PHYSICAL REVIEW A VOLUME 32, NUMBER 4 OCTOBER 1985

Quantum nondemolition measurement of the photon number via the optical Kerr effect

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(Received 30 April 1985)

This paper proposes a quantum nondemolition measurement scheme for the photon number. The signal and probe optical waves interact via the optical Kerr effect. The optical phase of the probe wave is selected as the readout observable for the measurement of the photon number of the signal wave. The measurement accuracy Δn and the imposed phase noise $\Delta\phi$ of the signal wave satisfy Heisenberg's uncertainty principle with an equality sign, $\langle(\Delta n)^2\rangle\langle(\Delta\phi)^2\rangle=\frac{1}{4}$.

I. INTRODUCTION

A quantum nondemolition (QND) measurement,[1–4] in which an observable is measured without perturbing its free motion, was first proposed in order to overcome the quantum limit in the detection of gravitational waves. In recent years, QND measurements were proposed in connection with quantum optics.[4,5] The QND theory for the gravitational wave detector (Weber bar), which consists of a mechanical harmonic oscillator, is directly applicable to the photon field, which is also described by an ensemble of harmonic oscillators. In a QND measurement, precise detection of an observable is accomplished at the expense of an increase in uncertainty of its canonical conjugate observable. This uncertainty imbalance is similar to the uncertainty relationship between two quadrature components for nonclassical boson states such as the squeezed states (or two-photon coherent states[6]) and number states.

Several schemes for a QND experiment have been proposed so far.[1–5] Unruh[7] has pointed out that the interaction should be quadratic for a QND measurement of an oscillator state. Milburn and Walls[5] have analyzed a QND measurement using a four-wave-mixing interaction. It seems, however, that little attention has been paid to possible physical realizations. The ultimate limit of the uncertainty product between the measurement accuracy and the imposed noise on the conjugate observable has not yet been investigated for QND measurements.

This paper proposes a QND measurement of the photon number using the optical Kerr effect, and obtains the uncertainty relationship between the measurement accuracy of the photon number and the imposed phase noise. The optical Kerr effect is used for an interaction which gives the product of the probe and signal intensities to be measured. The photon number of the signal wave is the QND observable, and the optical phase of the probe wave is the readout observable.

II. CONDITIONS FOR QND MEASUREMENTS

In a general quantum measurement, the observable of the signal system, A_s, is measured by detecting the change in the observable of the probe system, A_p, using the proper interaction between the signal and probe systems expressed by an interaction Hamiltonian H_I. The total Hamiltonian H is expressed as

$$H=H_s+H_p+H_I \ , \tag{1}$$

where H_s is the unperturbed Hamiltonian of the signal system and H_p is that of the probe system. Heisenberg's equations of motion for A_s and A_p are

$$-i\hbar\frac{dA_s}{dt}=[H_s,A_s]+[H_I,A_s] \ , \tag{2}$$

and

$$-i\hbar\frac{dA_p}{dt}=[H_p,A_p]+[H_I,A_p] \ . \tag{3}$$

The first commutators in Eqs. (2) and (3) contribute to the free motion of A_s and A_p, respectively, while the second commutators contribute to the interaction between the signal and probe systems. In order to measure A_s using A_p, $[H_I,A_p]$ in (3) should not be zero, and furthermore, it is necessary that H_I be a function of A_s.

In general, a measurement of A_s affects the motion of A_s itself in two ways. One is the change of A_s due to the second commutator in (2) during the time H_I is switched on for the measurement. This change should be avoided in order to ensure that the free motion of A_s remains unperturbed.

A broader definition of QND (Refs. 2 and 3) is a sequence of measurements such that the result of each measurement is predictable from the first measurement. In this broader sense, the change of A_s by H_I is allowable if it is predictable. We adopt the more limited but practical QND definition, however, in which a measurement should not perturb the motion of A_s, even deterministically. In this case $[H_I, A_s]=0$ is essential.

Another issue of interest is the uncertainty that is introduced by the measurement of A_s in the observable that does not commute with A_s. If the unperturbed Hamiltonian H_s contains this observable, the motion of A_s becomes unpredictable due to the uncertainty imposed on the conjugate observables by the measurement of A_s. To ensure that the measurement of A_s does not affect A_s itself, H_s should not be a function of the conjugate observable of A_s. A more general condition for continuous QND observables[4] is expressed as

$$[A_s(t_1), A_s(t_2)]=0 \tag{4}$$

in the absence of the interaction. It is clear that (4) is satisfied if H_s does not contain the conjugate observable of A_s, because A_s, which commutes with H_s, is a constant of motion.

