Heuristic stochastic model of mirrorless optical bistability

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A heuristic microscopic two-level-atom model with fluctuations is developed to explore, theoretically, the effect of fluctuations on mirrorless optical bistability. In stochastic models, bistability appears as a bimodal distribution function. We show that the predictions of bimodality and bistability are the same in the heuristic and semiclassical (nonstochastic) microscopic models.

I. INTRODUCTION

Over the last decade, there has been considerable interest in the problem of optical bistability[1,2] (OB), which has usually concentrated on Fabry-Perot devices[3] or ring cavity[4] devices. Recent experimental[5] and theoretical[6–9] interest has involved devices in which there are no specific mirrors or surfaces in the system to provide the feedback, i.e., "mirrorless" optical bistability (MOB). Since there are many systems in which MOB might occur, with a diversity of mechanisms, we simplify our discussion by confining it exclusively to models in which the medium consists of stationary, homogeneously broadened two-level atoms and in which certain common assumptions like the rotating-wave approximation are made. No experiment has as yet seen MOB under circumstances in which such a model is appropriate, and various theoretical treatments (see below for details) have been inconsistent on the issue of whether or not MOB exists.

In this paper we construct a heuristic stochastic microscopic model of an N-atom system. The purpose of studying this model is that it is statistical and is thus based on a set of approximations that are different from those that are used in the other models. It contains an explicit random modeling of decays which can be integrated numerically. We find that the results of this model are consistent with the semiclassical predictions of bistability, even to details of operation. We believe that this provides further credence to the idea that MOB in two-level-atom systems is a real phenomenon.

A. Background

In MOB the feedback is provided by the image charge of each atom that is induced in the polarization of the other atoms. The feedback results in a shift of the frequency of the atomic resonance. As the system saturates with increasing incident field, the dipoles and therefore the fed-back signal become small and the frequency shift decreases. Systems with field-dependent frequency shifts often have parameter ranges in which they are bistable, and our system is no different. The difference between OB in a cavity and MOB is that there is no trapped field in MOB; instead, the matter system exists in more than one state without necessarily having a strong effect on the field.

Semiclassical theories are derived from quantum electrodynamics (QED) through a factorization of binary products of operators. QED can always be formulated[10] such that, if the incident field is a coherent state, the factorizations read $\langle O_\alpha(t)O'_\beta(t_R)\rangle=\langle O_\alpha(t)\rangle\langle O'_\beta(t_R)\rangle$, where O and O' are atomic operators, $\alpha\neq\beta$ label atoms, and t and t_R denote time and a retarded time. In this usage all theories showing bistability use factorizations and all but one[6] are semiclassical. Thus MOB addresses the proper use of factorizations, especially those that relate semiclassical and quantum descriptions of optical systems.

Macroscopic semiclassical theory based on a self-consistent coupling of the density matrix (or Bloch equations) with the Maxwell equations[4] does not predict MOB. This failure results from using the self-consistent Maxwell field as the force field acting on the atoms.[11] The Maxwell field contains a fictitious self-field term[12] which exactly cancels out the feedback from the other atoms, thus preventing MOB. Macroscopic Maxwell-Bloch theories are usually applied to cases in which atomic densities are low, so that the self-field term is small. MOB is a high-density phenomenon and needs a more careful treatment. In a recent paper[8] we considered two alternative models, a microscopic semiclassical model in which the self-field term does not exist and a macroscopic model in which we include the local field correction[13] to cancel out the self-field term and restore the feedback. Thus MOB also addresses the issue of the passage between microscopic and macroscopic electrodynamics. When the frequency shift of MOB is evaluated far off resonance it yields the Lorenz-Lorentz correction to the index of refraction,[12] and MOB is one of the nonlinearities associated with this correction. MOB is also the dispersive, steady-state analog of the transient, resonant phenomena, superradiance and subradiance.[14,15]

In microscopic theory one is faced with a large number of different atom-atom interactions. The details of the spatial configuration do have important consequences for

the details of MOB. Macroscopic theory ignores these details by lumping them into a single coefficient. In order to make a comparison of the two cases we make an ansatz that the coefficients describing the spatial details can be taken to be the same for all atom-atom interactions. This ansatz eliminates all microscopic structural details of the N-atom system. The macroscopic and microscopic theories then can have steady states that are equivalent.[16] However, both models permit, in principle, a very large number of stationary states other than the ones involved in MOB. By numerical integration of the microscopic model we have tried, without success, to force the system to reach one of these states. We still are not certain whether these states do not exist in practice, or whether they are not stable. We have also done a few calculations on a seven-atom system in which the microscopic structural details have been retained. This case shows MOB, but it is not clear how to relate its threshold and other properties to the macroscopic coefficients. Hence our ansatz does not cause MOB (any more than it causes the Lorenz-Lorentz correction) and it permits generalizations.

