Mechanism for global features of chaos in a driven nonlinear oscillator

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Global features of chaos observed after an initial period-doubling route to chaos in a driven p-n junction oscillator are found to result from a simple mechanism which consists of a driven linear oscillator reset by a relaxation oscillator. The global features include staircases, period adding, replication, hopping, and higher-order periodic windows. A one-dimensional iterative map is constructed which models the initial quadratic behavior and subsequent global behavior. Other systems where this mechanism may be found are discussed.

Experimental studies of a driven oscillator consisting of a series inductor, resistor, and p-n junction[1] have shown that this system displays routes to chaos and chaotic behavior similar to one-dimensional iterative maps with a single quadratic maximum.[2] For example, period doubling, intermittency, noise scaling, and crises have been observed, and corresponding universal numbers measured. At large driving voltages, a periodic-chaotic sequence which appears globally and is characterized by staircases, period adding, replication, hopping, and higher-order periodic windows has been observed.[3] In this paper we present a simple mechanism which explains how the global features at large driving voltages are generated. The mechanism can be described as a driven linear oscillator which is periodically reset by a relaxation oscillator. A one-dimensional iterative map modeling this mechanism is constructed, which is quadratic at small driving voltages and predicts the observed global features at large driving voltages. Driven oscillators with a strong-weak restoring force and large damping on only the weak side may also display this mechanism.

The oscillator [Fig. 1(a)] consists of a series resistor, inductor, and p-n junction driven near resonance by a voltage $V(t)=A\sin(\omega t)$. The behavior is best summarized by a bifurcation diagram,[4] which in this case is a plot of the current maxima versus the amplitude of the driving voltage.[1] Figure 2 is a typical bifurcation diagram for the oscillator showing a periodic-chaotic sequence in which periodic regions of period $n=1,2,3,\ldots,$ are separated by chaotic regions. The behavior studied in Ref. 1 included only the period doubling and chaos before the start of the period 2 region, which were modeled by unimodal quadratic maps; the higher period regions, which usually appear at large driving amplitudes, are not part of the U sequence of unimodal maps.[5] The time dependence of the current in the period n region can be described as a repetition of a waveform which has the shape of a descending staircase with n steps, as in the period 5 region shown in Fig. 3(a). The voltage across the p-n junction is approximately constant except near the start of each current staircase when it momentarily becomes negative, as shown in Fig. 3(b). The current in the chaotic region between the period n and $n+1$ regions hops between staircases which have $n-1$, n, and $n+1$ steps. Higher-order periodic windows observed in this region consist of combinations of staircases with $n-1$, n, and $n+1$ steps. Also observed in Fig. 2 is hysteresis at the start of each periodic region. Unlike period adding sequences in unimodal[5] and phase-locking[6] maps, the sequence in Fig. 2 consists of period doubling orbits which have the same scale and orientation; i.e., the bifurcation diagram in Fig. 2 appears to grow by replication of a period doubling orbit. We emphasize that the behavior described is observed only at large driving amplitudes; below a certain amplitude staircases are not observed and the behavior is modeled by the quadratic logistic map.

The response of the p-n junction is modeled by the fol-

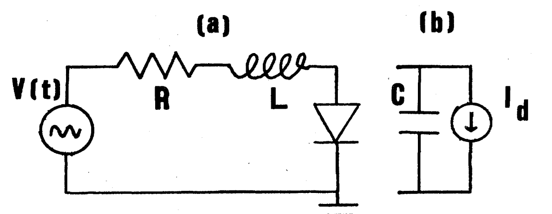

FIG. 1. (a) Driven nonlinear oscillator consisting of sinusoidal driver, resistor, inductor, and p-n junction. (b) Model for p-n junction consisting of current source I_d in parallel with a capacitance $C=C_j+C_s$.

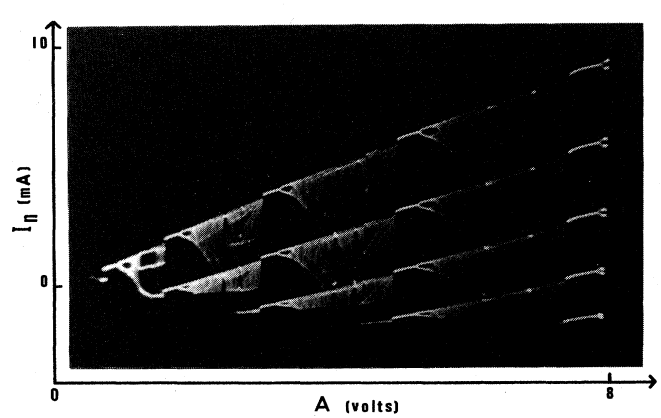

FIG. 2. Oscilloscope picture of birfurcation diagram for nonlinear oscillator obtained by strobing the current maxima I_n. The behavior at small driving amplitudes, which includes the small period 3 window, is similar to the quadratic logistic map. At large amplitudes, a periodic-chaotic sequence with global features appears. $R=100\,\Omega$, $L=10$ mH, $\omega=150$ kHz, and the p-n junction is type 1N4004.

