Damping of quantum coherence: The master-equation approach

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We solve the master equation for the coordinate-coordinate damped harmonic oscillator for initial superpositions of coherent states. In the zero-temperature case the solution remains a simple superposition of coherent states. While the underdamped oscillator evolves all initial superpositions into mixtures of coherent states the overdamped oscillator does so selectively. For finite temperatures coherent states are no longer preserved, and we find a decrease in the variance of the off-diagonal coordinate-basis density-matrix elements below the coherent-state value. This variance decreases with increasing bath temperature. In the overdamped case there is negligible associated spreading of the diagonal coordinate-basis density-matrix elements. Thus the coordinate basis is an example of Zurek's pointer basis and the coordinate damped oscillator models the coordinate-basis density-matrix diagonalization which occurs in a coordinate measurement.

I. INTRODUCTION

Recently there has arisen the prospect of experimentally producing superpositions of quantum states in macroscopic devices.[1] Macroscopic systems are inevitably weakly coupled to many degrees of freedom in their environment and it has long been recognized that such coupling may have a dissipative effect on the system.[2,3] The general objective of the work reported here is to discuss the behavior of quantum superposition states, i.e., quantum coherences, in dissipative systems. Measurement devices are of special interest in this context since their readouts must not develop superposition states and they also collapse superpositions in the systems upon which they perform measurements.

In particular this paper discusses the behavior of a harmonic oscillator coordinate-coordinate coupled to an environment of harmonic oscillators. Such a system has recently been analyzed by Caldeira and Leggett[4–6] using the Feynman-Vernon influence functional technique. They reported rapid decay of superpositions of coherent states having macroscopically distinct mean coordinates. Certain exact results obtained without eliminating the environmental oscillators have been presented by other authors.[7,8]

Our approach is based on the Markovian master equation for the coordinate-coordinate damped harmonic oscillator obtained by Agarwal and others.[9,10] Our results are in agreement with those obtained using the influence functional technique[6] in both the high- and low-temperature limits. The master-equation approach has previously been used to study superposition states of the zero-temperature damped harmonic oscillator in the high-frequency limit where the rotating-wave approximation is valid.[11,12] Our work extends those studies to finite temperatures and to high frequencies, or equivalently, to large damping.

Section II introduces Agarwal's master equation for the reduced density operator of the system and obtains an equivalent equation for the quantum characteristic function which we are able to solve. Section III analyzes the instructively simple zero-temperature case. This case is straightforward because an initial superposition of n coherent states remains as a superposition of n coherent states as the system evolves. We find that certain superpositions of coherent states are rapidly eliminated at a rate which increases with the separation of the superposed states. In the underdamped case all superpositions are eliminated; however, in the overdamped case the behavior depends on the arguments of the superposed coherent states. Section IV deals with finite temperatures. Focusing attention on the coordinate basis we find that the damping tends to diagonalize the density matrix in this basis, the diagonalization becoming more complete with increasing bath temperature. The underdamped case is associated with a spreading of the coordinate-basis density-matrix elements along the diagonal, which is essentially absent from the heavily overdamped case. In the concluding Sec. V we relate our results to the work of Zurek[13] concerning the environmentally induced diagonalization of the density matrix in the so-called pointer basis. We show that in the heavily overdamped case our system models the state reduction associated with a measurement of the harmonic-oscillator coordinate.

II. THE MASTER EQUATION

Consider a large number of harmonic oscillators, the kth oscillator having mass m_k and frequency ω_k. We choose the zeroth oscillator to be the system of interest and regard the remaining oscillators as a bath to which the system is weakly coordinate-coordinate coupled. The Hamiltonian of the system plus bath can then be written

$$H=\hbar\omega_0 a_0^\dagger a_0+\hbar\sum_{\mathbf{k}\,(\neq 0)}\omega_{\mathbf{k}}a_{\mathbf{k}}^\dagger a_{\mathbf{k}}+X_0\sum_{\mathbf{k}\,(\neq 0)}g_{\mathbf{k}}X_{\mathbf{k}}\,, \tag{1}$$

where $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$ are, respectively, the boson creation and annihilation operators and the $g_{\mathbf{k}}$ are coupling strengths. $X_{\mathbf{k}}$ is the harmonic-oscillator coordinate observable given by

$$X_k=\left[\frac{\hbar}{2m_k\omega_k}\right]^{1/2}(a_k^\dagger+a_k)\ . \tag{2}$$

The first two terms in the Hamiltonian (1) represent the free motion of the system plus bath while the third term represents the system-bath coupling. For a discussion of the generality of the Hamiltonian Eq. (1) as a model for dissipative systems see Ref. 5. We could now easily obtain an equation of motion for the total density operator of the system plus environment. However, our interest lies in the dynamics of the system rather than the bath so we seek an equation for the trace of the total density operator over the bath, that is, for the reduced density operator of the system alone, ρ.

