Squeezing spectra for nonlinear optical systems

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The squeezing spectra for the output fields of several intracavity nonlinear optical systems are obtained. It is shown that at critical points, e.g., the turning points for optical bistability, the threshold for parametric oscillation, and the self-pulsing instability in second-harmonic generation, perfect squeezing in the output field is, in principle, possible.

I. INTRODUCTION

Until recently, theoretical treatments of the generation of squeezed states inside a cavity have focused on the squeezing in the internal cavity mode, with less than dramatic results for the maximum obtainable squeezing. For instance, Milburn and Walls[1] found that the best squeezing attainable in the internal mode of a degenerate parametric oscillator was a reduction in the fluctuations by a factor of only 2. This result was confirmed by Lugiato and Strini[2] and appeared to be generic to intracavity systems, as analysis of several optical systems indicated, e.g., optical bistability,[2,3] second-harmonic generation,[4,5] and multiphoton absorption.[3,6]

The situation changed with the realization that the squeezing in the output field cannot be equated to that of the internal field; in fact, as the external field possesses a continuum of modes, there must be a complete spectrum of squeezing. Yurke[7] gave a single frequency component analysis of the parametric oscillator with a single-ended cavity configuration which showed that arbitrary squeezing was possible in the output field. Multimode analyses by Collett and Gardiner[8] and Gardiner and Savage[9] have confirmed this result, though only for a limited frequency band around the cavity resonance. The analysis has recently been extended to include frequencies corresponding to nearby cavity resonances by Yurke.[10]

In this paper we wish to study several nonlinear optical processes occurring inside cavities and calculate the squeezing in the output field. In particular, we will study the nonlinear polarizability model of optical bistability, two photon absorption, and sub-second-harmonic generation. We begin by outlining the basis of calculation, which is an application of the input-output theory of dissipative quantum systems developed by Collett and Gardiner[8] and particularly suited to problems of this kind.

II. SQUEEZING SPECTRUM OF THE OUTPUT FIELD

Our objective is to calculate the spectrum of squeezing in the output field of an optical cavity containing a nonlinear medium. In order to do this, we make use of the following expressions relating the two time-correlation functions of the output field to the internal field. For an optical cavity for which the input field is entirely coherent or vacuum, it is possible to express the moments of the output field quite simply in terms of time-ordered moments of the internal field.[8] In particular,

$$\langle a_{\mathrm{out}}^\dagger(t), a_{\mathrm{out}}(t')\rangle = 2\gamma\langle a^\dagger(t), a(t')\rangle \; , \tag{2.1}$$

$$\langle a_{\mathrm{out}}(t), a_{\mathrm{out}}(t')\rangle = 2\gamma\langle a[\max(t,t')], a[\min(t,t')]\rangle \; , \tag{2.2}$$

where γ is the damping constant of the internal field, and we have used the notation $\langle a,b\rangle = \langle ab\rangle - \langle a\rangle\langle b\rangle$. Now the behavior of quantum optical systems is frequently described using c-number representations. Implicit in such a representation is a favored ordering of the system operators. For example, in the P-function representation of the density operator[11,12]

$$\rho = \int P(\alpha)\,|\alpha\rangle\langle\alpha|\,d^2\alpha \; , \tag{2.3}$$

equal time moments of the c-number variables correspond to normally ordered moments of the operators. Two time moments imply precisely the time ordering of the internal operators that we need to evaluate the corresponding output moments. This can be demonstrated by noting that the evolution of the system will, in general, mix $a^\dagger$ and a. Hence, $a(t+\tau)$ contains elements of both $a(t)$ and $a^\dagger(t)$. In a normally ordered two time product $a(t+\tau)$ must, therefore, stand to the left of $a(t)$; similarly, $a^\dagger(t+\tau)$ must stand to the right of $a^\dagger(t)$. That is,

$$\langle \alpha(t+\tau)\alpha(t)\rangle = \langle a(t+\tau)a(t)\rangle \; , \tag{2.4}$$

$$\langle \alpha^*(t+\tau)\alpha^*(t)\rangle = \langle a^\dagger(t)a^\dagger(t+\tau)\rangle \; , \tag{2.5}$$

where the left-hand side of these equations represent the average of c-number variables using the P representation.