The two-atom problem has been explored theoretically using quantum electrodynamics[17] in which the factorization hypothesis is not made, and retardation is included, but without an external field. A short-time analysis has been given for a case with an external field.[18] The fluorescence of a spatially distributed many-body system of laser-driven three-level atoms has also been addressed,[19] where the factorization hypothesis has been used. These investigations have given no support to the idea that MOB exists. The expected correspondence between the prediction of semiclassical theory and QED are bistability with regard to the former and bimodality with regard to the latter. If written in the Heisenberg picture,[10] the operator equations of motion lead to an infinite hierarchy for the moments that need to be truncated in order to be tractable. Approximations leading to the imposition of equal-time commutation relations permit truncation at second moments. However, a second-moment analysis cannot address the issue of whether a distribution function is bimodal, so the current QED analysis is incapable of addressing the issues raised in semiclassical theory. The quantum analysis includes fluctuations, which are absent in the semiclassical model. The heuristic semiclassical model used here also addresses the role of fluctuations in MOB. Since our results are completely consistent with the semiclassical result, we have shown that if MOB turns out to be inconsistent with QED, it is not because of fluctuations alone.

Calculations using many-body theory in the single-mode approximation also predict MOB.[6] These involve factorizations that are different from the semiclassical factorizations. Such models ignore propagation, and we have not yet attempted to relate the results of this model to the ones discussed above. Hence they are not considered in any of the subsequent discussion.

The QED results[17–19] imply that the factorization procedure of semiclassical theory is invalid. While the heuristic model arises from a factorization of operators, it reintroduces the factorization question through the stochastic variables, insofar as one needs to make a factorization of the statistical average of binary products of atomic variables in order to derive the microscopic semiclassical model from the heuristic model. We thus examine the factorization approximation from the perspective of the heuristic model and find that its validity depends explicitly on the interpretation of the variables in semiclassical theory.

II. HEURISTIC MODEL

The heuristic model is based on the microscopic, semiclassical density-matrix and Maxwell equations of two-level atoms interacting with a radiation field. The slowly varying envelope approximation is made with respect to time. The convention for defining the slowly varying amplitude $\mathscr{E}$ of the electric field E reads

$$E(t,z)=\tfrac{1}{2}[\mathscr{E}(t,z)\exp(-i\nu t)+\text{c.c.}]\ , \tag{1}$$

and the slowly varying amplitude of the off-diagonal density-matrix element of the αth atom is $R_\alpha(t)$, defined by

$$\rho_{ab}(\alpha)=\tfrac{1}{2}[iR_\alpha(t)\exp(-i\nu t)+\text{c.c.}]\ . \tag{2}$$

The plane-wave field incident on the system is denoted E_I (amplitude $\mathscr{E}_I$) and is linearly polarized in the direction $\hat{\mathbf{e}}_I$. Here, ν is the frequency, ρ denotes the 2×2 density matrix, $i=\sqrt{-1}$, and c.c. denotes the complex conjugate. For simplicity we assume all atomic dipoles point in the direction $\hat{\mathbf{e}}_I$ and that all atoms lie in a plane perpendicular to it. We write the atom-atom coherent dipole interaction $A_{\alpha\beta}$ as

$$A_{\alpha\beta}=\zeta\exp(ikr_{\alpha\beta})\left[\frac{1}{kr_{\alpha\beta}}+\frac{i}{(kr_{\alpha\beta})^2}-\frac{1}{(kr_{\alpha\beta})^3}\right], \tag{3}$$

where $r_{\alpha\beta}=|r_\alpha-r_\beta|$ is the separation between atoms α (located at $\mathbf{r}_\alpha$) and β, $\mathbf{k}$ ($k=|\mathbf{k}|$) is the wave vector of the optical field, and ζ is a coefficient that is unity when the variables are made dimensionless (see below). The amplitude of the field incident on atom α is then

$$\mathscr{E}_\alpha=\mathscr{E}_I\exp(ikr_\alpha)+i\sum_{\beta(\neq\alpha)}A_{\alpha\beta}R_\beta(t)\ . \tag{4}$$

The time retardation of the slowly varying amplitudes of the polarizations is ignored, but the phases are preserved. This approximation is ostensibly similar to the neglect of retardation in the quantum theory. However, in the latter the approximation is imposed on operators, which effects the structural integrity of the theory, while here it is imposed on c-numbers and does not alter the equations in any fundamental way. Note that there is no term involving $R_\alpha(t)$ in Eq. (4), and hence there is no self-field. The atoms obey the equations

$$\frac{dR_\alpha(t)}{dt}=-\sum_q \delta(t-t_{q\alpha})R_\alpha(t) - i\Delta_\alpha R_\alpha(t)+\mathscr{E}_\alpha n_\alpha(t)\ , \tag{5}$$

$$\frac{dn_\alpha(t)}{dt}=-\sum_q \delta(t-t_{q\alpha})[1+n_\alpha(t)] - \tfrac{1}{2}[\mathscr{E}_\alpha R_\alpha(t)^*+\text{c.c.}]\ , \tag{6}$$

where Δ_α is the detuning (taken to be independent of α for all calculations discussed in this paper). The Dirac delta functions act on the discontinuous atomic variable $v_\alpha(t)$ by evaluating it at a time just prior to the time $t_{q\alpha}$, i.e.,

$$\int dt\,\delta(t-t_{q\alpha})v_\alpha(t)=\lim_{t\to t_{q\alpha}} v_\alpha(t)\ . \tag{7}$$