By taking a continuum limit for the number of bath oscillators and making the Born and Markov approximations, Agarwal[9] obtained the following Schrödinger-picture Markovian master equation for the reduced density operator:

$$\partial_t\rho=-i\omega[a^\dagger a,\rho]-\frac{i}{\hbar}\gamma[X,P\rho+\rho P]$$
$$-\frac{2\gamma}{\hbar}(\bar{n}+\tfrac{1}{2})m\omega[X,[X,\rho]]\ , \tag{3}$$

where we have dropped the subscript zero from the system quantities since no bath operators occur in the equation and to allow for possible renormalization of the oscillator frequency from its bare value ω_0.[4] γ is called the damping constant, P is the system momentum observable, and $\bar{n}$ is the expected number of quanta in a harmonic oscillator of frequency ω at equilibrium at temperature T:

$$\bar{n}=[\exp(\hbar\omega/k_BT)-1]^{-1}\ . \tag{4}$$

The approximations involved in obtaining the Markovian master equation (3) may be analyzed by choosing a particular distribution of the bath oscillator frequencies and a particular coupling strength distribution. For example, choosing the product of the frequency and strength distributions to be proportional to frequency squared, up to a certain cutoff frequency, Eq. (3) follows from the Hamiltonian (1) in the high-temperature limit, after suitable renormalization of the system frequency.[4] Models which are Markovian at low temperatures are more elusive and temperature-dependent couplings may be necessary.[3]

Instead of solving the master equation (3) directly we solve for the quantum characteristic function χ defined in terms of the density operator ρ by[14]

$$\chi(\lambda)=\mathrm{Tr}(\rho e^{\lambda a^\dagger}e^{-\lambda^* a})\ , \tag{5}$$

where λ is a complex variable and the trace is over the system. From the master equation (3) the following equation for χ is found:

$$\{\partial_t+[(-i\omega+\gamma)\lambda+\gamma\lambda^*]\partial_\lambda+[\gamma\lambda+(i\omega+\gamma)\lambda^*]\partial_{\lambda^*}\}\chi$$
$$=-\gamma\bar{n}(\lambda+\lambda^*)^2\chi\ . \tag{6}$$

This first-order partial-differential equation may be solved by the method of characteristics.

III. ZERO TEMPERATURE

At zero temperature, $\bar{n}=0$, and the solution of Eq. (6) is an arbitrary function $I(f,f^*)$ of the arguments f and f^*, where $f=u\lambda+v\lambda^*$ and

$$u=\frac{1}{2}\left[e^{-\mu_-t}+e^{-\mu_+t}+2i\frac{\omega}{\mu_--\mu_+}(e^{-\mu_+t}-e^{-\mu_-t})\right],$$
$$v=\frac{\gamma}{\mu_--\mu_+}(e^{-\mu_-t}-e^{-\mu_+t})\ . \tag{7}$$

The eigenvalues $\mu_\pm$ are given by

$$\mu_\pm=\gamma\pm(\gamma^2-\omega^2)^{1/2}\ . \tag{8}$$

The function $I(f,f^*)$ is chosen to fit the initial condition at time, $t=0$. We consider our system to initially be in a superposition of coherent states. The harmonic-oscillator coherent states $|\alpha\rangle$ are minimum uncertainty states having mean coordinate $\langle X\rangle$ and mean momentum $\langle P\rangle$ given by

$$\langle X\rangle=(2\hbar/m\omega)^{1/2}\mathrm{Re}(\alpha),\quad \langle P\rangle=(2\hbar m\omega)^{1/2}\mathrm{Im}(\alpha)\ . \tag{9}$$

The magnitude of the scalar product of coherent states $|\alpha\rangle$ and $|\beta\rangle$ is

$$|\langle\alpha|\beta\rangle|=\exp(-\tfrac{1}{2}|\alpha-\beta|^2) \tag{10}$$

so that $|\alpha\rangle$ and $|\beta\rangle$ are approximately orthogonal for large $|\alpha-\beta|$.

Consider an initial density operator representing a superposition of coherent states:

$$\rho(0)=\sum_{\alpha,\beta}N_{\alpha\beta}|\alpha\rangle\langle\beta|\ . \tag{11}$$

The quantum characteristic function corresponding to the operator $|\alpha\rangle\langle\beta|$ is from Eq. (5):

$$\chi=\mathrm{Tr}(|\alpha\rangle\langle\beta|e^{\lambda a^\dagger}e^{-\lambda^* a})=\langle\beta|\alpha\rangle\exp(\lambda\beta^*-\lambda^*\alpha)\ . \tag{12}$$

With this initial condition the zero-temperature solution to Eq. (6) is

$$\chi(\lambda,t)=\langle\beta|\alpha\rangle\exp(\beta^*f-\alpha f^*)$$
$$=\langle\beta|\alpha\rangle\exp[\lambda(u\beta^*-v\alpha)-\lambda^*(u^*\alpha-v\beta^*)]\ . \tag{13}$$

Comparing with Eq. (12) we find the initial density operator Eq. (11) evolves as

$$\rho(t)=\sum_{\alpha,\beta}N_{\alpha\beta}\langle\beta|\alpha\rangle\frac{|u^*\alpha-v\beta^*\rangle\langle u^*\beta-v\alpha^*|}{\langle u^*\beta-v\alpha^*|u^*\alpha-v\beta^*\rangle}\ . \tag{14}$$