Including the factor of 2γ going from inside to outside the cavity, we can, therefore, obtain the normally ordered output correlation matrix using

$$:C_{\rm out}(a(t+\tau),a(t)):=\begin{bmatrix}\langle a_{\rm out}(t+\tau),a_{\rm out}(t)\rangle & \langle a_{\rm out}^{\dagger}(t),a_{\rm out}(t+\tau)\rangle\\ \langle a_{\rm out}^{\dagger}(t+\tau),a_{\rm out}(t)\rangle & \langle a_{\rm out}^{\dagger}(t+\tau),a_{\rm out}^{\dagger}(t)\rangle\end{bmatrix}$$

$$=2\gamma\begin{bmatrix}\langle \alpha(t+\tau),\alpha(t)\rangle & \langle \alpha(t+\tau),\alpha^*(t)\rangle\\ \langle \alpha^*(t+\tau),\alpha(t)\rangle & \langle \alpha^*(t+\tau),\alpha^*(t)\rangle\end{bmatrix}$$

$$=2\gamma C_P(\alpha(t+\tau),\alpha(t))\ , \tag{2.6}$$

that is, the two time correlation functions for the output field may be calculated directly from correlation functions of the stochastic variables describing the internal field using the P representation.

Note that all different time output operators commute. From Eq. (2.6) it follows immediately that the corresponding relation for the spectra also holds,

$$\begin{bmatrix}\langle \tilde{a}_{\rm out}(\omega),a_{\rm out}\rangle & \langle a_{\rm out}^{\dagger},\tilde{a}_{\rm out}(\omega)\rangle\\ \langle \tilde{a}_{\rm out}^{\dagger}(\omega),a_{\rm out}\rangle & \langle \tilde{a}_{\rm out}^{\dagger}(\omega),a_{\rm out}^{\dagger}\rangle\end{bmatrix}=2\gamma\begin{bmatrix}\langle \tilde{\alpha}(\omega),\alpha\rangle & \langle \tilde{\alpha}(\omega),\alpha^*\rangle\\ \langle \tilde{\alpha}^*(\omega),\alpha\rangle & \langle \tilde{\alpha}^*(\omega),\alpha^*\rangle\end{bmatrix}, \tag{2.7}$$

where the tilde indicates a frequency-space variable.

For a linearized system $P(\alpha)$ obeys the Fokker-Planck equation

$$\frac{\partial P(\alpha)}{\partial t}=\left[\frac{\partial}{\partial\alpha_i}A\alpha_i+\tfrac{1}{2}D\frac{\partial^2}{\partial\alpha_i\alpha_j}\right]P(\alpha)\ , \tag{2.8}$$

where A is the drift and D the diffusion matrix. This is equivalent to the stochastic differential equation

$$\frac{\partial}{\partial t}\alpha(t)=-A\alpha(t)+D^{1/2}\epsilon(t)\ , \tag{2.9}$$

where $\epsilon(t)$ is a fluctuating force with

$$\langle \epsilon_i(t)\rangle=0\ ,\qquad \langle \epsilon_i(t)\epsilon_j(t')\rangle=\delta_{ij}\delta(t-t')\ . \tag{2.10}$$

The spectral matrix may be obtained directly from Eq. (2.9) as[13]

$$S(\omega)=(A+i\omega)^{-1}D(A^T-i\omega)^{-1}\ . \tag{2.11}$$

To study squeezing we need to look at the variances of the quadrature phases X_+ and X_-, where

$$a=e^{i\theta}(X_+ +iX_-)\ ,\qquad a^{\dagger}=e^{-i\theta}(X_+ -iX_-)\ . \tag{2.12}$$

The phase $e^{i\theta}$ will be chosen to maximize the squeezing. The quantities of interest are

$$:S_{\pm}^{\rm out}:=\langle :\tilde{X}_{\pm}^{\rm out}(\omega),X_{\pm}^{\rm out}:\rangle$$
$$=2\gamma\tfrac{1}{4}[\pm e^{-2i\theta}S_{11}(\omega)+S_{12}(\omega)+S_{21}(\omega)\pm e^{2i\theta}S_{22}(\omega)]\ . \tag{2.13}$$

This result holds for the most favorable configuration, that of a "single-ended cavity." What is really meant by this is that the only losses from the cavity are through the front mirror which acts as an input and output port, i.e., $\gamma=\gamma_{\rm out}$. If there are other significant losses from the cavity, the squeezing spectrum given by Eq. (2.13) is reduced by the factor $\gamma_{\rm out}/\gamma$ where now $\gamma=\gamma_{\rm out}+\gamma_{\rm loss}$.