The $t_{q\alpha}$ are sets of randomly chosen, statistically independent times such that on the average there is one $t_{q\alpha}$ for each unit of time and for each α. For all but a set of singular points, the atoms obey undamped Bloch equations. The delta functions, which give damping and fluctuations, formally represent a procedure implemented in the numerical integration in which the αth atom is returned to the ground state (with zero polarization) whenever a time $t_{q\alpha}$ is encountered. Physically, this is equivalent to removing the αth atom from the system at $t_{q\alpha}$ and replacing it with another atom in the ground state. The statistics of the $t_{q\alpha}$ imply that, on the average, each atom decays once in a unit time interval, which sets the decay time equal to unity. All Rabi frequencies and detunings are made dimensionless by this choice. Under this model of decay, the decay rates for the polarization and inversion are the same. In a proper quantum model the rates differ by a factor of 2. The model thus contains unrealistic elements and the results need to be treated with some caution.

A. Simplifications

The response of the N-atom system is very sensitive to geometry. However, the amount of parameter variation implicit in Eq. (4) is unmanageable. We therefore restrict our discussion to the case in which we take $r_{\alpha\beta}$ to be independent of α and β, and $kr_\alpha \ll 1$. We set $A_{\alpha\beta}=A$, and define a term $B=(N-1)A$. The real part of B corresponds to the factor that enters into the macroscopic theory through the local field correction[8,16] when the conventional formula[13] is used. We restrict our attention to the case in which all atoms have a detuning $\Delta_\alpha=-1.5$ in which case the macroscopic semiclassical threshold condition for bistability[8] reads $\mathrm{Re}B<-6$ or

$$\mathrm{Re}A<-\frac{6}{N-1}\ . \tag{8}$$

In the limit that $kr<1$, ReA goes as $1/r^3$ and hence varies rapidly with r, while ImA is independent of r and hence is independent of geometry. Unless stated otherwise, we use Im$A=-0.46$ which comes from a calculation in which real geometries were used. It turns out that the results are not greatly influenced by small changes in ImA. Note that ImA leads to superradiant[14] and subradiant[15] decays, both of which occur in our system. However, bistability is a phenomenon that occurs on timescales long compared to decay times, and it is not surprising that these coherent decay phenomena play a minor role in the bistability. However, we show below that these decays have an influence on passage times. The upper and lower bistable states have no associations with the Dicke states that enter into the analysis of superradiance.[14,20]

III. DIAGNOSTICS

The numerical analysis generates a time trace which we treat as data. In particular, we choose to examine the statistics of the inversions using the function

$$\bar{n}(t)=\frac{1}{N}\sum_{\alpha=1}^{N}\frac{1}{T_F}\int_{-\infty}^{t} dt'\,e^{-(t-t')/T_F}n_\alpha(t')\ . \tag{9}$$

In Eq. (9) the inversion is averaged over all atoms and is time averaged by a filter whose time constant is T_F (bandwidth $\sim 1/T_F$). The filter is important, since there are at least two timescales in the atomic response. On short timescales, there are Rabi cycles that do not interest us. We are concerned with fluctuations that occur on times greater than the decay time. Hence we choose $T_F=1$ to filter out the high-frequency response. This choice of T_F is justified by methods described below.

While we have not yet demonstrated bistability, we need to develop a method for examining its statistics should it occur. Accordingly, we define a constant $\bar{n}_c$ such that if $\bar{n}>\bar{n}_c$ the system is said to be in the upper state and for $\bar{n}<\bar{n}_c$ the system is said to be in the lower state. The choice $\bar{n}_c=-0.4$ is made using the same criterion that determines T_F. A passage time is measured from the time the system enters a state to the time it leaves it. Individual passage times are denoted t_u or t_d, where the subscript indicates the state to which the system goes. An up (down) passage time is the time spent in the down (up) state before changing. The mean passage time is denoted T_u (T_d).

A. Criteria

We are interested in examining events that are memoryless, i.e., in which the probability of an event is independent of previous occurrences of similar events. The time intervals in such processes are distributed exponentially. Hence, when we are looking at the passage times of interest, we should find distributions that go as

$$P(t_u)=\frac{1}{T_u}e^{-t_u/T_u} \tag{10a}$$

and

$$P(t_d)=\frac{1}{T_d}e^{-t_d/T_d}\ . \tag{10b}$$

Equations (10a) and (10b) must be simultaneously satisfied. The choice of T_F and $\bar{n}_c$ comes from meeting this requirement. The details of the calculations are given in the Appendix. We do not normally compute enough

statistics to check these distributions in detail. Instead, we use a simpler requirement, namely that the standard deviation in the passage times must equal the means to within 20%, which is the uncertainty in the results due to the stochastic method. This requirement is independently verified for every mean passage time reported.

