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Analysis of nonlinear mass and energy diffusion

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A study of nonlinear plasma mass and energy diffusion reveals that the diffusion coefficients uniquely determine one of two diffusion patterns. After a short initial transient time, either an organized universal pattern, described by a time-space separable solution, is formed or transport is inhibited, allowing only a partial organization of diffusion into a universal pattern. Consequently, in the second case, unlike the first, the asymptotic shape of the solution will depend to some extent on its initial state.

Before a complete analysis can be made of the radial transport of mass and energy in the presence of magnetic fields, the nonlinear diffusion of mass and energy across magnetic fields must be well understood.[1–6] In pursuit of this goal, this paper is concerned with a mathematical analysis of a simple model of slow diffusive self-relaxation of a magnetically confined plasma.

As prototype equations of heat and mass diffusion we consider

$$\rho_t=[D_1(\rho,T)\rho_x]_x,\quad D_1=d_{01}\rho^{\alpha_1}T^{\beta_1}\ , \tag{1}$$

$$\rho T_t=[\rho D_2(\rho,T)T_x]_x,\quad D_2=d_{02}\rho^{\alpha_2}T^{\beta_2}\ , \tag{2}$$

where the subscripts denote partial differentiation, α, β, d_{01}, and d_{02} are constants and $x\in[-1,1]$. We prescribe initial data for density $\rho(x,0)$ and temperature $T(x,0)$ together with homogeneous convective boundary conditions

$$\rho_x\pm h_1\rho=0,\quad T_x\pm h_2T=0\quad \text{at } x=\pm 1\ . \tag{3}$$

These convective boundary conditions are physically more relevant and mathematically more tractable than Dirichlet boundary conditions ($h=\infty$).

Despite the simple functional dependence of the diffusion coefficients, these equations are fair approximations to many of the current magnetically confined plasmas models. For example, in a fully collisional plasma the diffusion across magnetic field lines scales as $D\sim\rho T^{-1/2}$ and as $D=K/\rho\sim T^{3/2}q$ in the tokamak plateau regime. Here $q(x)$ is a safety factor that depends on the specifics of the model problem. We note that adding a spatially dependent function, such as $q(x)$, does not obstruct the space-time separability properties of the equation. A similar factor $(1+q^2)$ modifies the classical diffusion coefficient to account for the Pfirsh-Schluter effect. Our model only contains the simplified density and temperature equations; other effects, such as the poloidal magnetic field or the current, which enter the diffusion coefficients in the complete model, can be approximated by introducing a spatially varying adjustment function, such as $q(x)$. Other diffusion coefficients of interest are the Bohm coefficient $K=\rho D\sim\rho T$, the Alcator-A (tokamak) diffusion coefficient $D\sim\rho^{-1}$, and the Ohkawa anomalous diffusion coefficient $D\sim T^{1/2}\rho^{-1}$.

The mathematical analysis of even these simple mass-energy nonlinear diffusion prototype problems at first appears highly complex. Fortunately, if not surprisingly, the solution manifold of Eqs. (1) and (2) over a bounded domain with convective boundary conditions (3) is simple and can be easily classified. The solution evolves very quickly toward a universal diffusion mode which is almost independent of initial data. This highly organized diffusion pattern is mathematically represented by a time-space separable solution. (Similar separable solutions are known before to play a key role in the evolution of the solution to a single nonlinear diffusion equation.[7–10])

The analysis of these separable solutions is the central theme of this paper. While such solutions are special cases because they must satisfy special initial data, they attract all initial data and hence play the key role in the description of the general diffusion problem. While rigorously we can prove this proposition only for a subclass of the considered problem, extensive numerical experimentation has been used to give strong credence to them being global attractors.

Inserting the separable forms

$$\rho(x,t)=\phi_1(t)N(x),\quad T(x,t)=\phi_2(t)\psi(x)\ , \tag{4}$$

into Eqs. (1) and (2) leads to the following conditions:

$$\dot{\phi}_1=-\lambda_1\phi_1^{\alpha_1+1}\phi_2^{\beta_1},\quad \lambda_1\geq 0 \tag{5a}$$

$$\dot{\phi}_2=-\lambda_2\phi_1^{\alpha_2}\phi_2^{\beta_2+1},\quad \lambda_2\geq 0 \tag{5b}$$

$$d_{01}\frac{d}{dx}N^{\alpha_1}\psi^{\beta_1}\frac{dN}{dx}+\lambda_1N=0\ , \tag{6a}$$

$$d_{02}\frac{d}{dx}N^{\alpha_2+1}\psi^{\beta_2}\frac{d\psi}{dx}+\lambda_2N\psi=0\ , \tag{6b}$$

and the spatial part of Eq. (3).