Thus, we now have a general prescription to calculate the squeezing in the output field from an optical cavity, provided the internal field may be described by a Fokker-Planck equation of the form of Eq. (2.8). In the following sections we shall apply this formalism to some particular systems.

III. DISPERSIVE OPTICAL BISTABILITY AND TWO-PHOTON ABSORPTION

The first system we shall consider is the nonlinear polarizability model of optical bistability.[14] It is convenient to treat at the same time the rather less interesting case (from the point of view of squeezing) of two-photon absorption.

We consider a single-mode field in an optical cavity which is subject to cavity damping and is coherently driven by an external driving field. The cavity contains an intracavity medium with nonlinear optical properties. We shall consider two cases where (a) the medium is a two-photon absorber, and (b) the medium has a nonlinear polarizability. These systems may be described by the following Hamiltonian:

$$\begin{aligned}
H&=\sum_{j=1}^{5}H_j\ ,\\
H_1&=\hbar\omega_c a^{\dagger}a\ ,\\
H_2&=i\hbar(Ee^{-i\omega_L t}a^{\dagger}-E^*e^{i\omega_L t}a)\ ,\\
H_3&=a\Gamma_c^{\dagger}+a^{\dagger}\Gamma_c\ ,\\
H_4&=\hbar\chi'' a^{\dagger 2}a^2\ ,\\
H_5&=a^{\dagger 2}\Gamma_2+a^2\Gamma_2^{\dagger}\ ,
\end{aligned} \tag{3.1}$$

where H_1 describes the cavity mode a with frequency ω_c. H_2 describes the coupling with the coherent driving field with amplitude ϵ and frequency ω_L. H_3 describes the coupling to the cavity reservoir Γ_c. H_4 describes a nonlinear dispersive medium with nonlinear susceptibility χ''. H_5 describes an intracavity two-photon absorber with reservoir operators $\Gamma_2, \Gamma_2^\dagger$.

The linearized stochastic differential equations following from this Hamiltonian have been derived by Drummond and Walls.[14] They have the form of Eq. (2.9) with

$$D=\begin{bmatrix} -\epsilon e^{i\phi} & 0 \\ 0 & \epsilon^* e^{-i\phi} \end{bmatrix}, \tag{3.2}$$

$$A=\begin{bmatrix} \gamma+2\epsilon+i\delta & \epsilon e^{i\phi} \\ -\epsilon^* e^{-i\phi} & \gamma+2\epsilon^*-i\delta \end{bmatrix}, \tag{3.3}$$

where γ is the damping constant, $\delta=\omega_L-\omega_c$ is the detuning between the driving field and the cavity. $\epsilon=-2\chi\,|\,\alpha_0\,|^2$, where α_0 is the steady-state solution for α, and $\chi=\chi'+i\chi''$, where χ' (χ'') is the effective coupling constant for two-photon absorption (dispersive optical bistability). Hence ϵ is an effective driving parameter, real for two-photon absorption and imaginary for dispersive bistability, $e^{i\phi}$ is the phase of the steady-state solution and is of no material consequence in what follows. As we are principally interested in the limits on squeezing, thermal fluctuation terms have been dropped. The spectral matrix obtained directly using Eq. (2.11) is

$$S(\omega)=\frac{1}{|\,\lambda(\omega)\,|^2}\begin{bmatrix} -\epsilon e^{i\phi}[(\gamma+2\epsilon^*-i\delta)^2+\omega^2+|\,\epsilon\,|^2] & 2\,|\,\epsilon\,|^2(\gamma+\epsilon+\epsilon^*) \\ 2\,|\,\epsilon\,|^2(\gamma+\epsilon+\epsilon^*) & -\epsilon^* e^{-i\phi}[(\gamma+2\epsilon+i\delta)^2+\omega^2+|\,\epsilon^2] \end{bmatrix}, \tag{3.4}$$

where

$$\lambda(\omega)=(\gamma+2\epsilon^*-i\delta+i\omega)(\gamma+2\epsilon+i\delta+i\omega)-|\,\epsilon\,|^2 .$$