Summarizing the requirement discussed above, a measurement is the QND type when the observable A_s to be measured, the interaction Hamiltonian H_I, and the readout observable A_p, satisfy the following conditions:

(a) H_I is a function of A_s,

(b) $[H_I, A_s]=0$,

(c) $[H_I, A_p]\neq 0$, and

(d) H_s is not a function of the conjugate observable of A_s.

III. QND MEASUREMENT OF THE PHOTON NUMBER

In the proposed QND measurement scheme the photon number of the signal wave is measured via the phase of the probe wave using the optical Kerr effect. The measurement configuration is detailed in Fig. 1. The probe wave is sensitive to the refractive-index change, which is proportional to the signal intensity in the Kerr medium. The reflectivity of mirrors $M1$ and $M2$ is zero for the signal frequency ω_s and unity for the probe frequency ω_p so that an interferometer is formed only for the probe wave. The sine component of the phase shift for the probe wave passing through the Kerr medium is measured in terms of the photocurrent of the balanced-mixer detector. The sine component is regarded as the phase shift itself when the phase shift is small.

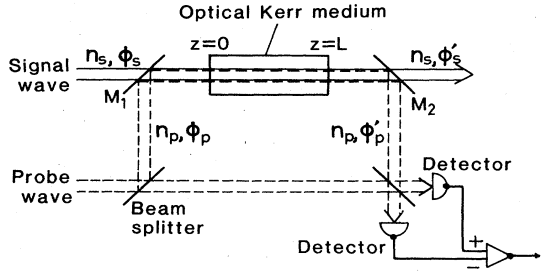

FIG. 1. Configuration for the QND measurement of the signal photon number. Transmissions of mirrors $M1$ and $M2$ are unity for signal frequency. Signal wave passes through the optical Kerr medium without changing its photon number. Phase of the probe wave is modulated by the signal photon number.

The electric field energy of the probe wave, H_p, is expressed by the dielectric constant ϵ and the electric field E_p as

$$H_p=\int\int\int \tfrac{1}{2}\epsilon E_p^2 dV \,. \tag{5}$$

The perturbation energy H_I resulting from changes in ϵ due to the presence of the signal electric field E_s becomes

$$H_I=\int\int\int (\tfrac{1}{2}\chi^{(3)}E_p^2E_s^2)dV \,. \tag{6}$$

Here, $\chi^{(3)}$ is the phenomenological third-order nonlinear susceptibility for the optical Kerr effect. The signal wave is assumed to be stationary (nondepletion approximation), and $\chi^{(3)}$ is defined as

$$(\Delta\epsilon)_{\text{probe wave}}=\chi^{(3)}E_s^2 \,. \tag{7}$$

Substituting the second-quantized formulas of E_p and E_s into (5) and (6), we obtain the operator formulas of the unperturbed Hamiltonian, H_p, for the probe wave (and similarly, H_s) and the interaction Hamiltonian H_I. The Hamiltonians H_p, H_s, and H_I, the QND observable A_s, as well as the readout observable A_p, for the scheme are

$$H_p=\hbar\omega_p(a_p^\dagger a_p+\tfrac{1}{2}) \,, \tag{8}$$

$$H_s=\hbar\omega_s(a_s^\dagger a_s+\tfrac{1}{2}) \,, \tag{9}$$

$$H_I=\frac{\hbar^2}{2V\epsilon^2}\omega_p\omega_s\chi^{(3)}a_p^\dagger a_p a_s^\dagger a_s \,, \tag{10}$$

$$A_s=n_s\equiv a_s^\dagger a_s \,, \tag{11}$$

and

$$A_p=S_p\equiv\frac{1}{2i}\left[\frac{1}{\sqrt{n_p+1}}a_p-a_p^\dagger\frac{1}{\sqrt{n_p+1}}\right] . \tag{12}$$

Here, $a^\dagger$ and a are the creation and annihilation operators, V is the volume for mode normalization, n is the number operator, and S is the sine operator.[8] Subscript s and p denote signal and probe waves, respectively.