The relevant cases of the first integrals of motion for Eqs. (5) are the following:

Case 1. $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$

$$\frac{\lambda_1}{\beta_2-\beta_1}\phi_2^{\beta_1-\beta_2}+\frac{\lambda_2}{\alpha_2-\alpha_1}\phi_1^{\alpha_2-\alpha_1}=C_0 \ . \tag{7a}$$

Case 2. $\alpha \equiv \alpha_1 = \alpha_2$, $\beta \equiv \beta_1 = \beta_2$

$$\phi_1 = (\phi_2/T_0)^{\lambda_1/\lambda_2} \ , \tag{7b}$$

where C_0 and T_0 are constants.

Even though case 2 is mathematically speaking degenerate, it is of considerable practical interest in many applications where D_1/D_2 is assumed to be constant (such as for the diffusion of a fully collisional plasma across a magnetic field). Integration of Eq. (5a) yields

$$\phi_1(t) = [T_0^{\beta}(t_0+\Omega t)]^{-\lambda_1/\Omega}, \quad \Omega = \lambda_1\alpha + \lambda_2\beta \tag{8}$$

where t_0 is a constant. According to whether Ω is positive, zero, or negative, we refer to the solution ϕ_1 as decaying slowly (algebraic decay), exponentially, or fast (ϕ_1 vanishes in a finite time).

The time dependences of the solutions in case 1 are given implicitly as

$$\phi_1 = (\zeta_0 + \lambda_1\alpha_1\zeta)^{-1/\alpha_1}, \quad \phi_2 = (\tau_0 + \lambda_2\beta_2\tau)^{-1/\beta_2} \ , \tag{9a}$$

where ζ_0 and τ_0 are new constants of integration and

$$dt = d\zeta/\phi_2^{\beta_1} = d\tau/\phi_1^{\alpha_2} \tag{9b}$$

defines ζ and τ, the stretched time coordinates. Of course only two of the ζ_0 and τ_0 are independent, since they are related by Eq. (7a).

Unless either β_1 or α_2 vanishes, τ may be found only after the integration of Eq. (9a) and Eq. (7a). Though the resulting Euler-type integrals can be solved only implicitly, ϕ_1 and ϕ_2 can be evaluated asymptotically to determine the large time behavior. The results of this analysis are summarized in Fig. 1.

We can find important features of the solution's temporal part directly from the first integrals of motion. In the $(\tilde{x},\tilde{y}) = (\beta_1-\beta_2, \alpha_1-\alpha_2)$ plane, the two possible $\Delta \equiv \alpha_1\tilde{x} - \beta_1\tilde{y} = 0$ lines separate regimes of fast and slow diffusion. The behavior of the temporal part of the solution dramatically changes in each of the four quadrants. In general, only in the second quadrant do both $\phi_1(t)$ and $\phi_2(t)$ decay to zero, elsewhere one of the ϕ's converges to a positive constant. The decay to zero is algebraic as described by Eq. (8), everywhere but on the $\Delta = 0$ lines where it is exponential.

For large t, the asymptotic form of $\phi_i(t)$ in the second quadrant is given by

$$\phi_i(t) = (t_0 + \lambda_i\omega_i t)^{\omega_i}, \quad i = 1,2 \tag{10a}$$

where

$$\omega_1 = (\beta_2-\beta_1)/\Delta, \quad \omega_2 = (\alpha_1-\alpha_2)/\Delta; \quad \Delta \equiv \alpha_2\beta_1 - \alpha_1\beta_2 \ . \tag{10b}$$

The ω_1 and ω_2 which give the rate of the temporal decay are defined *a priori*, and are independent of the symmetry in which our problem is considered. This is an essential feature of the nonlinear diffusion which has no counterpart in the linear theory.