The spectrum of squeezing may then be calculated from Eq. (2.13) with the phase $e^{i\theta}$ chosen to maximize the squeezing. The situation is complicated by the fact that the best choice of phase varies with ω, while in any practical situation, only one choice of phase can be made at a time. If we want to optimize the squeezing at some particular frequency ω_0, the appropriate choice of phase in Eq. (2.13) is

$$e^{2i\theta}=\frac{S_{22}^*(\omega_0)}{|\,S_{22}(\omega_0)\,|}=e^{i\phi}\frac{\epsilon[(\gamma+2\epsilon^*-i\delta)^2+\omega_0^2+|\,\epsilon\,|^2]}{|\,\epsilon\,|\;|(\gamma+2\epsilon^*-i\delta)^2+\omega_0^2+|\,\epsilon\,|^2|} . \tag{3.5}$$

A. Dispersive optical bistability

For the case of dispersive bistability ($\epsilon=i\,|\,\epsilon\,|$) we then have for the squeezing spectrum in the output field where the phase θ has been chosen to give maximum squeezing at frequency ω_0,

$$:S_\pm^{\text{out}}(\omega):=\gamma\,|\,\epsilon\,|\left[\frac{2\gamma\,|\,\epsilon\,|\mp\dfrac{(\gamma^2+\omega^2-\Delta^2)(\gamma^2+\omega_0^2-\Delta^2)+4\gamma^2(\Delta^2+|\,\epsilon\,|^2)}{[(\gamma^2+\omega_0^2-\Delta^2)^2+4\gamma^2(\Delta^2+|\,\epsilon\,|^2)]^{1/2}}}{(\gamma^2+\omega^2-\Delta^2)^2+4\gamma^2\Delta^2}\right], \tag{3.6}$$

where

$$\Delta^2=(3\,|\,\epsilon\,|+\delta)(\,|\,\epsilon\,|+\delta)=(2\,|\,\epsilon\,|+\delta)^2-|\,\epsilon\,|^2 . \tag{3.7}$$

To find any points of perfect squeezing we can use the fact that perfect squeezing in one quadrature must be associated with an infinite uncertainty in the other, and hence can only occur at a critical point. At a critical point the fluctuations diverge to infinity, resulting in divergence in the fluctuation spectrum (3.6) for some frequency ω. It is readily seen from Eq. (3.6) that the critical points occur at

$$\gamma\pm i\Delta\pm i\omega=0 . \tag{3.8}$$

The accessible ones are with $\omega=0$ and

$$\gamma^2=-\Delta^2=-(3\,|\,\epsilon\,|+\delta)(\,|\,\epsilon\,|+\delta) . \tag{3.9}$$

These values of the driving field $|\,\epsilon\,|$ correspond to the turning points of dispersive optical bistability.[14]

Setting $\omega_0=0$ (to optimize the squeezing for $\omega=0$) and taking a critical combination of the driving and detuning [as given by Eq. (3.9)] gives

$$:S_+^{\text{out}}(\omega):=-\frac{\gamma^2}{4\gamma^2+\omega^2} \tag{3.10}$$

which reaches $-\frac{1}{4}$; perfect squeezing at $\omega=0$. The spectrum of the squeezed quadrature at the critical point is thus a simple Lorentzian of width 2γ and height (or perhaps one should say depth) of $-\frac{1}{4}$. The spectrum of the other quadrature is

$$:S_-^{\text{out}}(\omega):=\frac{\gamma^2}{4\gamma^2+\omega^2}\,\frac{4\,|\,\epsilon\,|^2+\omega^2}{\omega^2} \tag{3.11}$$

which diverges at $\omega=0$. Thus, we have a case of critical quantum fluctuations which are asymmetric in the two quadratures.