Conditions (a)—(d) described in the preceding section are easily checked for (8)—(12), which ensure that the present scheme provides a QND measurement for n_s. For instance, condition (c) is checked using the commutation relationship between the number operator and the sine operator, that is,

$$[a^\dagger a, S]=iC\neq 0 \,, \tag{13}$$

where C is the cosine operator.[8]

The change in the operator, S_p, in terms of n_s and the material parameters is derived by integrating Heisenberg's equation of motion. The usual time-evolution equation should be rewritten into the spatial-evolution form for the present traveling-wave problem. The localized operators[9] are used for this purpose. A traveling wave is expressed as a sequence of wave packets, which move at the velocity $v=c\sqrt{\epsilon_0/\epsilon}$. The operators a and $a^\dagger$ are regarded as constant within the length of one packet. The electric field operator for the probe wave is written as

$$E_p(t,z)=\sqrt{\hbar\omega_p/2V\epsilon}\{a_p(t,z)\exp[-i(\omega_p t-k_p z)] + a_p^\dagger(t,z)\exp[i(\omega_p t-k_p z)]\} , \tag{14}$$

where z is the coordinate of the direction of propagation, and k_p is the wave number of the probe wave.

The unperturbed and interaction Hamiltonians are the same as (9) and (10) except that a_p and a_s are the localized operators which are slowly varying functions of t and z. The localized operators depend only on z in many problems of nonlinear optics in which spatial evolution of the traveling wave is temporally stationary. In this case, the time derivative d/dt is replaced by the derivative $-(c\sqrt{\epsilon_0/\epsilon})d/dz$ for a traveling wave that propagates toward the $+z$ direction. Heisenberg's equation of motion for E_p then leads to the spatial-evolution equation for $a_p(z)$,

$$-i\hbar c\left[\frac{\epsilon_0}{\epsilon}\right]^{1/2}\frac{d}{dz}a_p(z)=\frac{1}{2V}\left[\frac{\hbar}{\epsilon}\right]^2\omega_p\omega_s\chi^{(3)}n_s a_p(z) . \tag{15}$$

The time during which the wave packet passes through the Kerr medium from $z=0$ to L corresponds to the time H_I is switched on.

Integrating (15) form $z=0$ to L, and using the fact that n_s is a constant of motion, we obtain

$$a_p(L)=\exp(i\sqrt{F}n_s)a_p(0) , \tag{16}$$

where

$$\sqrt{F}\equiv\left[\frac{\epsilon}{\epsilon_0}\right]^{1/2}\frac{\hbar}{2cV\epsilon^2}\omega_p\omega_s\chi^{(3)}L . \tag{17}$$

The operator $\sqrt{F}n_s$ in (16) corresponds to the phase shift in a_p. Using (12) and (16), the sine operator $S_p(L)$ for the probe wave at $z=L$ is written as

$$S_p(L)=\frac{1}{2i}\left[\exp(i\sqrt{F}n_s)\frac{1}{\sqrt{n_p+1}}a_p(0) -\exp(-i\sqrt{F}n_s)a_p^\dagger(0)\frac{1}{\sqrt{n_p+1}}\right] . \tag{18}$$

When the intensity of the probe wave is large enough, that is, $\langle n_p\rangle \gg 1$, we can adopt the "phase operator" ϕ_p of the probe wave[10] which allows us to write

$$S_p=\sin(\phi_p) \tag{19}$$

and

$$C_p=\cos(\phi_p) . \tag{20}$$

The fact that ϕ_p commutes with n_s leads to the equation

$$\sin(\phi_p+\sqrt{F}n_s)=\sin(\phi_p)\cos(\sqrt{F}n_s)+\cos(\phi_p)\sin(\sqrt{F}n_s) . \tag{21}$$

Using (12) and (18)—(21), we can derive

$$\sin[\phi_p(L)]=\sin[\phi_p(0)+\sqrt{F}n_s] ,$$

or

$$\phi_p(L)=\phi_p(0)+\sqrt{F}n_s . \tag{22}$$

The signal photon number to be observed is expressed by the readout observable $S_p(L)$ as

$$n_s^{\text{obs}}\equiv S_p(L)/\sqrt{F}=n_s+\phi_p(0)/\sqrt{F} . \tag{23}$$

In the interferometer—balanced-mixer combination shown in Fig. 1, the dc term $\langle\phi_p(0)\rangle$ is canceled out. Taking the expectation value and variance of (23), we obtain

$$\langle n_s^{\text{obs}}\rangle=\langle n_s\rangle \tag{24}$$

and

$$\langle(\Delta n_s^{\text{obs}})^2\rangle=\langle(\Delta n_s)^2\rangle+\langle[\Delta\phi_p(0)]^2\rangle/F . \tag{25}$$

