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Theoretical basis for self-similarity in the decay of incompressible fluid turbulence

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By employing space-time dilatation invariance of the Hopf equation for the characteristic functional, it is shown that statistical-dynamical self-similarity must be featured in the free decay of inertia-dominated incompressible fluid turbulence. The experimentally established decay law $u^2 \propto t^{-6/5}$ and integral scale dependence $L \propto t^{2/5}$ follow deductively and without any additive assumption from a Gaussian normal probability distribution over velocity fields at the initial instant $t=0$.

The self-similarity hypothesis for freely decaying incompressible fluid turbulence was first proposed in 1938 by von Kármán and Howarth.[1] In terms of the probability measure $dP[\mathbf{u}(\mathbf{x});t]$ assigned to the velocity field $\mathbf{u}(\mathbf{x})$ at any $t \geq 0$, self-similarity requires

$$dP[\lambda^\alpha \mathbf{u}(\lambda\mathbf{x});\lambda^{-\beta}t] = dP[\mathbf{u}(\mathbf{x});t] \quad (1)$$

for all real $\lambda > 0$ with α and β certain fixed constant indices. Since (1) applies to the larger-scale as well as the fine-scale eddy structure of the turbulent flow, self-similarity is a statistical-dynamical symmetry distinct from a possible Kolmogorov invariance[2] in the fine scale (inertial subrange) at high Reynolds numbers.

In theoretical and experimental applications authors have generally required one or more additional physical assumptions in combination with a self-similarity hypothesis. For example, Saffman[3] postulated self-similarity and made the additional assumption of Reynolds-number independence, which is basically equivalent to assuming that an inertial subrange exists, to deduce the decay law $u^2 \propto t^{-6/5}$ and integral scale dependence $L \propto t^{2/5}$ associated with experimental[4] turbulence at high Reynolds numbers. My purpose in the present Rapid Communication is to show that the self-similar character of freely decaying inertia-dominated turbulence and the associated decay law $u^2 \propto t^{-6/5}$ and integral scale dependence $L \propto t^{2/5}$ follow deductively and without any additive assumption from a Gaussian normal probability distribution at the initial instant $t=0$. Expressed in the functional calculus terminology first proposed by Hopf,[5] my analysis incorporates the important finding of Deissler[6] that viscous decay effects become negligible compared with inertial decay effects at high Reynolds numbers; hence the Hopf equation (6) for the characteristic functional is the $\nu=0$ (zero viscosity) specialization of the more general form required at low or moderate Reynolds numbers.[5]

Let $\mathbf{u}=(u_1(\mathbf{x},t),u_2(\mathbf{x},t),u_3(\mathbf{x},t))$ denote the velocity field of an incompressible fluid flow governed by the Euler equation

$$\partial\mathbf{u}/\partial t = -\mathbf{u}\cdot\nabla\mathbf{u} - \rho^{-1}\nabla p \quad (2)$$

in which ρ is a positive constant. For boundary-free flow with $\mathbf{x}=(x_1,x_2,x_3)$ in $\mathbb{R}_3$, the incompressibility condition $\nabla\cdot\mathbf{u}=0$ can be used to eliminate the pressure term from (2); the resulting integro-differential equation

$$\partial\mathbf{u}/\partial t = -(\mathbf{u}\cdot\nabla\mathbf{u})^{\mathrm{tr}} \quad (3)$$

features the transverse (solenoidal) part of the inertial term, where for any vector field in $\mathbb{R}_3$,

$$\mathbf{v}^{\mathrm{tr}}(\mathbf{x}) \equiv \mathbf{v}(\mathbf{x}) - \nabla(\nabla^{-2}\nabla\cdot\mathbf{v}(\mathbf{x})) = \mathbf{v}(\mathbf{x}) + \frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{v}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \ . \quad (4)$$