To gain an overview of the general behavior of the output squeezing, it is convenient to choose the phase of the quadrature operators so that the peak of the squeezing spectrum is maximized. Three regions of operation can be distinguished. For $\Delta^2<-\gamma^2$, there is no stable solution, and the linearized analysis used here cannot be applied. For $-\gamma^2<\Delta^2<\gamma^2$, the output squeezing spectrum has a single peak at $\omega=0$, of height

$$:S_{\pm}^{\text{out}}(0):=\frac{-\gamma|\epsilon|}{2\gamma|\epsilon|\pm[(\gamma^2+\Delta^2)^2+4\gamma^2|\epsilon|^2]^{1/2}}\ . \tag{3.12}$$

Finally, for $\Delta^2>\gamma^2$, the spectrum splits into two peaks at $\omega^2=\Delta^2-\gamma^2$ with a height

$$:S_{\pm}^{\text{out}}((\Delta^2-\gamma^2)^{1/2}):=-\frac{1}{2}\frac{|\epsilon|}{|\epsilon|\pm(|\epsilon|^2+\Delta^2)^{1/2}}\ . \tag{3.13}$$

The squeezing spectra for several values of the parameters are plotted in Fig. 1. The squeezing of the internal cavity mode may be calculated from the integration of the squeezing spectrum over all ω,[8]

$$\langle :X_{\pm},X_{\pm}:\rangle=\frac{1}{2\pi}\int d\omega\langle :S_{\pm}(\omega):\rangle\ . \tag{3.14}$$

The integral on the right-hand side (rhs) is a maximum for $\omega_0^2=\Delta^2+\gamma^2$ which corresponds to the optimum choice of phase θ for the internal quadratures. Then the best internal squeezing is

$$\langle :X_{\pm},X_{\pm}:\rangle=\frac{|\epsilon|}{4}\frac{|\epsilon|\pm(\gamma^2+\Delta^2+|\epsilon|^2)^{1/2}}{\gamma^2+\Delta^2}\ . \tag{3.15}$$

Even at the critical points this reaches only $-\frac{1}{8}$.

B. Two-photon absorption

For two-photon absorption ($\epsilon=|\epsilon|$) there are no accessible critical points, and hence perfect squeezing cannot be achieved. If δ is taken to be zero, the optimum phase is independent of frequency, and we have simple Lorentzian spectra

$$:S_{+}^{\text{out}}(\omega):=\frac{-\gamma|\epsilon|}{(\gamma+3|\epsilon|)^2+\omega^2}\ , \tag{3.16}$$

$$:S_{-}^{\text{out}}(\omega):=\frac{\gamma|\epsilon|}{(\gamma+|\epsilon|)^2+\omega^2}\ . \tag{3.17}$$

The best output squeezing that can be achieved is

$$:S_{+}^{\text{out}}(0):=-\tfrac{1}{12} \tag{3.18}$$

when $3|\epsilon|=\gamma$. Again for comparison, the internal squeezing is

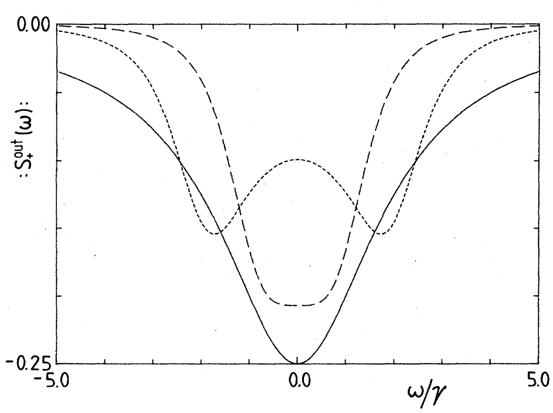

FIG. 1. $:S_{+}^{\text{out}}(\omega)$: for dispersive optical bistability with $|\epsilon|=\gamma$, and $\Delta^2=-\gamma^2$ (solid line), γ^2 (dashed line), $4\gamma^2$ (dotted line).

$$\langle :X_{+},X_{+}:\rangle=-\tfrac{1}{4}\frac{|\epsilon|}{\gamma+3|\epsilon|} \tag{3.19}$$

which is $-\frac{1}{24}$ when $3|\epsilon|=\gamma$, and has a minimum value of $-\frac{1}{12}$ when $|\epsilon|\gg\gamma$.