The explicit time dependence of the coherent-state arguments occurring here are

$$u^*\alpha - v\beta^* = \tfrac{1}{2}e^{-\mu_- t}\left[\alpha\left[1-\frac{i\omega}{\Delta}\right]+\beta^*\frac{\gamma}{\Delta}\right] + \tfrac{1}{2}e^{-\mu_+ t}\left[\alpha\left[1+\frac{i\omega}{\Delta}\right]-\beta^*\frac{\gamma}{\Delta}\right],$$

$$u^*\beta - v\alpha^* = \tfrac{1}{2}e^{-\mu_- t}\left[\beta\left[1-\frac{i\omega}{\Delta}\right]+\alpha^*\frac{\gamma}{\Delta}\right] + \tfrac{1}{2}e^{-\mu_+ t}\left[\beta\left[1+\frac{i\omega}{\Delta}\right]-\alpha^*\frac{\gamma}{\Delta}\right], \quad (15)$$

$$\Delta = \tfrac{1}{2}(\mu_+ - \mu_-) = (\gamma^2-\omega^2)^{1/2}.$$

Hence an initial superposition of coherent states remains as a superposition of the same number of coherent states under the evolution given by the master equation (3).

In the underdamped case, $\gamma < \omega$, the eigenvalues Eq. (8) are complex,

$$\mu_\pm = \gamma \pm i\Omega, \quad \Omega = (\omega^2-\gamma^2)^{1/2}, \quad (16)$$

and the coherent-state arguments Eqs. (15) become

$$u^*\alpha - v\beta^* = e^{-\gamma t}\left[\alpha\cos(\Omega t) + \left[i\frac{\omega}{\Omega}\alpha - \frac{\gamma}{\Omega}\beta^*\right]\sin(\Omega t)\right], \quad (17)$$

$$u^*\beta - v\alpha^* = e^{-\gamma t}\left[\beta\cos(\Omega t) + \left[i\frac{\omega}{\Omega}\beta - \frac{\gamma}{\Omega}\alpha^*\right]\sin(\Omega t)\right].$$

After n oscillations, that is, at time $t = 2\pi n/\Omega$, the density operator corresponding to the initial condition Eq. (11) is

$$\rho(t=2\pi n/\Omega) = \sum_{\alpha,\beta} N_{\alpha\beta}\langle\beta\,|\,\alpha\rangle^{[1-\exp(-2\gamma t)]} \times |\,e^{-\gamma t}\alpha\rangle\langle e^{-\gamma t}\beta\,|\,. \quad (18)$$

This result valid only for the discrete times, $t = 2\pi n/\Omega$, is otherwise identical to the result obtained in the small damping limit, $\gamma \ll \omega$, for which the initial condition Eq. (11) evolves to

$$\rho(t) = \sum_{\alpha,\beta} N_{\alpha\beta}\langle\beta\,|\,\alpha\rangle^{[1-\exp(-2\gamma t)]} \times |\,e^{-\gamma t}\alpha e^{-i\omega t}\rangle\langle e^{-\gamma t}\beta e^{-i\omega t}\,|\,, \quad (19)$$

a result which has previously been obtained by making the rotating-wave approximation on the master equation (3).[11,12] The rotating-wave approximation is commonly made in quantum optical systems, which have frequencies $\omega \approx 10^{15}\ \mathrm{s}^{-1}$, so that we may expect the small damping $\gamma \ll \omega$ approximation to be valid. Note that the results, Eqs. (18) and (19), contain an exponential relaxation to the vacuum state at rate γ. However, a point of particular interest is the rapid decay of the off-diagonal parts of the density operator. For short times, $2\gamma t \ll 1$, the decay occurs exponentially at the rate $-2\gamma\ln(\langle\beta\,|\,\alpha\rangle)$, a quantity which increases with decreasing overlap of the initial states. On taking diagonal coordinate-basis matrix elements one finds the solution Eq. (19) to be consistent with the result of Caldeira and Leggett[6] for the destruction of interference in their weakly damped low-temperature limit.

In the overdamped case,[15] $\gamma > \omega$, the eigenvalues Eq. (8) are real:

$$\mu_\pm = \gamma \pm (\gamma^2-\omega^2)^{1/2}. \quad (20)$$

As γ increases, μ_+ increases towards 2γ and μ_- decreases towards zero. Hence we have a fast and a slow eigenvalue.