The rate of the temporal decay is intimately related to the role played by the separation constants λ_1 and λ_2. To clarify this point consider first the case when Eq. (1) is a linear system whose solution decays as $\exp(-\lambda_i t)$, where λ_1 and λ_2 play the role of eigenvalues in Eqs. (6). In a nonlinear diffusive system the λ_i are nonessential constants in Eqs. (6) whose values depend on the normalization of ψ and N. Indeed, suppose that $\psi(0) = A$ and $N(0) = B$ with $\tilde{\psi}$ and $\tilde{N}$ being the solutions with $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ being their "eigenvalues." For any ψ_0, $N_0 > 0$, we then find that $\psi = \psi_0\tilde{\psi}$ and $N = N_0\tilde{N}$ are also solutions with $\tilde{\lambda}_i \to \tilde{\lambda}_i N_0^{\alpha_i}\psi_0^{\beta_i}$, $i = 1,2$. Alternatively, let $\Delta = \alpha_2\beta_1 - \alpha_1\beta_2$, then the choice

$$\psi_0^{\Delta} = \tilde{\lambda}_2^{\alpha_1}/\tilde{\lambda}_1^{\alpha_2}, \quad N_0^{\Delta} = \tilde{\lambda}_1^{\beta_2}/\tilde{\lambda}_2^{\beta_1} \ , \tag{11}$$

normalizes both λ_1 and λ_2 to one, with $\psi(0) = A\tilde{\Psi}_0$ and $N(0) = B\tilde{N}_0$.

Thus the λ's may be reshuffled from the spatial into the temporary part of the solution and are related to the amplitude of the diffusion mode [e.g., see Eqs. (10)]. Needless to say that this differs fundamentally from the linear case.

An exception occurs when Δ vanishes. The linear case is a trivial example. In the nonlinear case, where $\alpha_1/\alpha_2 = \beta_1/\beta_2 \neq 0$ (or ∞), only one λ can be eliminated from Eqs. (6); the other λ remains as an essential parameter. In this case, the solutions to Eqs. (1) and (2) are invariant with respect to the group of shifts; $T \to AT$, $\rho \to A^{-\beta_1}/\rho^{\alpha_1}$, and $t \to t + t_0$. If $A = \exp(-\lambda t_0)$, this invariance enables the construction of the invariant representation

$$T = e^{\lambda t}\psi(x), \quad \rho = e^{-\beta_1\lambda t/\alpha_1}N(x) \ , \tag{12}$$

where t is an eigenvalue that must be determined from the global-existence conditions of the separable solution. (A similar situation arises in the problem of imploding shock waves, where the λ is determined uniquely by requiring the existence of the self-similar solution in the large.[11])

A physically interesting case arises when D_1/D_2 is constant and $\alpha \neq 0$, $\beta \neq 0$ [case 2, Eq. (7b)]. Again (λ_1/λ_2)

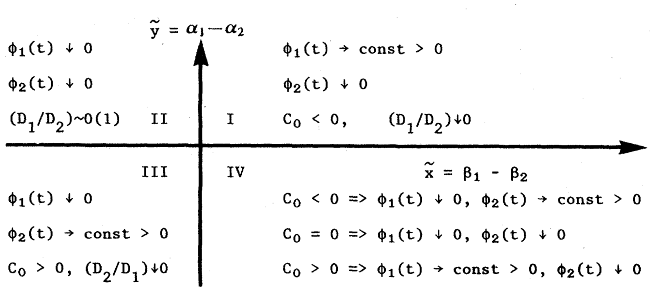

FIG. 1. Solution states in the $(\tilde{X},\tilde{Y}) = (\beta_1-\beta_2, \alpha_1-\alpha_2)$ plane. In the first and the third quadrant, the integration constant C_0 must have a definite sign, but its value is irrelevant for solutions in the second quadrant, and crucial in the fourth quadrant. Everywhere, but on the $\Delta \equiv \alpha_2\beta_1 - \alpha_1\beta_2 = 0$ line, the decay is either slow or fast.

plays the role of an eigenvalue with the exponential case being a transit solution between fast and slowly diffusing regimes. Here, both the mass and energy decay algebraically at a rate λ_i/Ω, $i=1,2$ [see (7b) and (8)] that must be found by solving Eqs. (6a) and (6b). The following homologous property:

$$\lambda_2 d_{01}\phi/\lambda_1 d_{02}=K \ , \tag{13}$$

where for given convective coefficients h_1 and h_2, K is a constant means that λ_1/λ_2 has to be only measured for one pair of d_{01} and d_{02} and then it may be calculated for any other d_{01} and d_{02}. Particularly, if $\alpha\beta<0$ (say, the fully collisional case wherein $\alpha_1=\alpha_2=1$ and $\beta_1=\beta_2=-\frac{1}{2}$), by changing the ratio of d_{01}/d_{02} we may transit from fast into a slow diffusion (or vice versa).