IV. SUB-SECOND-HARMONIC GENERATION

We consider a nonlinear medium with a significant second-order susceptibility which couples a light mode at frequency ω with its second harmonic at frequency 2ω; The nonlinear medium is placed within a Fabry-Perot interferometer and both modes are driven with external coherent phase locked driving fields. The modes are damped via cavity losses. This system may be described by the following Hamiltonian

$$\begin{aligned}H=&\hbar\omega a_1^\dagger a_1+2\hbar\omega a_2^\dagger a_2+\hbar\frac{\kappa}{2}(a_1^{\dagger 2}a_2-a_1^2a_2^\dagger)\\&+i\hbar(E_1a_1^\dagger e^{-i\omega t}-E_1^*a_1e^{i\omega t})\\&+i\hbar(E_2a_2^\dagger e^{-2i\omega t}-E_2^*a_2e^{2i\omega t})\\&+a_1\Gamma_1^\dagger+a_1^\dagger\Gamma_1+a_1\Gamma_2^\dagger+a_2^\dagger\Gamma_2\ .\end{aligned} \tag{4.1}$$

a_1 and a_2 are the boson operators for modes of frequency ω and 2ω, respectively. κ is the coupling constant for the interaction between the two modes and the spatial mode functions are chosen so that κ is real. Γ_1,Γ_2 are heat bath operators which represent cavity losses for the two modes and ϵ_1 and ϵ_2 are proportional to the coherent driving field amplitudes. The linearized stochastic differential equations resulting from the above Hamiltonian have been given by Drummond, McNeil, and Walls.[15] They have the form of Eq. (2.9) with

$$A=\begin{bmatrix}\gamma_1 & -\epsilon_2 & -\epsilon_1^* & 0\\ -\epsilon_2^* & \gamma_1 & 0 & -\epsilon_1\\ \epsilon_1 & 0 & \gamma_2 & 0\\ 0 & \epsilon_1^* & 0 & \gamma_2\end{bmatrix}, \tag{4.2}$$

$$D=\begin{bmatrix}\epsilon_2 & 0 & 0 & 0\\ 0 & \epsilon_2^* & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{4.3}$$

where γ_1,γ_2 are the cavity damping rates for the two modes and $\epsilon_2=\kappa\alpha_2^0$, $\epsilon_1=\kappa\alpha_1^0$, where α_1^0,α_2^0 are the steady-state values of α_1 and α_2. The subscripts 1 and 2 refer to low- and high-frequency modes, respectively. The spectral matrix is obtained by the same method as in the previous section. In this case the best choice of combination phase is not frequency dependent, so that for the spectra of the quadrature phase operators we obtain the comparatively straightforward expressions

$$:S_{1\pm}^{\text{out}}(\omega):=\pm\frac{\gamma_1|\epsilon_2|(\gamma_2^2+\omega^2)}{[\gamma_2(\gamma_1\mp|\epsilon_2|)+|\epsilon_1|^2-\omega^2]^2+\omega^2(\gamma_1\mp|\epsilon_2|+\gamma_2)^2}, \tag{4.4}$$

$$:S_{2\pm}^{\text{out}}(\omega):=\pm\frac{\gamma_2|\epsilon_2||\epsilon_1|^2}{[\gamma_2(\gamma_1\mp|\epsilon_2|)+|\epsilon_1|^2-\omega^2]^2+\omega^2(\gamma_1\mp|\epsilon_2|+\gamma_2)^2}. \tag{4.5}$$

There are two accessible critical points, the threshold for subharmonic generation and the instability point for second-harmonic generation,[16] giving four regions of operation: (a) subharmonic generation below threshold, with $|\epsilon_1|=0$, $|\epsilon_2|<\gamma_1$; (b) subharmonic generation above threshold, with $|\epsilon_1|>0$, $|\epsilon_2|=\gamma_1$; (c) second-harmonic generation below the instability point, with $|\epsilon_1|^2=2\gamma_2|\epsilon_2|$, $|\epsilon_1|<\gamma_1+\gamma_2$; and (d) second-harmonic generation above the instability point, $|\epsilon_1|>\gamma_1+\gamma_2$, where a linearized analysis is not applicable.