In the following we examine the limit of very large damping, $\gamma \gg \omega$, so that

$$\mu_+ \approx 2\gamma, \quad \mu_- \approx \frac{\omega^2}{2\gamma}, \quad \mu_- \ll \mu_+. \quad (21)$$

The time dependence of the coherent states, Eqs. (15), becomes

$$u^*\alpha - v\beta^* \approx e^{-(\omega^2/2\gamma)t}\left[\frac{\alpha+\beta^*}{2}\right] + e^{-2\gamma t}\left[\frac{\alpha-\beta^*}{2}\right],$$

$$u^*\beta - v\alpha^* \approx e^{-(\omega^2/2\gamma)t}\left[\frac{\alpha^*+\beta}{2}\right] + e^{-2\gamma t}\left[\frac{\beta-\alpha^*}{2}\right]. \quad (22)$$

A diagonal initial density operator, $\rho = |\,\alpha\rangle\langle\alpha\,|$, thus evolves as

$$\rho(t) \approx |\,\mathrm{Re}(\alpha)e^{-(\omega^2/2\gamma)t} + i\,\mathrm{Im}(\alpha)e^{-2\gamma t}\rangle \times \langle\mathrm{Re}(\alpha)e^{-(\omega^2/2\gamma)t} + i\,\mathrm{Im}(\alpha)e^{-2\gamma t}\,|\,. \quad (23)$$

Note that the imaginary part, proportional to the mean momentum, is damped at the rate 2γ, while the real part, proportional to the mean coordinate, is damped at the much slower rate $\omega^2/2\gamma$. Thus the heavily overdamped oscillator will respond sluggishly to forces trying to change its coordinate. A similar result is found for the strongly overdamped classical harmonic oscillator.

Next we consider the behavior of the off-diagonal parts of the initial density operator, $|\,\alpha\rangle\langle\beta\,|$. When $\alpha = a$ and $\beta = b$ are real, i.e., when the initial mean momenta are zero, the initial operator $|\,a\rangle\langle b\,|$ evolves as

$$(|\,a\rangle\langle b\,|)_t \approx \langle a\,|\,b\rangle^{[1-\exp(-4\gamma t)]}\left|e^{-(\omega^2/2\gamma)t}\left[\frac{a+b}{2}\right] + e^{-2\gamma t}\left[\frac{a-b}{2}\right]\right\rangle \times \left\langle e^{-(\omega^2/2\gamma)t}\left[\frac{a+b}{2}\right] + e^{-2\gamma t}\left[\frac{b-a}{2}\right]\right|. \quad (24)$$

There are two processes here. First, the density operator is being reduced by a factor which is 1 at $t = 0$ and $\langle a\,|\,b\rangle$ for

$4\gamma t \gg 1$. For a one-gram mass in a harmonic-oscillator potential of frequency $\omega = 1\ \mathrm{s}^{-1}$, the scalar product $\langle a \mid b \rangle$ is

$$\langle a \mid b \rangle = e^{-(a-b)^2/2} = \exp[-\tfrac{1}{2}(m\omega/2\hbar)(\langle X \rangle_a - \langle X \rangle_b)^2] \approx \exp[-(2.5\times 10^{30})(\langle X \rangle_a - \langle X \rangle_b)^2] , \tag{25}$$

where $\langle X \rangle_a$ and $\langle X \rangle_b$ are the mean coordinates of the coherent states $\mid a \rangle$ and $\mid b \rangle$, respectively. For separations of the mean coordinates of the superposed states exceeding nuclear dimensions $\langle a \mid b \rangle$ will be negligibly small, e.g., taking $\langle X \rangle_a - \langle X \rangle_b = 1$ Å we find $\langle a \mid b \rangle \approx \exp(-2.5\times 10^{10})$. Second, the initial superposition diagonalizes at rate 2γ, being replaced by a "classical" probability for finding the particle in the coherent state, $\mid \tfrac{1}{2}(a+b) \rangle$.

When $\alpha = ia$ and $\beta = ib$ are imaginary, i.e., when the initial coordinates are zero, the off-diagonal parts of the initial density operator, $\mid ia \rangle\langle ib \mid$, evolve as

$$(\mid ia \rangle\langle ib \mid)_t \approx \langle ia \mid ib \rangle^{1-\exp[-(\omega^2/\gamma)t]} \left| e^{-(\omega^2/2\gamma)t}\left[\frac{ia-ib}{2}\right] + e^{-2\gamma t}\left[\frac{ia+ib}{2}\right]\right\rangle \times \left\langle e^{-(\omega^2/2\gamma)t}\left[\frac{ib-ia}{2}\right] + e^{-2\gamma t}\left[\frac{ia+ib}{2}\right]\right| . \tag{26}$$

Again there are two processes. First, the multiplying factor decays from one to $\langle ia \mid ib \rangle$, but much more slowly than when α and β are real. Second, the superposition is reduced at rate 2γ to a superposition of two coherent states having opposite momenta and momentum difference proportional to $a-b$. This remaining superposition slowly decays towards the vacuum.

Thus superpositions of zero mean momentum-coherent states are rapidly diagonalized, while superpositions of zero mean coordinate-coherent states are preserved in a modified form. In other words, all "coordinate coherence" rapidly decays away while "momentum coherence" may still be found. This asymmetry is a result of the coordinate-coordinate system to bath coupling, Eq. (1).