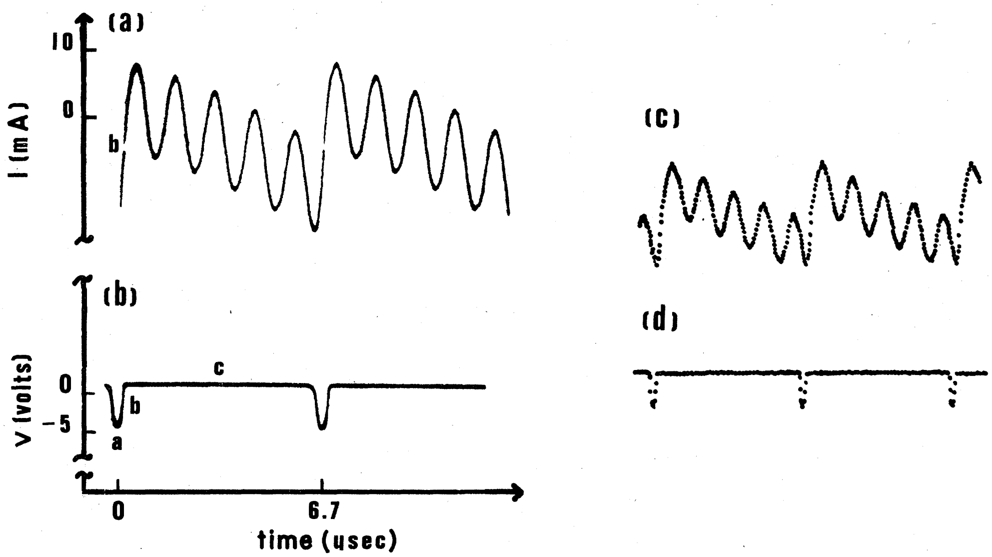

FIG. 3. (a) Observed time dependence of current in the period 5 region describable as descending staircases with 5 steps. The height of the staircase is determined by the large current, near point b, which injects minority charge into the p-n junction. (b) Observed voltage across the p-n junction in the period 5 region. The reverse bias at point a adds with the driver voltage to inject a large minority charge near point b; this charges the storage capacitance to a small positive voltage, region c, and permits the p-n junction to conduct in both directions. (c) and (d) Computed I and V, respectively, from model shown in Fig. 1(b), to be compared with observed behavior (a) and (b).

lowing processes:[7] the dc rectification described by $I_d = I_0(e^{qV/kT}-1)$, the depletion of charge during reverse bias described by a junction capacitance $C_j = C_1/(1-V/\phi)^{1/2}$, and the injection of minority charge during forward bias described by a charge storage capacitance $C_s = C_0 e^{qV/kT}$. In this model, the p-n junction is equivalent to a current source $I_s = I_d$ in parallel with a capacitance $C = C_j + C_s$, as shown in Fig. 1(b), and the oscillator is three dimensional, described by differential equations:

$$\dot{I} = \frac{A\sin\Omega - V - IR}{L}, \tag{1a}$$

$$\dot{V} = \frac{I - I_0(e^{qV/kT}-1)}{C_0 e^{qV/kT} + C_1/(1-V/\phi)^{1/2}}, \tag{1b}$$

$$\dot{\Omega} = \omega, \tag{1c}$$

where I is the current, V the voltage across the p-n junction, and Ω the phase of the driving voltage. Numerical integration gives good agreement with the observed behavior: The computed current and voltage across the p-n junction in the period 5 region are shown in Figs. 3(c) and 3(d), to be compared with the observed behavior, Figs. 3(a) and 3(b); higher-order periodic windows and hopping between staircases with different steps are also numerically observed.

The current's staircase waveshape and simple relation with the voltage across the p-n junction suggest a simple description of the behavior in terms of the response of the equivalent circuit, Fig. 1(b). At point a in Fig. 3(b), the p-n junction is reverse biased; the large negative voltage is stored in the junction capacitance and adds with the driver voltage to produce a large current through the circuit, as shown near point b in Fig. 3(a). This current injects minority charge into the p-n junction and charges the storage capacitance to a positive voltage, as shown near point b in Fig. 3(b). The stored charge permits the p-n junction to conduct in both directions, a transition known as the reverse recovery;[7] during this time the p-n junction acts as a conductor with a small positive emf, as shown in region c in Fig. 3(b). The stored charge begins to recombine in the p-n junction and to discharge through the inductor and resistor producing a counterclockwise discharge current which linearly increases with time. The discharge current adds with the sinusoidal response of the inductor and resistor to produce the observed descending staircase. When most of the stored charge has disappeared, the p-n junction is reverse biased and the cycle is repeated. Another way of describing this behavior is to say that a new current staircase is started with a new height by the large current which injects the minority charge. The system is therefore a driven linear oscillator which also resets itself after a period of time to a new starting position. The new starting position is determined by the amount of charge injected into the p-n junction and the period of time between resets by the time it takes the forward biased p-n junction to discharge through the inductor and resistor. The rapid charging and gradual discharging of the p-n junction is an example of the relaxation oscillator.[8] At small driving voltages, injection occurs, but staircases are not produced because too little minority charge is injected.