A system which is predicted to be bistable in a deterministic theory is bimodal in a stochastic theory. The bimodal condition and its passage statistics change with operating conditions. When we compare cases, we try to get as close as possible to the condition $T_u = T_d$, to make the comparison meaningful. When we refer to the influence or lack thereof of some parameter on the results we refer to the statistics at this operating point, not to the choice of incident field that generates it. The latter nearly always depends on specific parameters.

IV. RESULTS

In Fig. 1 we illustrate the points made above for the case of two atoms, which is the only one for which we can compute enough statistics to obtain detailed distributions of passage times. In Fig. 1(a) we give the statistics for $\bar{n}$, and in Fig. 1(b) we give the passage time statistics for a calculation that was continued for about 10^4 decay times. The distribution of inversions is clearly bimodal, and the distribution of decay times is roughly exponential. The mean up (down) passage time is 2.68 (3.35) and its standard deviation is 2.64 (3.30). Hence the standard deviations are nearly equal to the means.

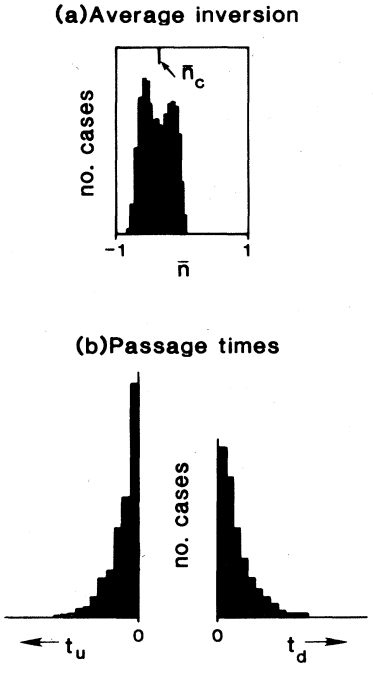

FIG. 1. Statistics for (a) mean inversion and (b) passage times (left, up times increasing to left, right, down times increasing to right), for a two-atom calculation lasting 10^4 decay times. One point per decay time is examined for the statistics of $\bar{n}$. Here $E_I = 3.60$, $\mathrm{Re}A = -8$. The value of $\bar{n}_c$ is shown in (a).

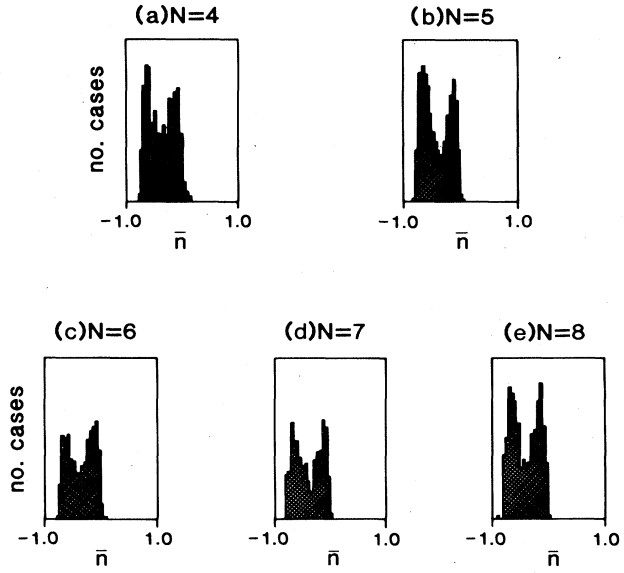

FIG. 2. Statistics of $\bar{n}$ for $N = 4, 5, 6, 7$, and 8.

A. N-atom calculations

Figure 1 establishes that bimodality is possible with as few as two atoms. We now consider how the bimodal condition varies with the number of atoms. The condition $T_u = T_d$ is extremely difficult to determine precisely. In practice, we scan stepwise across the region in which the condition occurs and pick the calculation that meets it most closely. We then report the geometric mean passage time $(T_u T_d)^{1/2}$, which varies slowly in the neighborhood of the point $T_u = T_d$. The geometric mean also varies slowly with different choices of $\bar{n}_c$. In all cases we have investigated the parametric variation of the geometric mean is small compared to the estimated 10% errors of the numerical analysis. These errors are comparable to the uncertainties in the passage times due to the stochastic method.