Still in the strongly overdamped limit, $\gamma \gg \omega$, we now consider the long time limit, $\gamma t \gg 1$. We restrict our attention to $\alpha = a$, $\beta = b$, real. Then Eq. (24) yields

$$(\mid a \rangle\langle b \mid)_t \approx \langle a \mid b \rangle \mid e^{-(\omega^2/2\gamma)t\frac{1}{2}(a+b)} \rangle \times \langle e^{-(\omega^2/2\gamma)t\frac{1}{2}(a+b)} \mid . \tag{27}$$

Consider an initial pure-state density operator corresponding to a superposition of two zero mean momentum-coherent states $\mid a \rangle$ and $\mid b \rangle$:

$$\rho(0) = N^{-1}(\mid a \rangle\langle a \mid + \mid b \rangle\langle b \mid + \mid a \rangle\langle b \mid + \mid b \rangle\langle a \mid) , \tag{28}$$

where N is a normalization constant. Using the result Eq. (27) for $\gamma \gg \omega$ and $\gamma t \gg 1$ this evolves as

$$\rho(t) \approx N^{-1}\Big[\mid e^{-(\omega^2/2\gamma)t}a \rangle\langle e^{-(\omega^2/2\gamma)t}a \mid + \mid e^{-(\omega^2/2\gamma)t}b \rangle\langle e^{-(\omega^2/2\gamma)t}b \mid + 2\langle a \mid b \rangle \mid e^{-(\omega^2/2\gamma)t\frac{1}{2}(a+b)} \rangle \times \langle e^{-(\omega^2/2\gamma)t\frac{1}{2}(a+b)} \mid \Big] . \tag{29}$$

For the example following Eq. (24) the coherent-state coordinate variance is $2\times 10^{-31}\ \mathrm{m}^2$ so that the coordinate is localized over nuclear dimensions. Thus the mixture of coherent states, Eq. (29), may for practical purposes be regarded as a mixture of well-defined coordinate states of a macroscopic oscillator. The interaction with the environment Eq. (1) has resulted in the collapse of the pure-state superposition Eq. (28) to a mixture of well-defined coordinate states of the macroscopic oscillator.

Taking diagonal matrix elements of Eq. (29), in the basis of the system coordinate X, we find

$$\langle x \mid \rho(t) \mid x \rangle \approx N^{-1}(\rho_a^2 + \rho_b^2 + 2\rho_a\rho_b \langle a \mid b \rangle^{1-\exp[-(\omega^2/\gamma)t]}) , \tag{30}$$

where

$$\rho_a = \langle x \mid e^{-(\omega^2/2\gamma)t}a \rangle , \quad \rho_b = \langle x \mid e^{-(\omega^2/2\gamma)t}b \rangle , \tag{31}$$

which agrees with a result obtained, using a path-integral approach, by Caldeira and Leggett in their strongly damped low-temperature limit.[6] The first two terms of Eq. (30) reflect uncertainty as to whether the oscillator is excited in the coherent states $\mid a \rangle$ or $\mid b \rangle$. The third term is a result of the fact that these states were initially superposed. However, for the times under consideration, $\gamma t \gg 1$, it is apparent from Eq. (29) that this superposition has been replaced by the coherent excitation $\mid \tfrac{1}{2}(a+b)\exp(-\omega^2 t/2\gamma) \rangle$. Note that since ρ_a and ρ_b are Gaussians centered at $\langle X \rangle_a$ and $\langle X \rangle_b$, respectively, the product $\rho_a\rho_b$ will be small when the overlap $\langle a \mid b \rangle$ is small.

IV. FINITE TEMPERATURE

As in the zero-temperature case we take our initial density operator to be a superposition of coherent states of the form Eq. (11). Solving Eq. (6) for the corresponding initial quantum characteristic function we find

$$\chi(\lambda,t) = \sum_{\alpha,\beta} N_{\alpha\beta}\langle \beta \mid \alpha \rangle \exp\{\lambda(u\beta^* - v\alpha) - \lambda^*(u^*\alpha - v\beta^*)\} \times \exp\{\bar{n}[\lambda^2 uv + \lambda^{*2}u^*v + \mid\lambda\mid^2(\mid u\mid^2 + v^2 - 1)]\} , \tag{32}$$

where u and v were defined in Eq. (7). Unlike the zero-temperature case this quantum characteristic function does not have the simple form of Eq. (12). However, the quantum characteristic function and the diagonal coherent-state density-matrix elements are related by[14]

$$\langle z|\rho|z\rangle=\frac{1}{\pi}\int\chi(\lambda)\exp(-|\lambda|^2-\lambda z^*+\lambda^* z)d^2\lambda\ , \tag{33}$$

where as usual $d^2\lambda=d[\mathrm{Re}(\lambda)]d[\mathrm{Im}(\lambda)]$. Now the density operator ρ can be expressed in terms of these diagonal coherent-state matrix elements by[16]

$$\rho=\frac{1}{\pi^2}\int\int\frac{|z+\delta\rangle\langle z-\delta|}{\langle z-\delta|z+\delta\rangle}e^{-|\delta|^2}\langle z|\rho|z\rangle d^2z\,d^2\delta\ . \tag{34}$$

We first examine the extremely underdamped case, $\gamma\ll\omega$, so that

$$v\approx 0,\quad u\approx\exp[(-\gamma+i\omega)t]\ . \tag{35}$$

The diagonal coherent-state matrix elements found from Eqs. (32) and (33) are then

$$\langle z|\rho|z\rangle=\sum_{\alpha,\beta}N_{\alpha\beta}\langle\beta|\alpha\rangle[\bar{n}(1-e^{-2\gamma t})+1]^{-1}\exp\left[-\frac{(z-\alpha e^{-i\omega t}e^{-\gamma t})(z^*-\beta^* e^{i\omega t}e^{-\gamma t})}{\bar{n}(1-e^{-2\gamma t})+1}\right]\ . \tag{36}$$