To explore and confirm this mechanism we construct as a model a one-dimensional iterative map, i.e., return map. The iterates of the map represent the current maxima. Figure 4(a) shows such a map:

$$x_{n+1} = \begin{cases} Ax_n + F \text{ for } x_n \geq K, \\ AK + F - L\left[(x_n - B)^2 - (K-B)^2\right] \text{ for } x_n < K. \end{cases}$$

The straight line for $x_n \geq K$ represents the linear behavior of the system when the p-n junction is charged and relax-

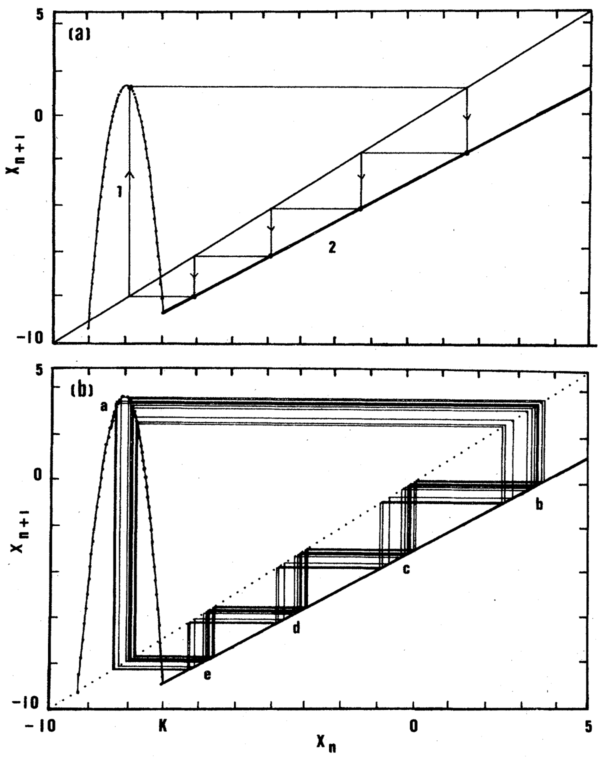

FIG. 4. (a) One-dimensional iterative map consisting of parabola and straight line. Diagonal line $x_{n+1}=x_n$ locates successive iterates. Shown is period 5 cycle consisting of five iterates relaxing along region 2 per injection in region 1 (corresponding to a staircase with five steps). (b) Chaos between period 5 and 6 cycles. Chaotic band a is replicated in bands $b-e$.

ing. The parabola for $x_n < K$ represents the return map at small driving voltages which, as previously observed,[1] is accurately modeled by the logistic map. The amount of injected minority charge is determined by this parabola. The control parameter of the map is L. As L is increased, the parabola steepens and increases in height, modeling the increase in injected minority charge as the amplitude of the driving voltage is increased. For small L, the iterates stay on the parabola, modeling the behavior at small driving voltages. For L sufficiently large, the iterates on the parabola are mapped on the straight line where they linearly relax to a value where reinjection occurs. Figure 4(a) shows a period 5 staircase with minority charge injection occurring in region 1 and linear relaxation over many driving periods in region 2. As L is increased past the value for the period 5 region, the parabola eventually reaches a height where six iterates are accommodated in a staircase. In between these two periodic regions there is chaos, as shown in Fig. 4(b). The four chaotic bands, $b-e$ in Fig. 4(b), are scaled copies of the top band a, so the attractor in this model replicates. After these five bands merge, hopping between staircases which have 3, 4, and 5 steps occurs as experimentally observed. Figure 5 is a bifurcation diagram for the map to be compared with the observed behavior, Fig. 2. The similarity is excellent: In addition to period adding and replication of period doubling orbits, there are the higher-order periodic windows such as the period 5 window, shown in Fig. 5, consisting of a combination period 3 staircase followed by a period 2 staircase. All experimentally observed windows were found in the computed bifurcation diagram. Hopping occured in all the chaotic regions as observed. Hysteresis can be obtained in the computed diagram by adding a dimension to the map. The observed period 3 window before the period 2 region can be obtained by increasing K, which extends the initial chaotic region.