In Fig. 2 we show the bimodal distributions that are obtained when N is varied from 4 to 8. We keep ReB fixed by simultaneously varying ReA. In Fig. 3 we show the geometric mean passage times. What is evident in Fig. 2 is that the bimodal feature of all distributions is more dis-

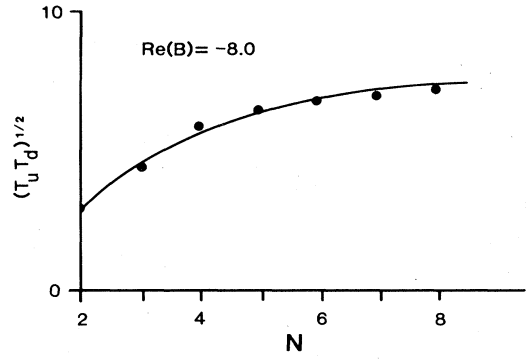

FIG. 3. Geometric mean passage time vs N for Re$B = -8$.

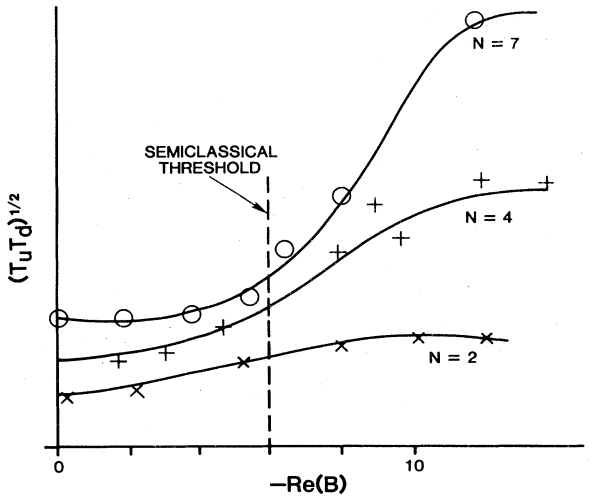

FIG. 4. Geometric mean passage time vs ReB for $N=2$ (crosses), 4 (plusses), and 7 (circles). Location of semiclassical threshold Re$B<-6$ shown as vertical dashed line.

tinct than $N=2$, but there is no rapid rise for larger N. We cannot tell from Fig. 2 whether the degree of bimodality saturates with increasing N or just increases slowly (e.g., as $\sqrt{N}$). The study given below on hysteresis loops suggests that the latter is correct. The passage times in Fig. 3 are consistent with the distributions in Figs. 1 and 2. They rise rapidly up to $N=4$ and then level off.

B. Bimodality versus coupling

We next consider how the bimodal condition varies with changing ReB. In Fig. 4 we show the passage times as a function of ReB for $N=2$, 4, and 7. A vertical line is drawn at Re$B=-6$, the value at which semiclassical theory predicts bistability. All results to the right of the point Re$B=-6$ and none to the left are bimodal. Hence the semiclassical prediction is necessary and sufficient for bimodal distributions. In each case the passage times increase with ReB but saturate at larger values. However, it appears that a simultaneous increase of ReB and N could lead to quite large passage times. Our numerical routine is inadequate to examine such regimes.

The condition Re$A=0$ implies Im$A=0$. Since the latter is fixed in Fig. 4, the case Re$B=0$ does not correspond to any achievable set of parameters, but does provide a calculation in which the only effects of coupling involve superradiant and subradiant decays. We computed Im$A=$Re$A=0$ for $N=2$, 4, and 7 and find that all give mean passage times near 1.5. Since the times in Fig. 4 are substantially larger than 1.5, the superradiant and subradiant effects are appreciable.

C. Hysteresis loops

While we are unable to compute passage times beyond the regimes shown in Figs. 3 and 4, we can study hysteresis. Hysteresis loops are convenient for comparing the results of the heuristic model with semiclassical theory. The microscopic semiclassical equations are the same as Eqs. (5) and (6) except that the factors $\sum_q \delta(t-t_{qa})$ are replaced by a unit constant. In these calculations we take $\mathscr{E}_I$ to vary in time as a sawtooth and make a parametric plot of $\bar{n}(t)$ versus $\mathscr{E}_I(t)$. In Fig. 5(a) we show a hysteresis loop for Re$B=-14$, $N=7$ generated from a dynamical calculation of the semiclasssical theory. In this calculation the cycle time used to complete a loop is 40