We now substitute this result into the expansion Eq. (34) and take matrix elements in the coordinate basis, $|x\rangle$:

$$\begin{aligned}\langle x-y|\rho|x+y\rangle=&(2\pi\sigma_x^2)^{-1/2}\sum_{\alpha,\beta}N_{\alpha\beta}\langle\beta|\alpha\rangle\exp[\sigma_y^2(m\omega/\hbar)\delta^2]\\&\times\exp\{-\tfrac{1}{2}\sigma_x^{-2}[x-(\hbar/2m\omega)^{1/2}\delta]^2\}\exp\{-\tfrac{1}{2}\sigma_y^{-2}[y-\sigma_y^2(2m\omega/\hbar)^{1/2}\delta]^2\}\\ \delta=&e^{-\gamma t}(\alpha e^{-i\omega t}+\beta^* e^{i\omega t})\ .\end{aligned} \tag{37}$$

The variances of the diagonal, σ_x^2, and off-diagonal, σ_y^2, variables are

$$\sigma_x^2=\frac{\hbar}{2m\omega}[2\bar{n}(1-e^{-2\gamma t})+1]\ ,\qquad \sigma_y^2=\left[\frac{\hbar}{2m\omega}\right]^2\frac{1}{\sigma_x^2}=\frac{\hbar}{2m\omega}[2\bar{n}(1-e^{-2\gamma t})+1]^{-1}\ . \tag{38}$$

These results have previously been derived for the case $\alpha=\beta$ by Milburn and Walls.[12] Starting from the initial coherent-state value the variance of the diagonal part increases from $\hbar/2m\omega$ to $(2\bar{n}+1)\hbar/2m\omega$ in a time of a few γ^{-1}. The off-diagonal variance decreases from $\hbar/2m\omega$ to about $(2\bar{n}+1)^{-1}\hbar/2m\omega$ after a similar time. Thus the environment tends to eliminate the off-diagonal coordinate-basis density-matrix elements. Although the degree of elimination achieved increases with temperature so too does the degree of diagonal spreading.

Consider an initial density operator of the form Eq. (28). From Eq. (37) the evolution of its diagonal coordinate basis matrix elements is found to be

$$\begin{aligned}\langle x|\rho|x\rangle=&N^{-1}[\rho_a^2+\rho_b^2+2\rho_a\rho_b\cos(\theta)\langle b|a\rangle^\eta]\ ,\\ \theta=&\tfrac{1}{2}\sigma_x^{-2}[(\hbar/2m\omega)^{1/2}e^{-\gamma t}2x(b-a)\sin(\omega t)\\&+(\hbar/2m\omega)e^{-2\gamma t}\sin(2\omega t)(a^2-b^2)]\ ,\\ \eta=&1-\sigma_x^{-2}(\hbar/2m\omega)e^{-2\gamma t}=1-\frac{e^{-2\gamma t}}{2\bar{n}(1-e^{-2\gamma t})+1}\ ,\end{aligned} \tag{39}$$

where

$$\begin{aligned}\rho_a^2=&(2\pi\sigma_x^2)^{-1/2}\\&\times\exp\{-\tfrac{1}{2}\sigma_x^{-2}[x-(\hbar/2m\omega)^{1/2}e^{-\gamma t}2a\cos(\omega t)]^2\}\ ,\\ \rho_b^2=&(2\pi\sigma_x^2)^{-1/2}\\&\times\exp\{-\tfrac{1}{2}\sigma_x^{-2}[x-(\hbar/2m\omega)^{1/2}e^{-\gamma t}2b\cos(\omega t)]^2\}\ .\end{aligned} \tag{40}$$

ρ_a^2 and ρ_b^2 are the contributions of the diagonal parts of the initial density operator and the third term is an oscillating interference resulting from the initial superposition. The exponent of the scalar product $\langle b|a\rangle$ is zero at $t=0$, and tends to 1 for $\gamma t\gg 1$, at which time the third term is decreased by the overlap $\langle b|a\rangle$ of the initially superposed states. Recalling the example following Eq. (24), $\langle b|a\rangle$ will be negligible for macroscopically separated initial coherent states of a macroscopic oscillator. For small times $\gamma t\ll 1$ and for high temperatures, $\bar{n}\approx kT/\hbar\omega\gg 1$, we find the exponent $\eta\approx(4k_BT\gamma/\hbar\omega)t$, and we recover a result of Caldeira and Leggett.[6]

Next we examine the heavily overdamped finite-temperature case. For $\gamma\gg\omega$ Eq. (7) yields

$$u\approx\tfrac{1}{2}(e^{-2\gamma t}+e^{-(\omega^2/2\gamma)t})\ ,\qquad v\approx\tfrac{1}{2}(e^{-(\omega^2/2\gamma)t}-e^{-2\gamma t})\ . \tag{41}$$