With dominating inertial and negligible viscous effects, freely decaying incompressible fluid turbulence in $\mathbb{R}_3$ is described by a Gibbs ensemble of solenoidal velocity fields that evolve dynamically according to (3). All equal-time multipoint velocity correlation tensors are contained in the complex-valued Fourier transform of the probability measure (1), the Hopf characteristic functional[5]

$$\Phi[\mathbf{y}(\mathbf{x});t] \equiv \int_{\text{all }\mathbf{u}(\mathbf{x})}\left[\exp\left(i\int\mathbf{y}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})d^3x\right)\right]dP[\mathbf{u}(\mathbf{x});t]$$
$$= 1 + i\int\langle u_j(\mathbf{x}',t)\rangle y_j(\mathbf{x}')d^3x' - \frac{1}{2}\int\langle u_j(\mathbf{x}',t)u_k(\mathbf{x}'',t)\rangle y_j(\mathbf{x}')y_k(\mathbf{x}'')d^3x'd^3x''$$
$$- \frac{i}{6}\int\langle u_j(\mathbf{x}',t)\mathbf{u}_k(\mathbf{x}'',t)u_l(\mathbf{x}''',t)\rangle y_j(\mathbf{x}')y_k(\mathbf{x}'')y_l(\mathbf{x}''')d^3x'd^3x''d^3x''' + \cdots \ . \quad (5)$$

In (5) the spatial integrals are over unbounded $\mathbb{R}_3$, and the real parameter field $\mathbf{y}=(y_1(\mathbf{x}),y_2(\mathbf{x}),y_3(\mathbf{x}))$ is required to be continuous, infinitely differentiable, and of compact support (i.e., a vector test function) but is otherwise unrestricted. Since the correlation tensors inherit the solenoidal quality of $\mathbf{u}$, the characteristic functional depends exclusively on the transverse part of $\mathbf{y}$: $\Phi[\mathbf{y};t] \equiv \Phi[\mathbf{y}^{\mathrm{tr}};t]$. The reality and nonnegativity of the normalized probability measure implies that $(\Phi[\mathbf{y};t])^* \equiv \Phi[-\mathbf{y};t]$ and $|\Phi[\mathbf{y};t]| \leq 1$. Furthermore, since all $\mathbf{u}$ satisfy (3), it follows that Φ satisfies the time-evolution equation[5]

$$\partial\Phi/\partial t + \Omega\Phi = 0 \ , \quad (6)$$

$$\Omega \equiv -i\int y_j^{\mathrm{tr}}(\mathbf{x})\frac{\delta}{\delta y_k(\mathbf{x})}\nabla_k\frac{\delta}{\delta y_j(\mathbf{x})}d^3x \ , \quad (7)$$

in which $\delta/\delta y_j(\mathbf{x})$ denotes the Volterra functional derivative with respect to $y_j(\mathbf{x})$. The parameter field $\mathbf{y}$ is not required to be solenoidal in order for the three functional derivative operators $\delta/\delta y_j(\mathbf{x})$, $j=1,2,3$, to be unconstrained and mutually independent in (7).

By transforming the functional integration variable $\mathbf{u}(\mathbf{x}) \rightarrow \lambda^{\alpha}\mathbf{u}(\lambda\mathbf{x})$ and making the replacements $\mathbf{y}(\mathbf{x}) \rightarrow \lambda^{3-\alpha}\mathbf{y}(\lambda\mathbf{x})$ and $t \rightarrow \lambda^{-\beta}t$ in the definition part of (5), it follows that self-similarity of the probability measure (1) is expressed equivalently as

$$\Phi[\lambda^{3-\alpha}\mathbf{y}(\lambda\mathbf{x});\lambda^{-\beta}t] \equiv \Phi[\mathbf{y}(\mathbf{x});t] \tag{8}$$

for all $\lambda > 0$. Because it relates the form of the characteristic functional for all $t > 0$ in terms of its form at one instant of time (say $t=1$), the self-similarity relation (8) must be compatible with the Hopf equation (6), and this imposes the index relation

$$\beta = \alpha + 1 \ . \tag{9}$$

Equation (9) is derived from the following steps. Since the functional derivative operators transform as[7]

$$\frac{\delta}{\delta y_j(\mathbf{x})} \rightarrow \lambda^{\alpha}\frac{\delta}{\delta y_j(\lambda\mathbf{x})} \quad \text{under } \mathbf{y}(\mathbf{x}) \rightarrow \lambda^{3-\alpha}\mathbf{y}(\lambda\mathbf{x}) \ , \tag{10}$$

a change of the integration variable $\mathbf{x} \rightarrow \lambda^{-1}\mathbf{x}$ in the time-evolution operator (7) shows that

$$\Omega \rightarrow \lambda^{\alpha+1}\Omega \quad \text{under } \mathbf{y}(\mathbf{x}) \rightarrow \lambda^{3-\alpha}\mathbf{y}(\lambda\mathbf{x}) \ . \tag{11}$$

Thus (9) is obtained as a consequence of (8) by making the replacements $\mathbf{y}(\mathbf{x}) \rightarrow \lambda^{3-\alpha}\mathbf{y}(\lambda\mathbf{x})$ and $t \rightarrow \lambda^{-\beta}t$ in (6).

