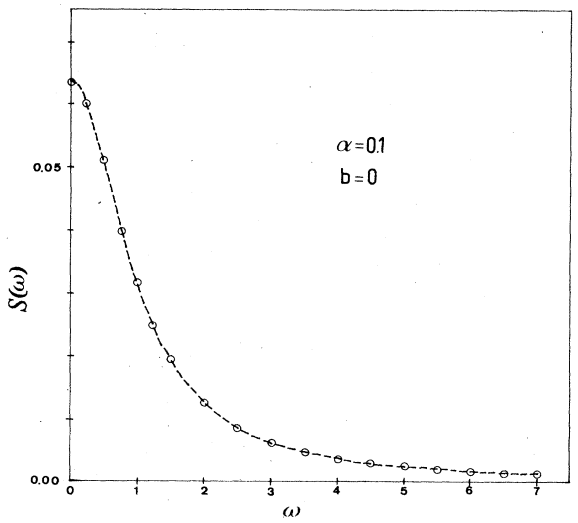


FIG. 1. Spectral density versus frequency for the linear case. ○, Monte Carlo; — — —, exact.

We have considered 19 equally spaced values of $x'$ between $-4\alpha^{1/2}$ and $4\alpha^{1/2}$ and $M=500$. As we will see this is adequate to obtain sufficiently accurate results for small values of $\alpha$.

Now the evaluation of $C(t)$ is straightforward. Equation (10) is transformed into a sum of the contributions corresponding to each of the 19 intervals of the variable $x'$. Finally, the spectral density is obtained by numerically transforming $C(t)$ according to Eq. (12).

To test the accuracy of the numerical simulation method, we have first considered the linear case, i.e., $b=0$ in Eqs. (1) and (4), for which an exact analytical expression for the spectral density is available, namely,

$$S(\omega)=\frac{2}{\pi}\frac{\alpha}{1+\omega^2}\,. \tag{16}$$

The plot in Fig. 1 shows the very good agreement between the numerical simulation values and Eq. (16). The data correspond to a diffusion constant $\alpha=0.1$. Actually, the relative difference is less than 1%.

## III. VAN KAMPEN'S EXPANSION

One of the most widely used methods to deal with fluctuations in nonlinear systems is van Kampen's expansion.[1] Within our present context, it corresponds to an expansion in powers of $\alpha$. A new stochastic variable $\xi$ is defined by means of the time-dependence transformation

$$x=\phi(t)+\alpha^{1/2}\xi\,, \tag{17}$$

where $\phi$ and $\xi$ are independent of $\alpha$. The idea behind this transformation has been extensively discussed. What we want to stress here is the fact that Eq. (17) only holds for some initial distributions and, of course, for some kinds of evolution equations. If the initial distribution is very flat, in the sense that

$$\langle[\Delta x(0)]^2\rangle\equiv[\langle x^2(0)\rangle-\langle x(0)\rangle^2]\gg\alpha\,,$$

then Eq. (17) is not suitable at least for very short times.

Therefore, it seems that the right distribution to which decomposition (17) should be applied is the joint probability distribution. From Eqs. (4) and (7) it is clear that $P(x,t;x',t')$ also obeys the FP equation, namely,

$$\frac{\partial}{\partial t}P(x,t;x',t')=\frac{\partial}{\partial x}[U'(x)P(x,t;x',t')] +\alpha\frac{\partial^2}{\partial x^2}P(x,t;x',t') \tag{18}$$

with the initial condition

$$P(x,t';x',t')=\delta(x-x')P(x',t') , \tag{19}$$

where in turn $P(x',t')$ is the solution of

$$\frac{\partial}{\partial t'}P(x',t')=\frac{\partial}{\partial x'}[U'(x')P(x',t')]+\alpha\frac{\partial^2}{\partial x'^2}P(x',t') \tag{20}$$

for a given initial distribution $P(x',0)$.