Equation (24) shows that our measurement is ideal in the sense that the true expectation value $\langle n_s\rangle$ is measured. Equation (25) indicates that the measurement accuracy is $\langle[\Delta\phi_p(0)]^2\rangle/F$.

The observable n_s^{obs} is defined in (23) rather intuitively. More detailed derivations for Eqs. (23)—(25) are described in the Appendix.

IV. SELF-PHASE-MODULATION EFFECT

Equations (8)—(10) are idealized in the sense that they do not include the self-modulation of the phase caused by the signal and probe waves. In order to treat the Kerr medium more realistically, we must consider the full Hamiltonian. We shall then show that it is possible to arrive at a QND measurement arrangement which is describable in terms of the ideal Hamiltonians (8)—(10).

The perturbation energy due to the third-order nonlinear effect is

$$H'=\int\int\int\left[\int E\,dP_{\text{NL}}\right]dV = \tfrac{3}{4}\int\int\int\sum\chi^{(3)}_{ijkl}E_iE_jE_kE_l dV . \tag{26}$$

Here, $\chi^{(3)}$ is defined not only for the optical Kerr effect but also for every process in which four photons are emitted or absorbed. In contrast, it should be noted that $\chi^{(3)}$ in (6) is phenomenologically defined for the optical Kerr effect, especially for the phase modulation of the probe wave by the signal wave.

In the presence of two optical frequencies, ω_s and ω_p, and using (14) and a similar expression for the signal wave, the perturbation Hamiltonian H' is expressed as

$$H'=H_s'+H_p'+H_I , \tag{27}$$

where

$$H_s'=D\omega_s^2[\chi^{(3)}(\omega_s;\omega_s,-\omega_s,\omega_s)a_s^\dagger a_s a_s^\dagger a_s + 5\,\mathscr{T}(a_s^\dagger,a_s)] , \tag{28}$$

$$H_p'=D\omega_p^2[\chi^{(3)}(\omega_p;\omega_p,-\omega_p,\omega_p)a_p^\dagger a_p a_p^\dagger a_p + 5\,\mathscr{T}(a_p^\dagger,a_p)] , \tag{29}$$

$$H_I=d\omega_s\omega_p[\chi^{(3)}(\omega_p;\omega_p,-\omega_s,\omega_s)a_p^\dagger a_p a_s^\dagger a_s + 23\,\mathscr{T}(a_s^\dagger,a_s,a_p^\dagger,a_p)] , \tag{30}$$

where $\mathscr{T}(a_s^\dagger, a_s)$ represents terms in which the order of $a_s^\dagger$ and a_s is interchanged, and

$$D = \frac{3\hbar^2}{16V\epsilon^2} . \tag{31}$$

H_s' and H_p' express the self-phase-modulation for the signal and probe, respectively. The products of the creation and annihilation operators can be reordered using the commutation relationship, which yields excess quadratic and constant terms like $aa^\dagger aa^\dagger \to a^\dagger aa^\dagger a + 2a^\dagger a + 1$. These excess terms are not essential since they only provide a constant phase shift.

When $\chi^{(3)}$ is caused by a nonresonant electric state transition, all of the $\chi^{(3)}$ coefficients can be set equal in each of Eqs. (28)–(30) as Kleinmann[11] did for the second-order nonlinear optical constants. H' is then expressed as

$$H' = \tfrac{1}{4}\tilde{\chi}_s a_s^\dagger a_s a_s^\dagger a_s + \tfrac{1}{4}\tilde{\chi}_p a_p^\dagger a_p a_p^\dagger a_p + \tilde{\chi}_{\rm int} a_s^\dagger a_s a_p^\dagger a_p , \tag{32}$$

where $\tilde{\chi}_s$, $\tilde{\chi}_p$, and $\tilde{\chi}_{\rm int}$ are defined as

$$\tilde{\chi}_s = 24D\omega_s^2 \chi^{(3)}(\omega_s;\omega_s,-\omega_s,\omega_s) , \tag{33}$$

$$\tilde{\chi}_p = 24D\omega_p^2 \chi^{(3)}(\omega_p;\omega_p,-\omega_p,\omega_p) , \tag{34}$$

$$\tilde{\chi}_{\rm int} = 24D\omega_p\omega_s \chi^{(3)}(\omega_p;\omega_p,-\omega_s,\omega_s) . \tag{35}$$