A. Subharmonic generation: The low-frequency mode

This is just the degenerate parametric amplifier. Below threshold Eq. (4.4) simplifies for the squeezed quadrature to

$$:S_{1-}^{\text{out}}(\omega):=\frac{-\gamma_1|\epsilon_2|}{(\gamma_1+|\epsilon_2|)^2+\omega^2} \tag{4.6}$$

a simple Lorentzian increasing in height and width with $|\epsilon_2|$, until at threshold

$$:S_{1-}^{\text{out}}(\omega):=\frac{-\gamma_1^2}{4\gamma_1^2+\omega^2} \tag{4.7}$$

with a peak height of $-\frac{1}{4}$ (perfect squeezing). The same result has been obtained by different methods by Yurke,[7] Collett and Gardiner,[8] and Gardiner and Savage.[9] Above threshold the spectrum is more complicated. The peak's height reduces for increasing $|\epsilon_1|$, and at

$$|\epsilon_1|^2=\gamma_2^2\{[\gamma_2^2+(\gamma_2+2\gamma_1)^2]^{1/2}-(\gamma_2+2\gamma_1)\} \tag{4.8}$$

the spectrum becomes double peaked. The spectrum at this point is

$$:S_{1-}^{\text{out}}(\omega):=\frac{-\gamma_1^2}{\{\gamma_2-[\gamma_2^2+(\gamma_2+2\gamma_1)^2]^{1/2}\}^2+\dfrac{\omega^4}{\gamma_2^2+\omega^2}}. \tag{4.9}$$

If the high-frequency losses from the cavity are insignificant ($\gamma_2\ll\gamma_1$), this splitting occurs immediately above threshold, with the greatest squeezing being at $\omega=\pm|\epsilon_1|$, and remaining close to $-\frac{1}{4}$ even for large $|\epsilon_1|$. The squeezing spectrum for the low frequency mode is shown in Fig. 2 on and above threshold.

B. Subharmonic generation: The high-frequency mode

Below threshold this mode is not squeezed. Above threshold, the peak squeezing (at $\omega=0$) increases to a maximum value of $-\frac{1}{8}$ when $|\epsilon_1|=2\gamma_1\gamma_2$, and falls off thereafter. Slightly later, at $|\epsilon_1|=2\gamma_1^2+\frac{1}{2}\gamma_2^2$, we again find a splitting into a double peak.

For the special case of $2\gamma_1=\gamma_2$, the point of splitting coincides with the point of maximum squeezing, $|\epsilon_1|=\gamma_2$. The spectrum at this point becomes

$$:S_{2-}^{\text{out}}(\omega):=\frac{-\frac{1}{2}\gamma_2^4}{\omega^4+4\gamma_2^4}. \tag{4.10}$$

The greatest squeezing for higher $|\epsilon_1|$ is then at $\omega=\pm(|\epsilon_1|^2-\gamma_2^2)^{1/2}$ and remains $-\frac{1}{8}$. The squeezing spectrum for the high-frequency mode above threshold is shown in Fig. 2.

C. Second-harmonic generation

For second-harmonic generation there is a less marked difference in the overall shape of the squeezing spectra of the two modes than for subharmonic generation, and the two may usefully be considered together. In both cases, the peak squeezing increases monotonically as $|\epsilon_2|$ increases from zero to the critical value $|\epsilon_2|=\gamma_1+\gamma_2$. Above the critical point the system exhibits self-sustained oscillations.[16] In a similar fashion to those of the subharmonic spectrum, we have divisions into two peaks, first for the low-frequency mode, and then (provided $\gamma_2^2>\frac{1}{2}\gamma_1^2$) for the high-frequency mode.

At the critical point itself, with the critical frequency being $\omega_c^2=\gamma_2(\gamma_2+2\gamma_1)$ (which is in fact the initial frequency of the hard mode oscillations) we have for the low-frequency mode

$$:S_{1-}^{\text{out}}(\omega_c):=-\frac{1}{4}\frac{\gamma_1}{\gamma_1+\gamma_2}, \tag{4.11}$$

and for the high-frequency mode

$$:S_{2-}^{\text{out}}(\omega_c):=-\frac{1}{4}\frac{\gamma_2}{\gamma_1+\gamma_2}. \tag{4.12}$$