Associated with the thorough randomization of the flow from three to five mesh distances downstream from a turbulence-generating grid, the initial statistical state at $t=0$ is represented approximately by a Gaussian normal form for the characteristic functional

$$\Phi[\mathbf{y}(\mathbf{x});0] = \exp\left[-\frac{c^2}{2}\int y_j^{\mathrm{tr}}(\mathbf{x})y_j(\mathbf{x})d^3x\right] \ , \tag{12}$$

in which c^2 is a positive constant associated with the level of turbulent fluctuations. By comparing (12) with (5), one obtains the associated initial values for the correlation tensors, e.g.,

$$\langle u_j(\mathbf{x},0)\rangle = 0 \tag{13}$$

$$\langle u_j(\mathbf{x}',0)u_k(\mathbf{x}'',0)\rangle = c^2\delta_{jk}^{\mathrm{tr}}(\mathbf{x}'-\mathbf{x}'') \ , \tag{14}$$

with zero correlation distance manifest at $t=0$. It is important to observe that the initial value for the characteristic functional (12) features the invariance symmetry

$$\Phi[\lambda^{3/2}\mathbf{y}(\lambda\mathbf{x});0] \equiv \Phi[\mathbf{y}(\mathbf{x});0], \quad \text{for all } \lambda > 0 \ , \tag{15}$$

which is compatible with the self-similarity condition (8) at $t=0$ if and only if[8]

$$\alpha = \tfrac{3}{2} \ . \tag{16}$$

Hence the index values $\{\alpha,\beta\} = \{\tfrac{3}{2},\tfrac{5}{2}\}$ are fixed by (16) and (9), and the self-similarity condition (8) becomes

$$\Phi[\lambda^{3/2}\mathbf{y}(\lambda\mathbf{x});\lambda^{-5/2}t] \equiv \Phi[\mathbf{y}(\mathbf{x});t] \tag{17}$$

for all $\lambda > 0$ by virtue of the Hopf equation (6) and initial condition (12).

Deviating rapidly from the Gaussian initial value (12), the functional form of (17) is non-Gaussian and complicated for $t > 0$ because of inertial distortion and skewness produced in the time evolution governed by (6).[9] However, the dilatation scaling invariance of (6) and (12) gurantees that the (unique) Φ prescribed by the latter equations has the self-similarity property shown in (17): Necessary and sufficient conditions for the self-similarity expressed by (17) are given by (6) with (7) and (15).[10] Therefore, statistical-dynamical self-similarity is a rigorous consequence of the Hopf time-evolution equation and the Gaussian initial condition.

If the disposable parameter λ (which may assume any positive value) is set equal to $t^{2/5}$ in (17), the characteristic functional is given for all $t > 0$ in terms of its form at $t=1$:

$$\Phi[\mathbf{y}(\mathbf{x});t] = \Phi[t^{3/5}\mathbf{y}(t^{2/5}\mathbf{x});1] \ . \tag{18}$$

All equal-time velocity correlation tensors are then related to their values at $t=1$ by expressing both sides of (18) according to the expansion shown in (5) and making a change of the integration variables $\mathbf{x} \rightarrow t^{-2/5}\mathbf{x}$ on the right-hand side of the resulting equation. In particular, one obtains

$$\langle u_j(\mathbf{x}';t)u_k(\mathbf{x}'';t)\rangle = t^{-6/5}\langle u_j(t^{-2/5}\mathbf{x}';1)u_k(t^{-2/5}\mathbf{x}'';1)\rangle \tag{19}$$

which gives the experimentally established decay law $u^2 \propto t^{-6/5}$ and integral scale dependence $L \propto t^{2/5}$ by inspection.

In summary, the statistical-dynamical self-similarity property of the characteristic functional (17) follows without any additional assumption from the Hopf equation (6) for incompressible fluid turbulence dominated by inertial effects and the Gaussian initial condition (12), the analysis being based on dilatation-transformation considerations.[11] The initial-value invariance (15) in combination with (6) is necessary and sufficient for a characteristic functional expressible for all $t > 0$ according to (18), with associated self-similar correlation tensors such as (19).

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[1]T. von Kármán and L. Howarth, Proc. R. Soc. London, Sect. A **164**, 192 (1938); T. von Kármán and C. C. Lin, in *Advances in Applied Mechanics*, edited by R. von Mises and T. von Kármán (Academic, New York, 1951), Vol. 2, p. 1.