The self-phase-modulation of the signal wave is allowable for a QND measurement of n_s, because n_s is the constant of motion in spite of the self-phase-modulation term in (32). The self-phase-modulation of the probe wave, however, should be removed since the motion of ϕ_p is affected by the second term in (32).

One scheme for the avoidance of the self-modulation effect is to use a resonant $\chi^{(3)}$ medium. If the nonlinear medium has an actual level of $\hbar(\omega_s+\omega_p)$, as is shown in Fig. 2, only the process in Fig. 2(a) is dominant while the processes in Figs. 2(b) and 2(c) are off resonance. This allows us to pick up only the process resonant with the level in (28)–(30), favoring it over the remaining interaction terms. It should be noted that perfect resonance should be avoided since the imaginary part of $\chi^{(3)}$ dominates at resonance, causing optical loss.

Another scheme is to cancel the effect by means of a negative $\chi^{(3)}$ medium. The self-modulation term can be canceled out if the probe light passes through another negative $\chi^{(3)}$ medium. It has been pointed out that a negative $\chi^{(3)}$ is physically possible in a medium having an electronic nonlinearity.[12]

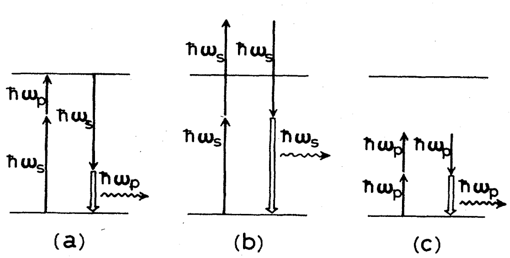

FIG. 2. Resonant process where $\chi^{(3)}$ for mutual phase modulation is enhanced compared with the self-modulation effect; (a) $\chi^{(3)}(\omega_p;-\omega_s,\omega_p,\omega_s)$ process, (b) $\chi^{(3)}(\omega_s;\omega_s,-\omega_s,\omega_s)$ process, and (c) $\chi^{(3)}(\omega_p;\omega_p,-\omega_p,\omega_p)$ process.

V. MEASUREMENT ACCURACY AND THE IMPOSED PHASE NOISE

In general quantum measurements, the product of the measurement accuracy and the additional uncertainty imposed on the conjugate observable is expected to satisfy the inequality of Heisenberg's uncertainty principle. However, whether the equality sign is achievable or not in a QND measurement has not yet been investigated. We will show that the proposed QND measurement scheme provides the minimum uncertainty product of measurement accuracy for photon number and imposed phase noise.

Consider the case without the self-phase-modulation effect for both the signal and probe waves. The output phase of the signal is, in analogy with (22),

$$\phi_s' = \phi_s + \sqrt{F} n_p , \tag{36}$$

where a prime denotes an observable after passage through the Kerr medium. Taking the variance of (36), the imposed phase uncertainty for the signal wave is

$$\langle(\Delta\phi_s)^2\rangle_{\rm imp} \equiv \langle(\Delta\phi_s')^2\rangle - \langle(\Delta\phi_s)^2\rangle = F\langle(\Delta n_p)^2\rangle . \tag{37}$$

The measurement uncertainty for the signal photon number is derived from (25) as

$$\langle(\Delta n_s)^2\rangle_{\rm meas} \equiv \langle(\Delta n_s^{\rm obs})^2\rangle - \langle(\Delta n_s)^2\rangle = \langle(\Delta\phi_p)^2\rangle / F . \tag{38}$$

Multiplying (37) and (38), we obtain

$$\langle(\Delta n_s)^2\rangle_{\rm meas}\langle(\Delta\phi_s)^2\rangle_{\rm imp} = \langle(\Delta n_p)^2\rangle\langle(\Delta\phi_p)^2\rangle = \tfrac{1}{4} . \tag{39}$$

The last equality stands for the probe wave in a minimum uncertainty state.

