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Harmonic oscillator with strongly pulsating mass under the action of a driving force

M. Sebawe Abdalla
Department of Mathematics, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia

R. K. Colegrave*
Department of Mathematics, Chelsea College, University of London, 552 King's Road, London SW10, England
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The exact solution for the problem of a harmonic oscillator of frequency ω_0 and mass $M_0\cos^2(\nu t)$ is extended to include the effect of a driving force $M_0 f_0\cos(\lambda t+\phi)$. With $\phi\neq 0$, catastrophic resonances occur when $\lambda=(\omega_0^2+\nu^2)^{1/2}\pm\nu$. Pseudoperiodic states exist, provided that $\lambda=\omega_0$ and λ,ν are commensurate. The corresponding quasienergies are finite in the cases $\phi=\pm\pi/2$.

I. INTRODUCTION

The effect on an oscillator of an input or removal of energy may be represented by suitably varying the mass parameter. The best known example is the case of damping, when the mass is given a growth factor $\exp(2\gamma t)$, $\gamma>0$.[1–7] Remaud and Hernandez[8] remark that whenever energy is supplied to an oscillating system in a periodic cycle, the resulting dynamics may be described by letting the mass be a periodic function of the time. These authors compute the fluctuations in the energy and position of a Gaussian wave packet when the oscillator mass is taken as

$$M(t)=\begin{cases}1, & t<0\\ 1+\alpha\sin(\lambda t), & t\geqq 0,\ |\alpha|<1\ .\end{cases} \tag{1}$$

A well-known resonance is found in the case $\lambda=2\omega_0$, where ω_0 is the natural frequency of the oscillator. Dodonov and Man'ko,[6] Landovitz *et al.*,[9,10] and Leach[11] have extended the discussion to variable ω_0 as well as variable mass.

Our aim in the present series of papers[12–17] is to examine mass laws which are of interest in quantum optics, or other branches of quantum-field theory, and which are mathematically tractable. As discussed in Refs. 12 and 15 the field in a Fabry-Pérot cavity in contact with a reservoir of two-level atoms can be represented by the Hamiltonian

$$H(t)=\tfrac{1}{2}p^2/M(t)+\tfrac{1}{2}M(t)\omega_0^2q^2\ . \tag{2}$$

Under suitable conditions the photon population in the cavity and hence the field intensity can vary quite appreciably with the time, often in a periodic or nearly periodic manner.[18,19] To simulate such situations the mass may be taken of a form similar to Eq. (1)

$$M(t)=M_1+M_0\cos^2(\nu t)\ . \tag{3}$$

The case $M_1=0$ is considered in Refs. 14 and 15 and in the present paper. The case $M_0\ll M_1$ is considered in Ref. 20, and also in Ref. 15 when coupled with the adiabatic condition $\nu\ll\omega_0$. In order to represent more fully the conditions in a cavity or waveguide in contact with a resonant atomic reservoir, it is desirable to add a periodic driving force. The general problem of a driving force, coupled with variable mass, has been considered already in Refs. 4–6, 8, and 21. As far as the authors are aware, the present paper is the first in which an exact solution is obtained for periodic mass and periodic driving force. In Secs. II, III, and IV we work in the Heisenberg picture; in Sec. V the Green function is calculated and is applied in Sec. VI to calculate the wave function for a quasicoherent state at any time $t>0$. Finally, in Sec. VI, an exact solution of the full Schrödinger equation is found and the concept of quasienergy is discussed.

II. THE SOLUTION IN THE HEISENBERG PICTURE

Under the action of a force $F(t)$ the Hamiltonian becomes[4–6]

$$H(t)=\tfrac{1}{2}p^2/M(t)+[M(t)/M(0)][\tfrac{1}{2}M(0)\omega_0^2q^2-F(t)q]\ . \tag{4}$$

We introduce the scaling transformation, as in Refs. 12–14

$$Q=[M(t)/M(0)]^{1/2}q,\quad P=[M(t)/M(0)]^{-1/2}p\ , \tag{5}$$

then with

$$M(t)=M_0\cos^2(\nu t)\ , \tag{6}$$

the Hamiltonian [Eq. (4)] assumes the form

$$H(Q,P,t)=\tfrac{1}{2}P^2/M_0+\tfrac{1}{2}M_0\omega_0^2Q^2-\tfrac{1}{2}\nu\tan(\nu t)(QP+PQ)-F(t)\cos(\nu t)Q\ . \tag{7}$$