We now introduce the distribution functions for the stochastic variable $\xi$. It is readily shown that the two-time joint distribution $\Pi(\xi,t;\xi',t')$ obeys the equation

$$\frac{\partial\Pi}{\partial t}-\alpha^{-1/2}\dot{\phi}\frac{\partial\Pi}{\partial\xi}=\alpha^{-1/2}\frac{\partial}{\partial\xi}[(\phi+\alpha^{1/2}\xi)\Pi] +\alpha^{-1/2}b\frac{\partial}{\partial\xi}[(\phi+\alpha^{1/2}\xi)^3\Pi]+\frac{\partial^2\Pi}{\partial\xi^2} \tag{21}$$

with $\Pi(\xi,t';\xi',t')=\delta(\xi-\xi')\Pi(\xi',t')$. Identification of the coefficients of $\alpha^{-1/2}$ leads to

$$\dot{\phi}=-\phi-b\phi^3 . \tag{22}$$

This is the deterministic equation in van Kampen's terminology. Taking into account Eq. (22), we get from Eq. (21)

$$\frac{\partial\Pi}{\partial t}=\frac{\partial}{\partial\xi}[(1+3b\phi^2)\xi+\alpha^{1/2}3b\phi\xi^2+\alpha b\xi^3]\Pi+\frac{\partial^2\Pi}{\partial\xi^2} , \tag{23}$$

and from here it is a simple task to derive the following hierarchy of equations:

$$\frac{d}{dt}\langle\xi(t)\xi(t')\rangle=-(1+3b\phi^2)\langle\xi(t)\xi(t')\rangle -3b\alpha^{1/2}\phi\langle\xi^2(t)\xi(t')\rangle -b\alpha\langle\xi^2(t)\xi(t')\rangle , \tag{24a}$$

$$\frac{d}{dt}\langle\xi^2(t)\xi(t')\rangle=2-2(1+3b\phi^2)\langle\xi^2(t)\xi(t')\rangle -6b\alpha^{1/2}\phi\langle\xi^2(t)\xi(t')\rangle -2b\alpha\langle\xi^4(t)\xi(t')\rangle , \tag{24b}$$

$$\frac{d}{dt}\langle\xi^3(t)\xi(t')\rangle=6\langle\xi(t)\xi(t')\rangle-3(1+3b\phi^2)\langle\xi^3(t)\xi(t')\rangle -9b\alpha^{1/2}\langle\xi^4(t)\xi(t')\rangle -3b\alpha\langle\xi^5(t)\xi(t')\rangle . \tag{24c}$$

The same analysis can also be carried out for the one-time distribution $\Pi(\xi,t)$. The corresponding hierarchy is formally equivalent to the set (24). For further reference we will write the first few equations,

$$\frac{d}{dt}\langle\xi(t)\rangle=-(1+3b\phi^2)\langle\xi(t)\rangle -3b\alpha^{1/2}\phi\langle\xi^2(t)\rangle-\alpha b\langle\xi^3(t)\rangle , \tag{25a}$$

$$\frac{d}{dt}\langle\xi^2(t)\rangle=2-2(1+3b\phi^2)\langle\xi^2(t)\rangle -6b\alpha^{1/2}\phi\langle\xi^3(t)\rangle-2\alpha b\langle\xi^4(t)\rangle , \tag{25b}$$

$$\frac{d}{dt}\langle\xi^3(t)\rangle=6\langle\xi(t)\rangle-3(1+3b\phi^2)\langle\xi^3(t)\rangle -9b\alpha^{1/2}\phi\langle\xi^4(t)\rangle-3\alpha b\langle\xi^5(t)\rangle , \tag{25c}$$

$$\frac{d}{dt}\langle\xi^4(t)\rangle=12\langle\xi^2(t)\rangle-4(1+3b\phi^2)\langle\xi^4(t)\rangle -12b\alpha^{1/2}\phi\langle\xi^5(t)\rangle-4\alpha b\langle\xi^6(t)\rangle . \tag{25d}$$

Let us consider an equilibrium situation. From Eqs. (25) one easily obtains the following by making $\dot{\phi}=\phi=0$ and by setting all the time derivatives equal to zero:

$$\langle\xi\rangle_{eq}=0 , \tag{26a}$$

$$\langle\xi^2\rangle_{eq}=1-3b\alpha+O(\alpha^{3/2}) , \tag{26b}$$

$$\langle\xi^3\rangle_{eq}=0+O(\alpha^2) , \tag{26c}$$

$$\langle\xi^4\rangle_{eq}=3+O(\alpha^{1/2}) . \tag{26d}$$

It also follows from Eqs. (24), that by neglecting terms of order $\alpha^{3/2}$ and higher, we obtain

$$\frac{d}{dt}\langle\xi(t)\xi(t')\rangle_{eq}=-\langle\xi(t)\xi(t')\rangle_{eq} -\alpha b\langle\xi^3(t)\xi(t')\rangle_{eq} , \tag{27a}$$

$$\frac{d}{dt}\langle\xi^3(t)\xi(t')\rangle_{eq}=-3\langle\xi^3(t)\xi(t')\rangle_{eq} +6\langle\xi(t)\xi(t')\rangle_{eq} . \tag{27b}$$