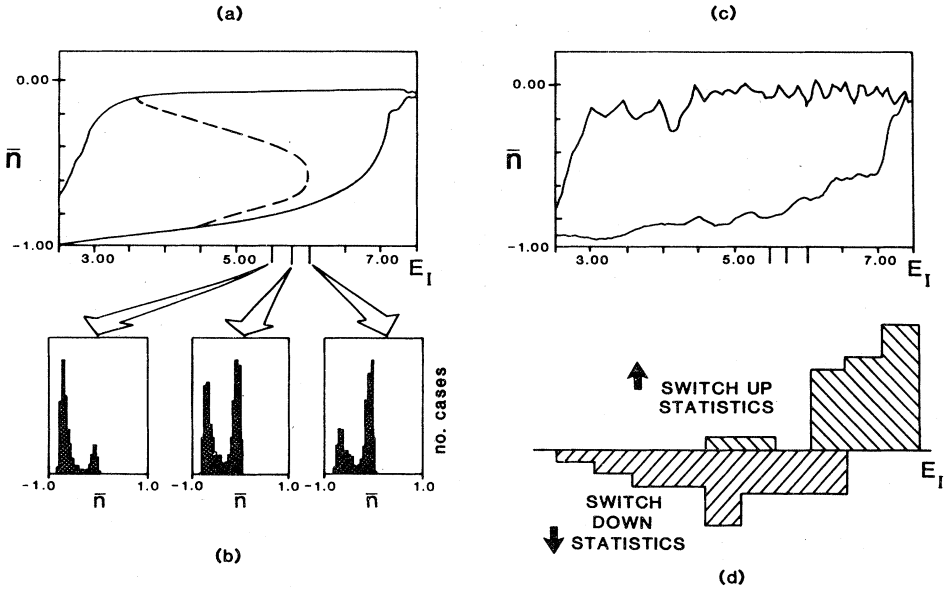

FIG. 5. (a) Semiclassical steady-state loop (— — —) and dynamical loop (solid curve). (b) Histograms of the inversions at $E_I=5.5$, 5.75, and 6.0 [arrows indicate position on axis of (a), bars indicate position in (c)]. (c) Widest of 30 hysteresis loops generated from the heuristic model under conditions identical to those in (a). (d) Statistics of switching in 28 of the 30 loops generated under conditions identical to those in (c). Upper histogram: number of loops that switched up at that value of the incident field. Lower histogram (plotted downwards): number of loops that switched down at that value of the incident field.

decay times (see below for rationale). The dashed curve in Fig. 5(a) is the semiclassical S curve generated from a steady-state analysis (it and the numerical run coincide over much of the plot). Because of the rapid sweep of $\mathscr{E}_I$, the dynamical loop is much wider than the S curve. We have computed a loop with a cycling time of 1000 decay times which does follow the S curve adiabatically. In Fig. 5(b) we show the statistics of $\bar{n}$ for three positions in the loop around the point $\mathscr{E}_I = 5.75$ for which $T_u = T_d$. This point is close to the upper turning point of the S curve which suggests that the upper state is more susceptible to fluctuations than the lower state.

We made 30 hysteresis loops with the heuristic model using the same parameters as in the calculation of Fig. 5(a). In Fig. 5(c) we show the widest hysteresis loop obtained among these runs. This loop agrees well with the semiclassical loop in Fig. 5(a). Note that we need a cycle time that is comparable to the passage times (about 10 decay times). Otherwise, there would be many noise-driven transitions in the course of each cycle. These times are too short for the system to track the S curve adiabatically. We expect that the observed hysteresis loops should be narrower than the one in Fig. 5(c), insofar as the rapid passage time should cause transitions between the states before the turning point is reached. This expectation is verified in 28 of the calculated loops, one of which made two extra noise-driven transitions and is dropped from further consideration. The remaining calculation failed to make a transition to the upper branch. No loop was formed and it is also dropped. In Fig. 5(d) we summarize the statistics of the values of the field at which the transitions occurred for the 28 calculations that resulted in a single loop. While in all cases the up passage occurred at a higher field than the down passage, the passage statistics show that anomalous loops should be possible. The results in Fig. 5 show that the heuristic and semiclassical results are largely consistent as to the details of bistability.

In Fig. 6 we show one of two nearly identical loops generated for $N = 40$. In this calculation we take $\mathrm{Re}A = -0.4$ and $\mathrm{Im}A = -0.2$ (the value of $\mathrm{Im}A$ used earlier is no longer realistic for the interatomic separations giving this $\mathrm{Re}A$) and we have used a cycle time of 100 decays. In this case the choice of cycle time is set by the need to complete the calculation in finite time. For these parameters the upper (lower) turning point of the S curve is at $\mathscr{E}_I = 8$ (5) and the hysteresis loops are in very good agreement with the semiclassical predictions. Hence the bistable condition continues to improve with increasing N.

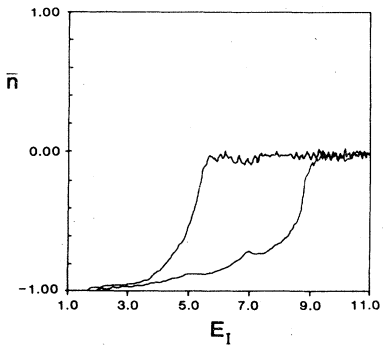

FIG. 6. Hysteresis loops for $N = 40$, $A = -0.4 - 0.2i$, cycle times = 100 decay times.