Let us consider a periodic driving force

$$F(t)=M_0f_0\cos(\lambda t+\phi)\ . \tag{8}$$

The Heisenberg equations of motion are

$$\dot{Q}=P/M_0-\nu\tan(\nu t)Q\ , \tag{9a}$$

$$\dot{P}=\nu\tan(\nu t)P-M_0\omega_0^2Q+F(t)\cos(\nu t)\ , \tag{9b}$$

or, in decoupled form

$$\ddot{Q}+\Omega^2Q=[F(t)/M_0]\cos(\nu t)\ , \tag{10a}$$

$$\ddot{P}+[\Omega^2-2\nu^2\sec^2(\nu t)]P=\dot{F}(t)\cos(\nu t)\ , \tag{10b}$$

$$\Omega^2=\omega_0^2+\nu^2\ . \tag{10c}$$

From Eqs. (5) and (6) the physical coordinate and momentum are given by

$$q=Q\sec(\nu t),\quad p=P\cos(\nu t)\ . \tag{11}$$

The equation of motion (10a) may be written in the form

$$\ddot{q}-2\nu\tan(\nu t)\dot{q}+\omega_0^2q=F(t)/M_0\ . \tag{12}$$

In classical terms Eq. (12) describes the motion of a particle of mass $M(t)=M_0\cos^2\nu t$ attracted to $q=0$ by a force $M(t)\omega_0^2q$ together with the driving force $[M(t)/M(0)]F(t)$. The solution corresponding to the force (8) is [cf. Eqs. (4.3a) and (4.3b) of Ref. 14]

$$Q(t)=Q(0)\cos(\Omega t)+[P(0)/M_0\Omega]\sin(\Omega t)+R(t)\ , \tag{13a}$$

$$\begin{aligned}P(t)=&P(0)[(\nu/\Omega)\tan(\nu t)\sin(\Omega t)+\cos(\Omega t)]\\&+M_0\Omega Q(0)[(\nu/\Omega)\tan(\nu t)\cos(\Omega t)-\sin(\Omega t)]\\&+S(t)\ ,\end{aligned} \tag{13b}$$

where the responses R,S in Q,P are given by

$$R(t)=\tfrac{1}{2}f_0\left\{\frac{\cos[(\lambda+\nu)t+\phi]-\cos(\Omega t)\cos\phi}{\Omega^2-(\lambda+\nu)^2}+\frac{\cos[(\lambda-\nu)t+\phi]-\cos(\Omega t)\cos\phi}{\Omega^2-(\lambda-\nu)^2}\right\} \tag{13c}$$

$$S(t)=M_0[\dot{R}+R\nu\tan(\nu t)]\ . \tag{13d}$$

For a constant driving force ($\lambda\to0,\phi\to0$), Eqs. (13c) and (13d) reduce to

$$R(t)=(f_0/\omega_0^2)[\cos(\nu t)-\cos(\Omega t)]\ , \tag{14a}$$

$$S(t)=(M_0f_0\Omega/\omega_0^2)[\sin(\Omega t)-(\nu/\Omega)\tan(\nu t)\cos(\Omega t)]\ . \tag{14b}$$

We see from Eqs. (11), (13c), and (13d) that the responses in Q and the physical momentum p are finite at all times for all driving frequencies except possibly for resonances at

$$\lambda=\lambda_1,\quad \lambda_2\equiv(\omega_0^2+\nu^2)^{1/2}\pm\nu\ . \tag{15}$$

If $\phi\neq0$ and λ takes one of the values given by Eq. (15), then a catastrophic resonance in Q occurs immediately (i.e., at $t=0$). If, however, $\phi=0$ and λ takes one of the values λ_1 or λ_2, then from Eqs. (13a) and (13c) the response $R(t)$ contains a secular term proportional to $t\sin[(\lambda\pm\nu)t]$. From Eq. (11) we see that the physical coordinate q becomes infinite when $M\to0$ at $t=\pi/2\nu$, but this is an unavoidable feature of the model.

III. THE ENERGY OPERATOR

We extend the expression for the energy given in Refs. 13 and 14 to include the work done by the driving force; thus from Eq. (4) the energy operator is

$$E=[M(t)/M(0)][T+V-F(t)q]\ , \tag{16a}$$

where

$$T=\tfrac{1}{2}p^2/M(t)=\tfrac{1}{2}P^2/M(0)\ ,\qquad V=\tfrac{1}{2}M(t)\omega_0^2q^2=\tfrac{1}{2}M(0)\omega_0^2Q^2\ . \tag{16b}$$

The mass law (6) coupled with transformation (11) gives

$$E=\cos^2(\nu t)(T+V)-\cos(\nu t)F(t)Q(t)\ . \tag{17}$$

Using the solution given by Eqs. (13), we find

$$\begin{aligned}E=E_0&+M_0Q(0)\Big\{[\nu\tan(\nu t)\cos(\Omega t)-\Omega\sin(\Omega t)]\dot{R}\\&+\{[\omega_0^2+\nu^2\tan^2(\nu t)]\cos(\Omega t)-\Omega\nu\tan(\nu t)\sin(\Omega t)\}R-\frac{F}{M_0}\cos(\Omega t)\cos(\nu t)\Big\}\\&+\frac{P(0)}{\Omega}\Big\{[\nu\tan(\nu t)\sin(\Omega t)+\Omega\cos(\Omega t)]\dot{R}\\&+\{[\omega_0^2+\nu^2\tan^2(\nu t)]\sin(\Omega t)+\Omega\nu\tan(\nu t)\cos(\Omega t)\}R-\frac{F}{M_0}\sin(\Omega t)\cos(\nu t)\Big\}\\&+\tfrac{1}{2}M\omega_0^2R^2+\tfrac{1}{2}S^2/M_0-FR\cos(\nu t)\ ,\end{aligned} \tag{18}$$

where E_0 denotes the expression for the energy in the absence of a driving force, given by Eq. (4.5) in Ref. 14.