This is a linear system of ordinary first-order differential equations. Solving formally the second one and inserting the result into the first one, we readily get

$$\frac{d}{dt}\langle\xi(t)\xi(t')\rangle_{eq}=-(1+3b\alpha)\langle\xi(t)\xi(t')\rangle_{eq} . \tag{28}$$

Taking into account that

$$C(\tau)=\alpha\langle\xi(\tau)\xi\rangle_{eq} , \tag{29}$$

we have that the equilibrium time correlation function satisfies the linear differential equation

$$\frac{dC(\tau)}{d\tau}=-(1+3b\alpha)C(\tau)\ , \tag{30}$$

which is to be solved with the initial condition

$$C(0)=\alpha\langle\xi^2\rangle_{\rm eq}=\alpha(1-3b\alpha)+O(\alpha^{5/2})\ . \tag{31}$$

Before comparing the solution of Eq. (30) with the Monte Carlo simulation results, some comments seems to be adequate. First, we notice that the dynamical coupling between ensemble averages and ensemble fluctuations introduces a correction term in the relaxation of the equilibrium time correlation function for a nonlinear system,[3,4] and this happens even at the lowest order in the nonlinearity parameter $b$. Nonetheless, the evolution equation for $C(\tau)$ is linear.

In the last few years several theories have been proposed to relate the transport equations for the averages and the evolution equations for the time correlation functions.[5] In many cases, the result is that the evolution equation for the fluctuations is the linearization of the transport equation. We think it is worthwhile to check whether this is also the case in the present context. Then, we come back to Eqs. (25) and consider a general nonequilibrium situation. We have

$$\frac{d}{dt}\langle\xi(t)\rangle=-(1+3b\phi^2)\langle\xi(t)\rangle -3b\alpha^{1/2}\phi\langle\xi^2(t)\rangle+O(\alpha)\ , \tag{32}$$

and using again transformation (17) it is a matter of simple manipulations to derive

$$\frac{d}{dt}\langle x(t)\rangle=-[1+3b\alpha\langle[\Delta\xi(t)]^2\rangle]\langle x(t)\rangle -b\langle x(t)\rangle^3+O(\alpha^{3/2})\ . \tag{33}$$

In fact, the same equation can be proved to be correct to order $\alpha^2$ if one assumes that $\langle(\Delta\xi(t))^3\rangle$ can be neglected to zeroth order in Eq. (25a). Comparison of Eqs. (30) and (33) shows that within our level of approximation, if we formally write Eq. (33) as

$$\frac{d}{dt}\langle x(t)\rangle=\Phi(\langle x(t)\rangle)\ , \tag{34}$$

then

$$\frac{d}{d\tau}C(\tau)=\left.\frac{\delta\Phi(\langle x(t)\rangle)}{\delta\langle x(t)\rangle}\right|_{\rm eq}C(\tau)\ , \tag{35}$$

which is the result we are looking for.

## IV. LOCAL EQUILIBRIUM APPROXIMATION

We have recently[2] formulated a local equilibrium approximation within the context of FP models and at the level of the two-time joint distribution function. The idea was to extend the usual LE approximation for the one-time distribution function to allow the calculation of time correlation functions.

The LE one-time distribution function $P_{\rm LE}(x,t)$ for our system is given by

$$P_{\rm LE}(x,t)=A(t)\exp\left[-\frac{1}{\alpha}U(x)+\gamma(t)x\right]\ , \tag{36}$$

where $A(t)$ is a time-dependent normalization factor and $\gamma(t)$ is chosen in such a way that

$$\langle x(t)\rangle=\int dx\,x\,P_{\rm LE}(x,t)\equiv\langle x(t)\rangle_{\rm LE}\ . \tag{37}$$

In Ref. 2 we have found that a suitable generalization of Eq. (36) for the two-time joint distribution is

$$P_{\rm LE}(x,t;x',t')=P_{\rm LE}(x,t)P_{\rm LE}(x',t') +\frac{\partial}{\partial\langle x(t)\rangle}P_{\rm LE}(x,t) \times\int dx(x-\langle x\rangle_t)P(x,t;x',t')\ . \tag{38}$$