Having delineated the temporal part of the solution, we still need to interpret the fact that in a diffusive process for α and β not belonging to the second quadrant, one of the solutions (i.e., either ϕ_1 or ϕ_2) does not decay to zero. The time evolution of a particular example is shown in Fig. 2. This behavior is very different from what is expected from a single diffusion equation.

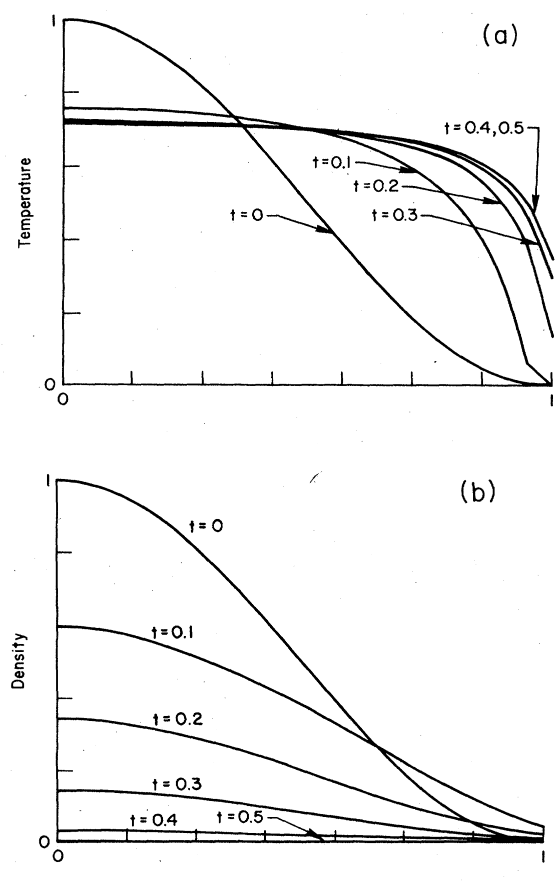

FIG. 2. For these initial data and these parameters in the third quadrant, $\alpha_1=-\frac{1}{2}$, $\alpha_2=1$, $\beta_1=0$, $\beta_2=\frac{3}{2}$, $d_{01}=1$, $d_{02}=5$, $h_1=h_2=10$, the decay and diffusion of heat [panel (a)] is inhibited by the rapid decay of density [panel (b)].

To understand the principle mechanism involved in this somewhat unexpected process, consider the case where β_1 is zero causing Eqs. (1) and (2) to decouple and allowing them to be solved separately. The separable solution of Eq. (1) is a global attractor;[8-10] it represents a universal mode of diffusion with the temporal behavior

$$\phi_1(t)=(t_0+\lambda_1\alpha_1 t)^{-1/\alpha_1}, \quad t_0=\text{const} \tag{14}$$

and $\phi_2(t)$ is given by Eq. (7a). If α_1 is positive, the solution asymptotically converges to the separable form. In numerical tests, the general solution becomes indistinguishable from the separable one after a relatively short time. For a single equation t_0 is important only in the case of fast diffusion because $t_0/(\lambda_1|\alpha_1|)$ defines the finite extinction time of the process.

Although $\phi_2(t)$ is known from Eq. (7a), addressing the solution of Eq. (2) directly is instructive. Using the asymptotic form of ρ known from Eq. (1) we can treat Eq. (2) as a separate equation in T with a variable diffusion coefficient. The solution of this equation rapidly converges to a separable form asymptotically. With this expectation, we substitute $\rho=\phi_1(t)N(x)$ and obtain

$$N(x)\phi_1^{-\alpha_2}T_t=N(x)T_\tau=(N^{\alpha_2+1}T^{\beta_2}T_x)_x \ , \tag{15}$$

where

$$\tau=\int_0^t \phi_1^{\alpha_2}(\eta)d\eta \ . \tag{16}$$

When $0<N<\infty$, Eq. (15) is a standard diffusion equation similar to Eq. (1) with $\beta_1=0$, but is measured in τ units.

If $\beta_1=0$ and $\alpha_1\geq\alpha_2$, then $\tau\to\infty$ as $t\to\infty$. For large-τ time, the temperature converges to the separable solution $T=\phi_2(\tau)\psi(x)$, for whatever initial condition. Here,

$$\phi_2(t)=\tilde{\phi}_2(\tau(t))=(\tau_0+\lambda_2\beta_2\tau)^{-1/\beta_2}, \tag{17}$$

with τ_0 again an unknown function of the initial conditions.