The uncertainty product for $n_s^{\rm obs}$ and ϕ_s' gives the uncertainty relationship of a simultaneous measurement for conjugate variables. If we suppose both the signal and probe waves are minimum-uncertainty states, the uncertainty product is given by

$$\begin{aligned}\langle(\Delta n_s^{\rm obs})^2\rangle\langle(\Delta\phi_s')^2\rangle &= [\langle(\Delta n_s)^2\rangle + \langle(\Delta\phi_p)^2\rangle/F][\langle(\Delta\phi_s)^2\rangle + F\langle(\Delta n_p)^2\rangle] \\ &= \tfrac{1}{4}\left[\frac{1}{4F\langle(\Delta n_s)^2\rangle\langle(\Delta n_p)^2\rangle} + 4F\langle(\Delta n_p)^2\rangle\langle(\Delta n_s)^2\rangle\right] \\ &\quad + \tfrac{1}{2} \geq 1 .\end{aligned} \tag{40}$$

The equality stands for the case when

$$F\langle(\Delta n_s)^2\rangle = \frac{1}{4\langle(\Delta n_p)^2\rangle} = \langle(\Delta\phi_p)^2\rangle \; . \tag{41}$$

In this case the phase variance of the probe wave (right-hand side) is matched with the photon-number uncertainty for the signal wave (left-hand side). The inequality relationship (40) coincides with the general uncertainty relationship of the simultaneous measurement for conjugate variables by Arthurs and Kelly.[13]

VI. CONCLUSION

A QND measurement scheme of the photon number using the optical Kerr effect has been proposed. The optical phase of the probe wave is sensitive to the refractive index change due to optical intensity (photon number) of the signal wave without affecting it. The process is formulated quantum mechanically. The measurement accuracy and the imposed phase noise on the signal wave satisfies the Heisenberg uncertainty principle. This demonstrates that the minimum product of the measurement accuracy and the imposed uncertainty on the conjugate observable is achievable in the proposed QND measurement.

ACKNOWLEDGMENTS

The authors wish to thank Dr. F. Kanaya of Musashino Electrical Communication Laboratories (ECL), NTT for his encouragement. They are also indebted to Professor O. Nilsson of the Royal Institute of Technology for his useful discussion about a practical scheme of the photon-number QND. Critical readings by Dr. A. Sugimara and M. Kumagai of ECL, NTT are also greatly appreciated.

APPENDIX

In this appendix the output of the proposed interferometer—balanced-mixer detector is derived. The observed photon number is defined as the output current divided by a normalized factor which changes the current into the photon number. Equations (23)—(25) are derived by the obtained formula for the observed photon-number operator.

Figure 3 shows the present scheme in which the annihilation operator for each part of the interferometer is specified. The probe laser output a is divided by beam splitter 1 into the probe wave a_p and the reference wave a_r. In this process, the zero-point fluctuation (vacuum-state quantum noise) b is introduced as[14]

$$a_p = \sqrt{\eta}a + \sqrt{1-\eta}b \; , \tag{A1}$$

$$a_r = \sqrt{\eta}b - \sqrt{1-\eta}a \; , \tag{A2}$$

where η is the reflection coefficient of beam splitter 1. The value η is assumed to be small aiming at the ideal homodyne detection. The phase of the probe wave is shifted by the Kerr medium as Eq. (16):

$$a_p' = a_p \exp[i(\sqrt{F}n_s + \pi/2)] \; , \tag{A3}$$

where the phase shift $\pi/2$ is added by adjusting the interferometer configuration. With the value $\frac{1}{2}$ for the reflection coefficient of beam splitter 2, the waves d and f are written as

$$d = (a_p' + a_r)/\sqrt{2} \; , \tag{A4}$$

$$f = (a_r - a_p')/\sqrt{2} \; . \tag{A5}$$

The electrical current is the difference in the photocurrents of the two detectors, and *measures* the following operator:[14]

$$I = \frac{e}{\tau}(f^\dagger f - d^\dagger d) \; . \tag{A6}$$

Here, e is the electron charge and τ is the time constant for the traveling-wave quantization, which is related to the cross-section area of the beam, A, and the normalization volume V, as

$$V = A\tau c\sqrt{\epsilon_0/\epsilon} \; . \tag{A7}$$

The photon-number operator, n_s^{obs}, which is actually observed is defined by I as

$$n_s^{\text{obs}} \equiv \frac{\tau}{e}\frac{1}{2\sqrt{\eta(1-\eta)}\sqrt{F}\langle a^\dagger a\rangle}I = \frac{f^\dagger f - d^\dagger d}{2\sqrt{\eta(1-\eta)}\sqrt{F}\langle a^\dagger a\rangle} \; . \tag{A8}$$