We conjecture that other physical systems may display this mechanism of injection and relaxation. When Eqs. (1) are rewritten as a second-order differential equation, they describe a driven oscillator with a very strong restoring force and little damping on one side of equilibrium and a very weak restoring force with large damping on the other side.[9] Injection and relaxation can be explained as follows. On the side with a strong restoring force and little damping, energy

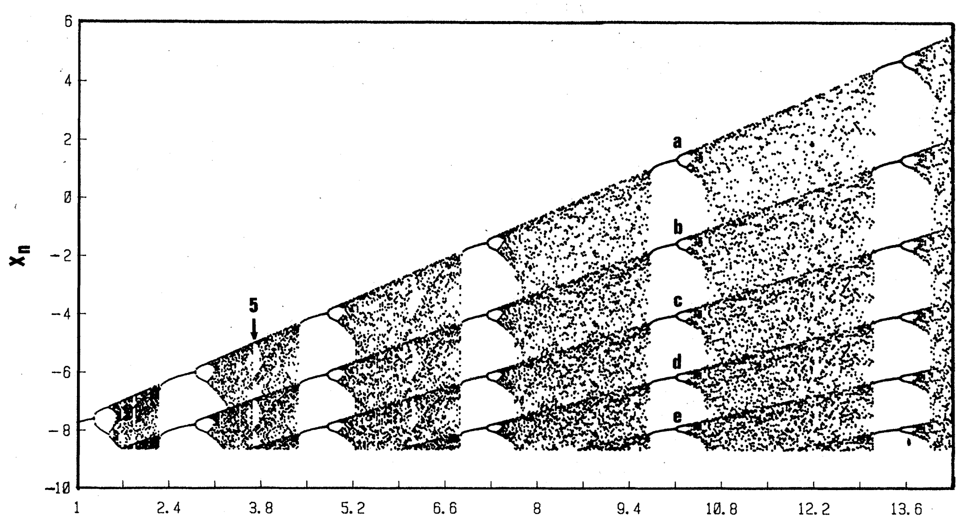

FIG. 5. Computed bifurcation diagram for the map shown in Fig. 4, to be compared with observed behavior, Fig. 2. Higher-order periodic windows such as the period 5 window consisting of period 3 and 2 staircases are experimentally observed. Other higher-order windows can be seen by holding figure at eye level with line of sight parallel to plane of paper. Regions $a-e$ correspond with Fig. 4(b). The parameters are $A=0.85$, $B=-8$, $F=-2.7$, $Q=2$, and $K=-7$.

is quickly and efficiently stored and then released, together with the energy of the driver, into the side with a weak restoring force and large damping. On this side, the oscillator quickly comes to rest due to the damping and then slowly relaxes over many driving periods back to equilibrium, etc. The staircase behavior, period adding, replication, hopping, and higher-order periodic windows displayed by our model also characterize numerical solutions of driven oscillators with a similar strong-weak restoring force and damping.[9] Oscillators with strong-weak restoring forces include, for example, extrinsic photoconductors.[10] Finally, since staircases are a direct consequence of injection and relaxation, systems where period adding is observed[11] should also be examined for staircases. The observation of staircases and, if experimentally possible, replication, hopping, and the higher-order periodic windows would strongly support the mechanism we have described.

In conclusion, global features of chaos observed at large driving voltages in a driven p-n junction oscillator result from a mechanism consisting of a driven linear oscillator with a relaxation type of reset. A simple one-dimensional iterative map models the transition from quadratic behavior at small driving voltages to injection and relaxation at large driving voltages where the global features are observed: staircases, period adding, replication, hopping, and higher-order periodic windows. We conjecture that this mechanism can be found in driven oscillators with a similar strong-weak restoring force and damping. In systems where period adding is observed, the observation of staircases would be the most direct additional evidence for this mechansim.

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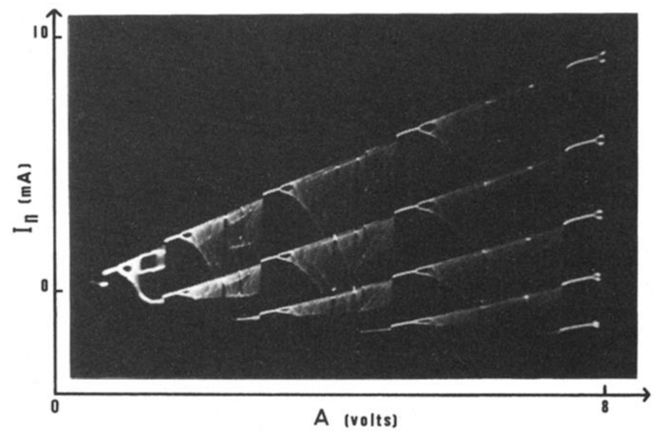

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