If $\beta_1=0$ and $\alpha_2>\alpha_1$, however, the integral in Eq. (16) converges, and

$$\tau=\tau_D[1-(1+\lambda_1\alpha_1 t/t_0)^{1-\alpha_2/\alpha_1}] \ , \tag{18a}$$

where

$$\tau_D=t_0^{1-\alpha_2/\alpha_1}/[\lambda_1(\alpha_2-\alpha_1)] \ . \tag{18b}$$

Thus, $\tau\to\tau_D$ as $t\to\infty$. If τ is bounded, the time needed to attain the separable solution is not available, and $\phi_2(t\to\infty)$ converges to a positive constant. Thus, while $\rho(x,t\to\infty)$ decays to zero, $T(x,t=\infty)=T(x,\tau_D)$ is a positive nonzero steady state.

Stated differently in such a case the solution in principle will remember its initial conditions. If also $\beta_1=\beta_2=0$ this follows at once by writing

$$T(x,t)=\sum_{j=1}^{\infty}a_j\exp[-\delta_j\tau(t)]\psi_j(x) \ .$$

Here δ_j and ψ_j are the jth eigenvalue and eigenfunction, respectively. Using Eqs. (18), we can see from

$$T(x,t=\infty)=T(x,\tau_D)=\sum_{j=1}^{\infty}a_j\exp(-\delta_j\tau_p)\psi_j$$

that none of the harmonics initially present vanish at $t \to \infty$. For the nonlinear case we show this property by taking $\psi(x)$, the spatial counterpart of (15) as the initial condition and perturbing it. The perturbed solution of Eq. (15) is

$$T(x,t)=\phi(t)\psi(x)[1+u(x,t)] .$$

It is easily seen that if $u=w(t)V(x)$, when ψ is the first eigenfunction of V. Again $w(\infty)=w(\tau_D)>0$ and u cannot return to $\phi_2\psi$.

Thus in the third quadrant, $\beta_2>\beta_1$, $\alpha_2>\alpha_1$, the diffusion of heat is always inhibited by the density—see Fig. 2 for a numerical example. In the fourth quadrant ($\beta_1>\beta_2$, $\alpha_1>\alpha_2$), depending on the initial data and the values of α_i and β_i, either temperature or density will inhibit the diffusion of the other. Numerical experiments have shown the density decays faster usually and inhibits the diffusion of heat, as in the third quadrant. If α_1 is negative, the process always terminates on the fast scale. If α_1 is positive, the process is *fast* if the *temperature vanishes* (the plasma becomes cold within a finite time), but it is *slow* if the *density decays to zero.*

When $\beta_1\neq 0$, the asymptotic analysis of the temporal part is more tedious but confirms the above conclusions. However for $\beta_1\neq 0$ we were unable to demonstrate analytically the attractive nature of the separable solution. It is at this point that an extensive numerical experimentation was used covering all of the four quadrants of the $(\tilde{x},\tilde{y})$ plane to ensure the attractive nature of the separable solution. This leads us to believe that the lack of rigorous mathematical proof is a technical rather than a fundamental obstacle. Moreover, if $\tau_D<\infty$, unlike the semicoupled case, either both T and ρ come close to their ideal counterparts ψ and N or neither comes close, as $\tau \to \tau_d$. In practice, however, for the many cases considered numerically, T and ρ approach their attractor very quickly, long before the process "runs out of time." That is, by the time the diffusion coefficient becomes suppressed, the process is extremely close to its universal mode.

The effect of an inhibited diffusion is expected to play a far more important role in the presence of sources and sinks. Particularly with respect to a stable plasma heating if thermal stability is to be observed, appropriate particle injection must take place to avoid the suppression of thermal conductivity by diffusing particles. With a suppressed thermal conductivity, the process cannot be thermally stable.

In summary, we have unfolded the structure of a system of prototype equations of heat and mass diffusion. The space-time separable solution was shown to play a key role in a later time description of the diffusion. We have shown the existence of two conceptually different diffusion processes. When $\alpha_1\geq\alpha_2$, $\beta_2\geq\beta_1$, both of the diffusion coefficients D_1 and D_2 are on an equal footing resulting in an ultimate decay of both plasma mass and energy. For other α and β one of the diffusion coefficients becomes suppressed to the extent that either the diffusion of heat or mass is blocked. This phenomenon is expected to have an important effect on the evolution of a heated and radiating plasma.

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