Substituting (A1)—(A5) into (A8), we obtain

$$n_s^{\text{obs}} = \frac{(a^\dagger a - b^\dagger b)n_s}{\langle a^\dagger a\rangle} + \frac{(e^{i\theta}-2\eta\cos\theta)a^\dagger b + (e^{-i\theta}-2\eta\cos\theta)b^\dagger a}{2\sqrt{\eta(1-\eta)}\sqrt{F}\langle a^\dagger a\rangle} \; , \tag{A9}$$

where the notation $\theta \equiv \sqrt{F}n_s + \pi/2$ and the approximation $\sin(\sqrt{F}n_s) \cong \sqrt{F}n_s$ are used.

Since the noise b is the zero-point fluctuation, the expectation value of (A9) is obtained as $\langle n_s\rangle$, which is Eq. (24).

Taking the variance of (A9) we obtain

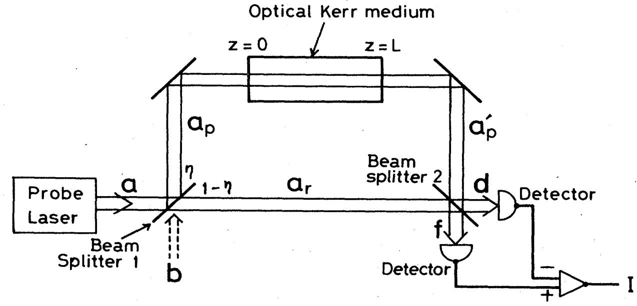

FIG. 3. Detailed description of the annihilation operators in the interferometer—balanced-mixer detector. Probe wave and reference wave are denoted as a_p and a_r, respectively. Zero-point fluctuation, b, is mixed at beam splitter 1.

$$\langle(\Delta n_s^{\rm obs})^2\rangle = \langle(\Delta n_s)^2\rangle + \frac{\langle[\Delta(a^\dagger a)]^2\rangle\langle n_s\rangle^2 + \langle[\Delta(a^\dagger a)]^2\rangle\langle(\Delta n_s)^2\rangle}{\langle a^\dagger a\rangle^2} + \frac{\langle(e^{i\theta}-2\eta\cos\theta)(e^{-i\theta}-2\eta\cos\theta)\rangle\langle a^\dagger a\rangle}{4\eta(1-\eta)F\langle a^\dagger a\rangle^2} \;. \tag{A10}$$

In the case that the probe laser radiates a coherent state, (A10) becomes

$$\langle(\Delta n_s^{\rm obs})^2\rangle - \langle(\Delta n_s)^2\rangle = \left[[\langle(\Delta n_s)^2\rangle + \langle n_s\rangle^2] + \frac{1-4\eta(1-\eta)\langle\cos^2\theta\rangle}{4\eta(1-\eta)} \right] \Big/ \langle a^\dagger a\rangle \;. \tag{A11}$$

The first term in the right-hand side is much smaller than the second term due to the assumption $\sqrt{F}\,\langle n_s\rangle \ll 1$. Using the relation $\eta\langle a^\dagger a\rangle = \langle a_p^\dagger a_p\rangle$ and the approximation $\eta \ll 1$, (A11) is rewritten as

$$\langle(\Delta n_s^{\rm obs})^2\rangle - \langle(\Delta n_s)^2\rangle = \frac{1}{4F\langle a_p^\dagger a_p\rangle} \;. \tag{A12}$$

Since the probe wave is assumed to be a coherent state, (A12) is rewritten in terms of the phase variance as

$$\langle(\Delta n_s^{\rm obs})^2\rangle - \langle(\Delta n_s)^2\rangle = \frac{\langle(\Delta\phi_p)^2\rangle}{F} \;, \tag{A13}$$

which is Eq. (25)

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