Using these expressions in Eqs. (32) and (33) we find the diagonal coherent-state matrix elements to be

$$\langle z|\rho|z\rangle=\sum_{\alpha,\beta}N_{\alpha\beta}\langle\beta|\alpha\rangle\frac{1}{\sqrt{rs}}\exp\left[\frac{1}{4r}[(\beta^*-\alpha)e^{-2\gamma t}+z-z^*]^2\right]\exp\left[-\frac{1}{4s}[(\beta^*+\alpha)e^{-(\omega^2/2\gamma)t}-z-z^*]^2\right],$$
$$r=1+\bar{n}(1-e^{-4\gamma t}),\quad s=1+\bar{n}(1-e^{-\omega^2 t/\gamma}). \tag{42}$$

Substituting this result into Eq. (34), taking coordinate-basis matrix elements and evaluating the integrals we find

$$\begin{aligned}\langle x-y|\rho|x+y\rangle=(2\pi\sigma_x^2)^{-1/2}\sum_{\alpha,\beta}N_{\alpha\beta}\langle\beta|\alpha\rangle&\exp[\tfrac{1}{2}\sigma_y^2(2m\omega/\hbar)(\beta^*-\alpha)^2e^{-4\gamma t}]\\&\times\exp\{-\tfrac{1}{2}\sigma_x^{-2}[x-(\hbar/2m\omega)^{1/2}(\beta^*+\alpha)e^{-(\omega^2/2\gamma)t}]^2\}\\&\times\exp[-\tfrac{1}{2}\sigma_y^{-2}y^2+(2m\omega/\hbar)^{1/2}(\beta^*-\alpha)e^{-2\gamma t}y].\end{aligned} \tag{43}$$

The variances of the diagonal, σ_x^2, and off-diagonal, σ_y^2, parts are

$$\sigma_x^2=\frac{\hbar}{2m\omega}[1+2\bar{n}(1-e^{-\omega^2 t/\gamma})],$$
$$\sigma_y^2=\frac{\hbar}{2m\omega}[1+2\bar{n}(1-e^{-4\gamma t})]^{-1}. \tag{44}$$

At high temperatures and after a time of a few γ^{-1} the off-diagonal variance becomes $\hbar^2/4mk_BT=(\lambda/4\pi)^2$, where λ is the de Broglie wavelength associated with the oscillator's mean kinetic energy at temperature T.[12] However, the spreading of the diagonal part occurs at a much slower rate determined by the quantity ω^2/γ. Thus for times t such that $\gamma/\omega^2\gg t\gg 1/4\gamma$ the density matrix has been substantially diagonalized in the coordinate basis without much thermal spreading of the diagonal matrix elements. The degree of diagonalization increases with temperature and the off-diagonal variance can be made arbitrarily small by making the temperature sufficiently high.

As in the underdamped case we can use the result Eq. (43) to calculate the diagonal coordinate-basis density-matrix elements for an initial density operator of the form Eq. (28). For short times such that $\omega^2 t/\gamma, 2\bar{n}\omega^2 t/\gamma\ll 1$ we find

$$\langle x|\rho|x\rangle=N^{-1}(\rho_a^2+\rho_b^2+2\rho_a\rho_b\langle b|a\rangle^{(\omega^2/\gamma)(1+2\bar{n})t}), \tag{45}$$

where as usual the first two terms arise from the diagonal parts of the initial density matrix. For higher temperatures the exponent of $\langle b|a\rangle$ is $(2\omega k_BT/\gamma\hbar)t$, Eq. (45) then agrees with the high-temperature strongly damped result of Caldeira and Leggett.[6]

V. DISCUSSION

Zurek has recently discussed the behavior of a quantum system coupled to a bath consisting of a large number of other quantum systems.[13] He investigated how a system-environment coupling may lead to a diagonalization of the system density matrix. The basis in which the density matrix becomes diagonalized is called the pointer basis and is determined by the form of the system-environment interaction Hamiltonian. Specifically, the pointer basis will be the eigenstates of the system observable which commutes with both the free and interaction parts of the Hamiltonian.

These observations are particularly relevant to the theory of quantum measurements.[17] In this context we have a quantum system on which a measurement is to be performed, a meter from which the measurement result is read, and an environment which together with the meter forms the classical measuring apparatus. Following Zurek, the meter-environment coupling determines the pointer basis. Due to its interaction with the environment the meter density matrix becomes diagonal in the pointer basis, corresponding to a classical probability distribution for it to be found in a given pointer eigenstate. The system-meter coupling is designed so that the eigenstates of the system observable being measured are correlated with pointer basis states of the meter. Thus the environmental reduction of the meter state also reduces the system state, as required in quantum measurement theory.

We have considered in detail only the cases of large underdamping, $\gamma\ll\omega$, and of large overdamping, $\gamma\gg\omega$. In the former case the system undergoes many oscillations before the damping effects are significant, whereas in the strongly overdamped case the system is prevented by the damping from completing even one oscillation. Now the coordinate X, which couples the system to the environment in the Hamiltonian Eq. (1), is not a constant of the motion. Hence as the system oscillates coordinate eigenstates evolve through various mixtures of coordinate and momentum eigenstates. Only in the overdamped case, when the damping can exert a significant effect before the system begins to oscillate, will the pointer basis be clearly defined as the coordinate basis.