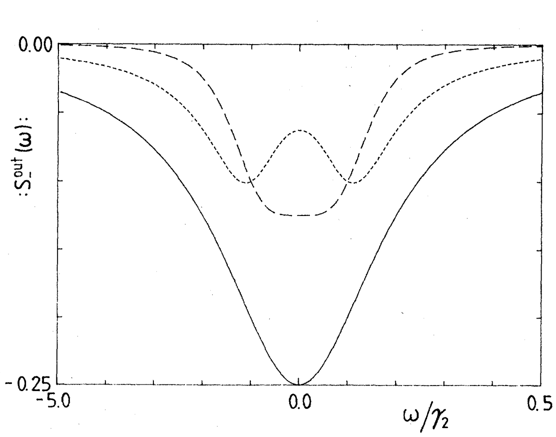

FIG. 2. Subharmonic generation with $\gamma_2=2\gamma_1$. $:S_{1-}^{\text{out}}(\omega)$: with $|\epsilon_1|=0, |\epsilon_2|=\gamma_1$ (solid line), $:S_{-1}^{\text{out}}(\omega)$: with $|\epsilon_1|=\gamma_2$ (dotted line), $:S_{2-}^{\text{out}}(\omega)$: with $|\epsilon_1|=\gamma_2$ (dashed line).

The critical frequency is not, in fact, the one giving the greatest squeezing, except in the following limiting cases, but is close to it. If $\gamma_1 \gg \gamma_2$ perfect squeezing is approached in the low-frequency mode at $\omega = 0$, while, conversely, if $\gamma_2 \gg \gamma_1$, it is approached in the high-frequency mode at $\omega = \pm\gamma_2$. The squeezing spectra for the two modes at the critical point are shown in Fig. 3.

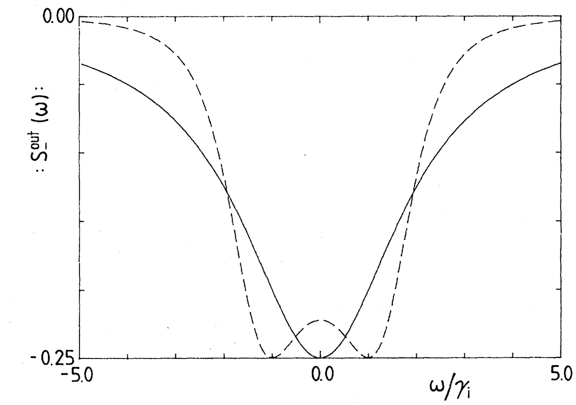

FIG. 3. Second-harmonic generation at the critical point. $:S_{1-}^{out}(\omega)$: against ω/γ with $\gamma_1 \gg \gamma_2$ (solid line), $:S_{2-}^{out}(\omega)$: against ω/γ_2 with $\gamma_2 \gg \gamma_1$ (dashed line).

V. SUMMARY

A direct relationship between the correlation functions of the output field and those for the internal variables averaged via the P function exists provided the normally ordered variance of the input field is zero. We have used this relation to obtain the linearized output squeezing spectra for the nonlinear polarizability model of optical bistability, two-photon absorption, and sub-second-harmonic generation. At the turning points of optical bistability, the threshold for parametric oscillation and the instability point for self-pulsing in second-harmonic generation, we have found that it is, in principle, possible to approach perfect squeezing in one quadrature of the output field.

This is an example of quantum critical fluctuations which are asymmetric in the two quadrature phases. It is clear that in order to approach zero fluctuations in one quadrature it is necessary to operate near a critical point where the fluctuations in the other quadrature approach infinity. We note that the linearization procedure we have used will break down in the vicinity of the critical point and in practice the systems will operate some distance from the critical point.

All these results are, for the most favorable configuration, that of a single-ended cavity. That is, it is assumed that the only losses from the cavity are through the front mirror which acts as an input and output port. If there are other significant losses from the cavity, either from the back mirror or the nonlinear medium, the maximum squeezing obtained is reduced. Considerable squeezing can still be obtained if the other losses are sufficiently small. However, if a quantized model of the nonlinear medium is taken (see, for example the two-level atom model for four-wave mixing[17]) quantum fluctuations in the medium limit the maximum squeezing attainable.

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