[2]For example, F. H. Champagne, J. Fluid Mech. **80**, 67 (1978).

[3]P. G. Saffman, Phys. Fluids **10**, 1349 (1967).

[4]R. W. Stewart and A. A. Townsend, Philos. Trans. R. Soc. London, Ser. A **243**, 359 (1951); G. K. Batchelor, *Homogeneous Turbulence* (Cambridge Univ. Press, New York, 1960), pp. 135–147; G. Comte-Bellot and S. Corrsin, J. Fluid Mech. **48**, 273 (1971); Z. Warhart and J. L. Lumley, in *Structure and Mechanisms of Turbulence II*, edited by H. Fiedler, Lecture Notes in Physics, Vol. 76 (Springer-Verlag, New York, 1978); K. R. Sreenivasan *et al.*, J. Fluid Mech. **100**, 597 (1980).

[5]E. Hopf, J. Ration. Mech. Anal. **1**, 87 (1952); E. Hopf and E. W. Titt, *ibid.* **2**, 587 (1953); G. Bosen, Phys. Fluids **3**, 519 (1960); **3**, 525 (1960); J. Math. Phys. **23**, 2582 (1982).

[6]R. G. Deissler, Phys. Fluids **22**, 1852 (1979); also, see L. D. Lan-

dau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, New York, 1959), pp. 118–119.

[7]The dilatation-transformation formula for the functional derivative operators follow from the basic commutation relation

$$[\delta/\delta y_j(\mathbf{x}'),y_k(\mathbf{x}'')]=\delta_{jk}\delta^{(3)}(\mathbf{x}'-\mathbf{x}'') \ ,$$

where $\delta^{(3)}$ is the three-dimensional Dirac distribution.

[8]The value $\alpha=\frac{3}{2}$ also reflects *dual-space dilatation symmetry,* in the sense that (1) becomes

$$dP[\lambda^{3/2}\mathbf{u}(\lambda\mathbf{x});0]\equiv dP[\mathbf{u}(\mathbf{x});0]$$

for $t=0$ with $\alpha=\frac{3}{2}$. Thus the relevant dilatation-transformation formulas for the velocity field and its dual are identical: $\mathbf{u}(\mathbf{x})\rightarrow\lambda^{3/2}\mathbf{u}(\lambda\mathbf{x})$ and $\mathbf{y}(\mathbf{x})\rightarrow\lambda^{3/2}\mathbf{y}(\lambda\mathbf{x})$.

[9]By substituting (18) into (6), performing the time differentiation via the chain rule, and introducing the similarity variable $\boldsymbol{\zeta}(\boldsymbol{\alpha})\equiv t^{3/5}\mathbf{y}(t^{2/5}\mathbf{x})$, one obtains a functional differential equation which prescribes the form of self-similar Φ, viz.,

$$\left[\int\left\{\frac{3}{5}\boldsymbol{\zeta}(\boldsymbol{\alpha})+\frac{2}{5}\boldsymbol{\alpha}\cdot\nabla_{\boldsymbol{\alpha}}\boldsymbol{\zeta}(\boldsymbol{\alpha})\right\}\cdot\frac{\delta}{\delta\boldsymbol{\zeta}(\boldsymbol{\alpha})}d^3\alpha+\Omega'\right]\Phi[\boldsymbol{\zeta}(\boldsymbol{\alpha});1]=0 \ ,$$

$$\Omega'\equiv-i\int\zeta_j^{\mathrm{tr}}(\boldsymbol{\alpha})\frac{\delta}{\delta\zeta_k(\boldsymbol{\alpha})}\frac{\partial}{\partial\alpha_k}\frac{\delta}{\delta\zeta_j(\boldsymbol{\alpha})}d^3\alpha \ .$$

[10]This is the functional differential extension of Morgan's theorem [A. J. A. Morgan, Q. J. Math. (Oxford) **2**, 250 (1952)], commonly applied to obtain self-similar solutions to partial differential equations invariant under one-parameter groups of transformations.

[11]It follows that the essentially correct analyses of nonlinear inertial transfer [R. H. Kraichnan, J. Fluid Mech. **47**, 525 (1971); **83**, 349 (1977); S. A. Orszag, *ibid.* **41**, 363 (1970); R. G. Deissler, Phys. Fluids **22**, 185 (1979); **22**, 1852 (1979)] maintain the dilatation invariance symmetry which underlies self-similarity.