D. Factorization

In this subsection we consider the validity of making a factorization of the average of products of statistical variables that allows us to derive a semiclassical model from the heuristic model. For the purpose of discussion, we use the notation $\bar{v}_{s\alpha}(t) = \langle v_{s\alpha}(t) \rangle_a$, where v denotes either n or R, the subscript s denotes the statistical variables in the heuristic model, and $\langle\ \rangle_a$ signifies a time average when evaluated numerically and a statistical average in the analysis of Eqs. (5) and (6). A variable without the subscript s denotes a variable in a conventional semiclassical model. When Eq. (4) is used to eliminate $\mathscr{E}_\alpha$ in Eqs. (5) and (6), there are then two products that must be factorized. The factorization of $n_{s\alpha}(t)R_{s\beta}(t)$ in Eq. (5) is essential, since it leads to the frequency shift that makes MOB possible. The factorization of $[R_{s\alpha}(t)]^* R_{s\beta}(t)$ in Eq. (6) is probably not essential since it makes a small contribution to bistability through the term $\mathrm{Im}A$.

We show in Table I the factorizations obtained from a seven-atom calculation for 2000 decay times whose histogram of $\bar{n}(t)$ is the middle figure of Fig. 5(b). We explicitly compute the statistical averages of the variables and products for $\alpha = 1$ and $\beta = 2$ (all atom pairs are equivalent so this choice involves no loss of generality). We also computed factorizations for the $N = 7$ and 8 calculations in Fig. 3 to check consistency. We estimate the numerical error with calculations using $\mathrm{Re}A = \mathrm{Im}A = 0$ and $N = 7$. Factorization must be valid in this case so all discrepancies are numerical. In addition, we computed factorizations for the below-threshold cases in Fig. 4 for $N = 7$. The factorization errors of $\langle [R_{s\alpha}(t)]^* R_{s\beta}(t) \rangle_a$ show no case-to-case consistency. However, the imaginary parts of $\langle n_{s\alpha}(t) R_{s\beta}(t) \rangle_a$ and $\langle n_{s\alpha}(t) \rangle_a \langle R_{s\beta}(t) \rangle_a$ differ by a factor of 2, so the factorization of this essential term is quite inaccurate. This factorization error is much

TABLE I. Numerical values for the average of the products (top) and the product of averages (bottom) from the calculation giving the middle plot in Fig. 5(b).

$\langle n_{s1} R_{s2} \rangle_a = 0.068 + i\,0.149$
$\langle n_{s1} \rangle_a \langle R_{s2} \rangle_a = 0.063 + i\,0.067$

$\langle n_{s1} R_{s2} \rangle_u = 0.034 + i\,0.029$
$\langle n_{s1} \rangle_u \langle R_{s2} \rangle_u = 0.020 - i\,0.002$

$\langle n_{s1} R_{s2} \rangle_l = 0.108 + i\,0.291$
$\langle n_{s1} \rangle_l \langle R_{s2} \rangle_l = 0.102 + i\,0.286$

$\langle R_{s1} R_{s2}^* \rangle_a = 0.121 + i\,0.001$
$\langle R_{s1} \rangle_a \langle R_{s2}^* \rangle_a = 0.063 - i\,0.003$

$\langle R_{s1} R_{s2}^* \rangle_u = 0.080 + i\,0.003$
$\langle R_{s1} \rangle_u \langle R_{s2}^* \rangle_u = 0.039 - i\,0.003$

$\langle R_{s1} R_{s2}^* \rangle_l = 0.171 - i\,0.001$
$\langle R_{s1} \rangle_l \langle R_{s2}^* \rangle_1 = 0.202 - i\,0.002$

larger than can be accounted for by numerical error, is consistent for all bimodal cases, and nearly vanishes for the cases below threshold. To understand these results, note that $\bar{R}_{s\alpha}(t)$ and $\bar{n}_{s\alpha}(t)$ have unique steady-state values. Hence, if factorization were exactly valid, then the resulting equations would necessarily deny bistability. This means that factorization must be invalid if the physical system shows bistability. For the same reasons, if such a system is validly described by semiclassical theory, the variables in the semiclassical theory are not $\langle v_{s\alpha}(t)\rangle_a$'s.

We adopt an *ad hoc* procedure for examining statistical averages that have multiple values. The numerics determine, at every instance, whether the system is in the upper or lower state, and it is a well-defined numerical procedure to take the time averages in the individual states. (We have no idea what statistical averages these correspond to.) We adopt these averages as $\langle\ \rangle_u$ and $\langle\ \rangle_l$ and report them and their factorizations in Table I. By and large the factorizations in the two states are quite accurate. The critical factorization $\langle R_{s\alpha}(t)n_{s\beta}(t)\rangle_l = \langle R_{s\alpha}(t)\rangle_l\langle n_{s\beta}(t)\rangle_l$ is nearly perfect. There is a 15% error in the factorization of $\langle[R_{s\alpha}(t)]^* R_{s\beta}(t)\rangle_l$, which is found in all cases, but is probably unimportant. The factorization in the upper state has large percent errors, but the terms are all nearly zero and the results can be attributed to numerical error. If any of the factorization errors in the upper state are real, they are largely irrelevant. Hence the averages that correspond to the dynamics of the semiclassical variables do factorize.