With this definition, it is

$$\langle x(t)x(t')\rangle_{\rm LE}=\langle x(t)x(t')\rangle\ . \tag{39}$$

In the LE approximation we substitute the actual joint distribution function by its LE approximation, Eq. (38), and write

$$\frac{d}{dt}\langle x(t)x(t')\rangle=\int dx\,dx'[D^\dagger(x)x]x'P_{\rm LE}(x,t;x',t')\ , \tag{40}$$

where

$$D^\dagger=-U'(x)\frac{\partial}{\partial x}+\alpha\frac{\partial^2}{\partial x^2}\ . \tag{41}$$

Inserting Eq. (38) into Eq. (40), we get

$$\frac{\partial}{\partial t}\langle\Delta x(t)\Delta x(t')\rangle=\frac{\delta\langle D^\dagger x\rangle_{\rm LE}}{\delta\langle x(t)\rangle}\langle\Delta x(t)\Delta x(t')\rangle\ . \tag{42}$$

Upon writing this expression we have taken into account that in the LE approximation the evolution equation for the average is given by

$$\frac{d}{dt}\langle x(t)\rangle=\langle D^\dagger x\rangle_{\rm LE}\ . \tag{43}$$

Particularizing Eq. (42) for an equilibrium situation, we get

$$\frac{d}{d\tau}C(\tau)=-\left.\frac{\partial\langle U'(x)\rangle_{\rm LE}}{\partial\langle x\rangle}\right|_{\langle x\rangle=0}C(\tau)\ . \tag{44}$$

A simple calculation shows that

$$\left.\frac{\partial\langle U'(x)\rangle_{\rm LE}}{\partial\langle x\rangle}\right|_{\langle x\rangle=0}=\frac{\langle xU'(x)\rangle_{\rm eq}}{\langle x^2\rangle_{\rm eq}} \tag{45}$$

and, then,

$$\frac{d}{d\tau}C(\tau)=-\left[1+b\frac{\langle x^4\rangle_{\rm eq}}{\langle x^2\rangle_{\rm eq}}\right]C(\tau)\ , \tag{46}$$

with $C(0)=\langle x^2\rangle_{\rm eq}$. It can be seem that if we expand the equilibrium average values in powers of $\alpha$ and keep the

lowest order, we recover Eq. (30). This shows that in the limit of $\alpha$ small the LE approximation and van Kampen's expansion agree. Nevertheless, it must be pointed out that the LE approximation does not correspond to a systematic expansion in powers of $\alpha$, and so, it can be used, at least in principle, even when $\alpha$ is not very small.

It can be shown that in the LE approximation for the equilibrium time correlation function one neglects memory effects associated with the nonlinearity of the system.[6] In this sense, the LE approximation can be viewed as a short-time approximation.

## V. RESULTS AND DISCUSSION

In this section we present a comparison between the numerical results provided by the Monte Carlo simulation discussed in Sec. II and the analytical results of the approximations considered in Secs. III and IV. We have seen that the $\alpha$ expansion leads to Eq. (30) with the initial condition (31), i.e., we have

$$C_{\mathrm{vK}}(\tau)=\alpha(1-3b\alpha)\exp[-(1+3b\alpha)\tau] . \tag{47}$$

On the other hand, in the LE approximation, integration of Eq. (46) yields

$$C_{\mathrm{LE}}(\tau)=\langle x^2\rangle_{\mathrm{eq}}\exp\left[-\left[1+b\frac{\langle x^4\rangle_{\mathrm{eq}}}{\langle x^2\rangle_{\mathrm{eq}}}\right]\tau\right] . \tag{48}$$