In Sec. IV we found the coordinate-basis matrix elements of the system density matrix, $\langle x-y|\rho|x+y\rangle$. The momentum-basis matrix elements are also of interest and may be obtained from the coordinate-basis elements by a Fourier transformation. We seek them in the form $\langle p-q|\rho|p+q\rangle$:

$$\begin{aligned}\langle p-q|\rho|p+q\rangle=(2\pi\hbar)^{-1}\int\int dx\,dy&\\\times\exp[-2i(py+qx)/\hbar]&\\\times\langle x-y|\rho|x+y\rangle.&\end{aligned} \tag{46}$$

Since the coordinate-basis density-matrix elements are a product of Gaussians in each of the variables x and y, Eqs. (37) and (43), the momentum-basis density-matrix

elements will be a product of Gaussians in the variables p and q. The variances in the diagonal, p, and off-diagonal, q, momentum variables are, in the highly underdamped case $\gamma \ll \omega$,

$$\sigma_p^2 = \frac{\hbar^2}{4}\sigma_y^{-2} = \tfrac{1}{2}\hbar m\omega[2\bar{n}(1-e^{-2\gamma t})+1] ,$$
$$\sigma_q^2 = \frac{\hbar^2}{4}\sigma_x^{-2} = \tfrac{1}{2}\hbar m\omega[2\bar{n}(1-e^{-2\gamma t})+1]^{-1} , \tag{47}$$

and in the highly overdamped case $\gamma \gg \omega$,

$$\sigma_p^2 = \frac{\hbar^2}{4}\sigma_y^{-2} = \tfrac{1}{2}\hbar m\omega[1+2\bar{n}(1-e^{-4\gamma t})] ,$$
$$\sigma_q^2 = \frac{\hbar^2}{4}\sigma_x^{-2} = \tfrac{1}{2}\hbar m\omega[1+2\bar{n}(1-e^{-(\omega^2/\gamma)t})]^{-1} . \tag{48}$$

The highly underdamped coordinate-basis variances and the momentum-basis variances are identical up to multiplying factors. After a time of a few γ^{-1} the variances of the diagonal variables in both representations are proportional to $2\bar{n}+1$ and the variances of the off-diagonal variables proportional to $(2\bar{n}+1)^{-1}$. For high temperatures $\bar{n}$ is large so the diagonal variances are also large while the off-diagonal variances are small. Recall that the eigenstates of the system observable X coupling to the bath oscillate through mixtures of coordinate and momentum eigenstates. In this case the interaction tends to eliminate the off-diagonal elements of the density matrix in both representations at the expense of spreading in the diagonal-matrix elements.

The zero-temperature highly underdamped result, Eq. (19), shows comparable behavior. The damping of the parts of the density operator off diagonal in the coherent-state basis depends not on the mean coordinate or momentum of the coherent states but rather on the scalar product of the initially superposed coherent states. Coherences between zero mean coordinate-coherent states are damped just as effectively by the highly underdamped coordinate-coordinate coupling as coherences between zero mean momentum-coherent states.

In the highly overdamped case differences between the coordinate-basis and momentum-basis density-matrix elements appear. For instance, comparing the coordinate-basis variances, Eq. (44), with the momentum basis variances, Eq. (48), we find that after a time of a few γ^{-1} the diagonal coordinate-basis variances and the off-diagonal momentum-basis variance remain at essentially their initial coherent-state values. However, the off-diagonal coordinate-basis variance has decreased by a factor of about $(1+2\bar{n})^{-1}$, which is small at high temperatures. Thus we find a diagonalization of the density matrix in the coordinate basis without an associated increase in the diagonal variance. The original coordinate distribution is unaltered by the diagonalization process.

Comparable behavior was found in the zero-temperature highly overdamped case. Comparing Eq. (24) with Eq. (26) we recall that a coherence between zero-momentum coherent states is damped much more rapidly than a coherence between zero-coordinate coherent states. We have already discussed how macroscopic oscillators are collapsed into a mixture of states of well-defined coordinate in the zero-temperature case. Microscopic oscillators, however, require a high-temperature bath to be diagonalized in the coordinate basis, with higher temperatures leading to more complete diagonalization. The resistance of the overdamped oscillator to changes in its coordinate, Eq. (23), is reminiscent of the watchdog or quantum Zeno effect in which continuous measurement of an observable inhibits its change.[18]

To summarize, we have demonstrated for the system-environment coupling given by the Hamiltonian (1) contentions of Zurek concerning environmental diagonalization of the density matrix. In particular we have found that when the system oscillator is heavily overdamped by a high-temperature bath the diagonalization in the coordinate basis occurs without substantial disturbance of the oscillator's coordinate probability distribution. The environmental interaction given by the Hamiltonian (1) thus provides a model of a coordinate measurement on a harmonic oscillator leading to diagonalization of the density matrix, otherwise referred to as collapse of the state vector.

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