V. SUMMARY

In this paper we investigate, numerically, a heuristic, microscopic N-atom model with fluctuations. We find the statistics of the system to be bimodal where the corresponding semiclassical theory predicts bistability. The hysteresis loops are in substantial agreement with semiclassical calculations. The bimodality condition improves slowly as the number of atoms and the interatom coupling increases. We also show that the semiclassical calculations agree on the details of MOB to within numerical error, and that there is a way of justifying the factorization of semiclassical theory.

APPENDIX

In this appendix we illustrate some of the results obtained in our verification of our initial choices of $T_F=1$ and $\bar{n}_c=-0.4$. The objective verification of these values is based on the criteria in Eqs. (10a) and (10b). Several independent tests were made to determine T_F, and we illustrate the method with a case using $N=7$, $E_I=5.75$, and $\mathrm{Re}A=-2$. For both this study and the verification of $\bar{n}_c$ we make an unfiltered time trace of 10^3 decay times. This trace is then filtered using various values of T_F until the mean passage times equal the variances. The results of this test are illustrated in Fig. 7 for $T_F=0$, 0.25, 0.5, and 1.0. The statistics of the up passage times are consistently discrepant with the exponential distribution of Eq. (10a) and the standard deviations are substantially different

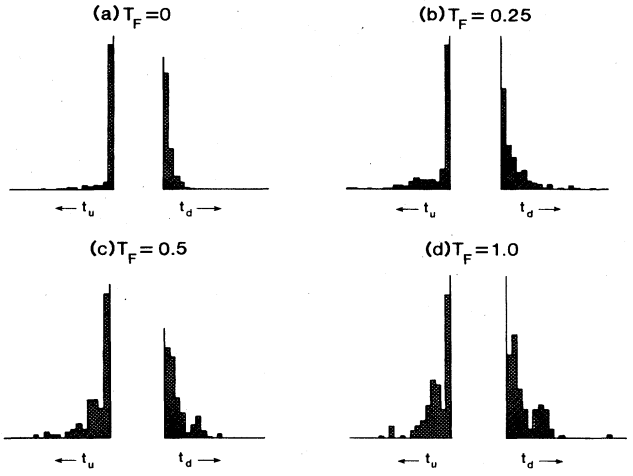

FIG. 7. Passage time statistics for various values of the filter time. See Fig. 1 for details. Denote the standard deviation of the times with ΔT, then (a) $T_u=2.22$, $\Delta T_u=5.22$, $T_d=1.94$, $\Delta T_d=1.77$; (b) $T_u=5.35$, $\Delta T_u=8.25$, $T_d=5.39$, $\Delta T_d=6.72$; (c) $T_u=9.21$, $\Delta T_u=11.03$, $T_d=9.75$, $\Delta T_d=9.66$; (d) $T_u=11.79$, $\Delta T_u=11.66$, $T_d=12.94$, $\Delta T_d=12.83$.

from the means for all values of T_F below 1.0. Hence for values of $T_F<1$, the results are a composite of short- (see the large peak at $t_u=0$ in the figures) and long-time-scale phenomena, indicating that T_F is too small. In other tests we choose parameters such that the down passage times are discrepant. While larger values of T_F are also consistent with the criteria, they simply damp out all temporal responses.

We illustrate in Fig. 8 part of the verification of $\bar{n}_c=-0.4$ with the case $N=4$, $\mathrm{Re}A=-4.67$, and $E_I=5.2$. We set $T_F=1$ and vary $\bar{n}_c$. We show $(T_u T_d)^{1/2}$ as a function of $\bar{n}_c$ as an example of tests that show that it is a slowly varying quantity in the neighborhood of the operating points that we report. For $\bar{n}_c=-0.4$ the mean up (down) time is 7.68 (8.09) and the variance is 7.22 (7.66), which fall within our criteria. For $\bar{n}_c=-0.2$ the mean and variance of the up time are 4.04 and 6.75 and

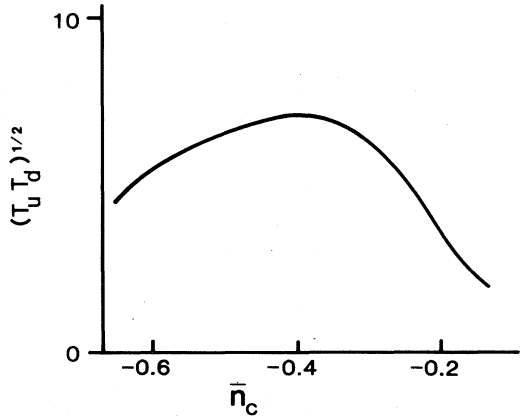

FIG. 8. Geometric mean passage time vs $\bar{n}_c$.

for $\bar{n}_c = -0.6$ the mean and variance of the down time are 6.73 and 10.18. Both are substantially discrepant with the criteria of Eqs. (10a) and (10b) and indicate a two-time-scale response. The second time scale involves the fluctuations about the upper or lower state, which can be quite large for small N. It turns out that at the operating point $T_u = T_d$, the long-time average inversion is within 5% of -0.4, which is how $\bar{n}_c$ was chosen initially.

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