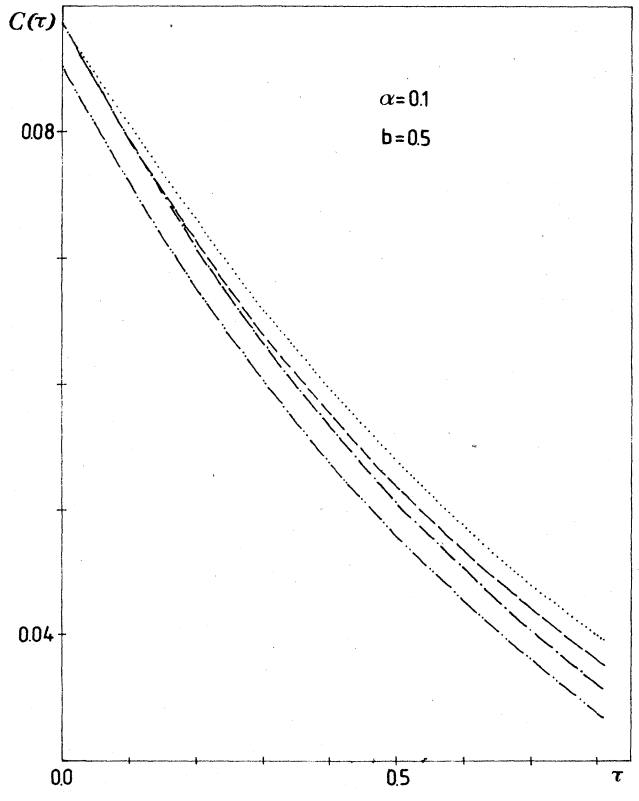


FIG. 2. Equilibrium time correlation function versus time. · · · ·, linear ($b=0$); — — —, Monte Carlo; —··—··—, van Kampen's expansion; —·—·—, LE approximation.

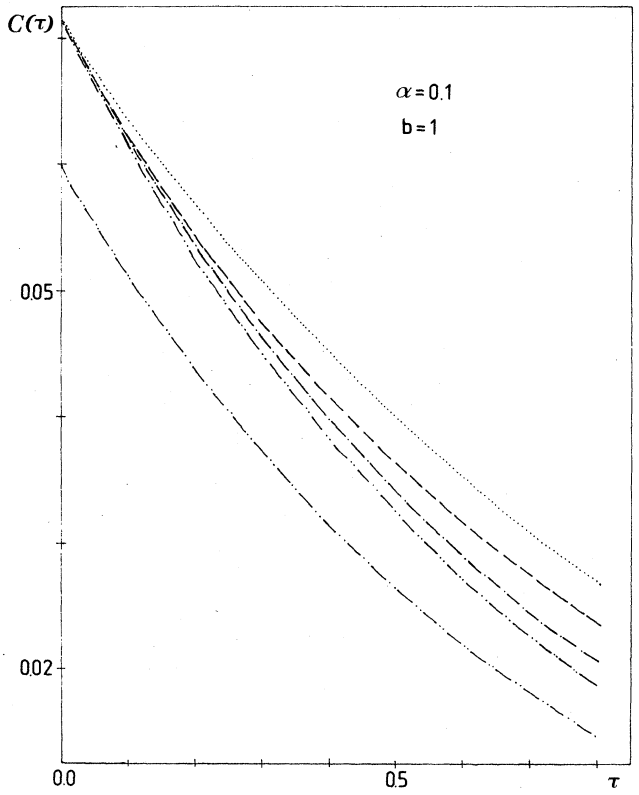


FIG. 3. Equilibrium time correlation function versus time. · · · ·, linear ($b=0$); — — —, Monte Carlo; —··—··—, van Kampen's expansion; —·—·—, LE approximation.

In Fig. 2 we have plotted the equilibrium correlation function corresponding to the values $\alpha=0.1$ and $b=0.5$ as a function of time. We have just plotted the behavior for short times as the behavior for all times will be discussed later on in terms of the spectral density. The Monte Carlo data shows an actual exponential decay with an exponent which is approximately equal to 1.07. It is seen that the LE approximation provides a quite good description of the relaxation of $C(\tau)$ for the times we are considering. Of course, as time increases, the difference with Monte Carlo results becomes more appreciable, as suggested in the discussion at the end of the preceding section.

The comparison of the $\alpha$ expansion method is not that good. One could think that most of the discrepancy is due to the initial value, but we will see in Fig. 3 that this is not the case. When discussing the applicability of the $\alpha$ expansion it should be kept in mind that the actual expansion parameter turns out to be the product $\alpha b$, according to Eq. (47). In Fig. 2 $\alpha b=0.05$. A check of the "smallness" of this parameter could be to consider the relaxation of a purely linear system, namely, to set $b=0$. The plot shows that, in fact, the linear approximation is not a bad one in this case.

In Fig. 3 we have considered the case $\alpha=0.1$, $b=1$. We have also plotted Eq. (47), but with the correct initial value instead of $\alpha(1-3b\alpha)$. Once again, it is seen that the LE approximation gives a good description of the relaxation of $C(\tau)$. On the other hand, for not very short

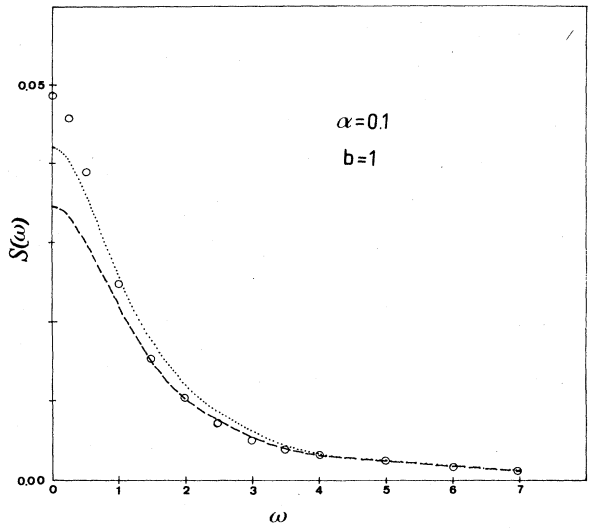


FIG. 4. Spectral density versus frequency. ○, Monte Carlo; · · · ·, LE approximation; — — —, van Kampen's expansion.

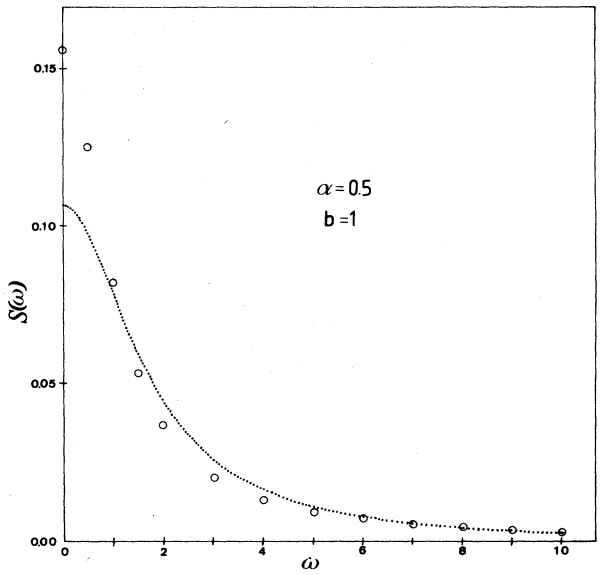


FIG. 5. Spectral density versus frequency. ○, Monte Carlo; · · · ·, LE approximation.

times, the linear approximation gives results at least as accurate as those of the $\alpha$ expansion method.

In Figs. 4 and 5 we present the results for the spectral density $S(\omega)$ defined in Eq. (12). In the $\alpha$ expansion approximation it is given by

$$S_{\mathrm{vK}}(\omega)=\frac{2\alpha}{\pi}\frac{1-9b^2\alpha^2}{(1+3b\alpha)^2+\omega^2}\ , \tag{49}$$

while in the LE approximation we have

$$S_{\mathrm{LE}}(\omega)=\frac{2}{\pi}\frac{\Omega^2\langle x^2\rangle_{\mathrm{eq}}}{\Omega^4+\omega^2} \tag{50}$$

with

$$\Omega^2=1+b\frac{\langle x^4\rangle_{\mathrm{eq}}}{\langle x^2\rangle_{\mathrm{eq}}}\ . \tag{51}$$

For $\alpha=0.1$, $b=1$ the low-frequency behavior is better described by the LE approximation than by the $\alpha$ expansion method. The fact that the LE approximation is increasingly less good as the frequency decreases is a direct consequence of its short-time character already discussed. To check whether the LE approximation remains useful for greater values of the diffusion constant, we have considered in Fig. 5 the case $\alpha=0.5$, $b=1$, for which the $\alpha$ expansion method cannot be applied, as correction terms are even larger than the leading ones. Again, the comparison is quite satisfactory except for the very-low-frequency region. Our final conclusion is that the LE approximation seems to be a useful tool to describe the relaxation in systems where the coupling between fluctuations and nonlinearity is relevant.

## ACKNOWLEDGMENT

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