IV. DIRAC OPERATORS, NUMBER STATES, AND QUASICOHERENT STATES

We proceed as in Ref. 14 and introduce Dirac operators $A,A^\dagger$ which satisfy $[A,A^\dagger]=1$ at all times. A suitable choice is

$$\begin{aligned}A(t)=&(2M_0\hbar\Omega)^{-1/2}\{M_0[\Omega-i\nu\tan(\nu t)]Q+iP\}\\&+L(t)\ ,\end{aligned} \tag{19a}$$

$$L(t)=-(M_0/2\hbar\Omega)^{1/2}\{\Omega R(t)+i[\dot{R}(t)-\dot{R}(0)]\}\ , \tag{19b}$$

where we note that, apart from cases of catastrophic resonance, $L(0)=0$. The variation of A with time is according to

$$A(t)=e^{-i\Omega t}\left[A(0)+\int_0^t e^{i\Omega\tau}\epsilon(\tau)d\tau\right]+L(t)\ , \qquad (20a)$$

$$\epsilon(\tau)=\tfrac{1}{2}\nu\tan(\nu\tau)(L-L^*)+i(\nu^2/2\Omega)\tan^2(\nu\tau)(L+L^*)$$
$$+i(2M_0\Omega\hbar)^{-1/2}F(\tau)\cos(\nu\tau)\ . \qquad (20b)$$

The scaled coordinate Q and momentum P may be expressed as

$$Q(t)=(\hbar/2M_0\Omega)^{1/2}[A(t)+A^\dagger(t)]+R(t)\ , \qquad (21a)$$

$$P(t)=(M_0\hbar\Omega/2)^{1/2}\{[(\nu/\Omega)\tan(\nu t)-i]A$$
$$+[(\nu/\Omega)\tan(\nu t)+i]A^\dagger\}$$
$$+M_0[\nu\tan(\nu t)R(t)+\dot{R}(t)-\dot{R}(0)]\ . \qquad (21b)$$

Using Eqs. (21a) and (21b) we may write the Hamiltonian (7) in the form

$$H=\hbar\Omega(A^\dagger A+\tfrac{1}{2})[1-(\nu^2/2\Omega^2)\sec^2(\nu t)]-(\hbar\nu^2/4\Omega)(A^2+A^{\dagger 2})\sec^2(\nu t)$$
$$+(A\{(M_0\hbar\Omega/2)^{1/2}[(\nu/\Omega)\tan(\nu t)-i]+\omega_0^2(M_0\hbar/2\Omega)^{1/2}R+(\hbar/2M_0\Omega)^{1/2}F\cos(\nu t)\}+\text{H.c.})$$
$$+\tfrac{1}{2}M_0\omega_0^2R^2-(\hbar\Omega/4)g^2(t)-FR\cos(\nu t)+\tfrac{1}{2}i\nu\tan(\nu t)(2M_0\hbar\Omega)^{1/2}Rg(t)\ , \qquad (22a)$$

$$g(t)=i(2M_0/\hbar\Omega)^{1/2}\{\nu\tan(\nu t)R(t)+[\dot{R}(t)-\dot{R}(0)]\}\ . \qquad (22b)$$

Time-dependent number states (eigenvectors of $A^\dagger A$) and quasicoherent states (eigenvectors of A) exist for $A,A^\dagger$ given by Eqs. (19a) and (19b). Since we have ensured that $[A,A^\dagger]=1$ at all times, it follows that

$$A^\dagger A\,|n(t)\rangle=n\,|n(t)\rangle,\quad n=0,1,2,\ldots \qquad (23a)$$

$$A\,|\alpha(t)\rangle=\alpha(t)\,|\alpha(t)\rangle\ , \qquad (23b)$$

where the connection is

$$|\alpha\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}\,|n\rangle\ . \qquad (23c)$$

From Eqs. (21a), (21b), and (23b)

$$\langle\alpha|Q(t)|\alpha\rangle=(\hbar/2M_0\Omega)^{1/2}[\alpha(t)+\alpha^*(t)]+R(t)\ , \qquad (24a)$$

$$\langle\alpha|P(t)|\alpha\rangle=(\hbar M_0\Omega/2)^{1/2}\{[\alpha(t)+\alpha^*(t)](\nu/\Omega)\tan(\nu t)-i[\alpha(t)-\alpha^*(t)]\}+S(t)-M_0\dot{R}(t)\ . \qquad (24b)$$

Clearly, the uncertainties in Q and P in state $|\alpha\rangle$ are not affected by the presence of a driving force and the expressions (9.2c), (9.2d), and (9.3) in Ref. 14 still hold. In a number state $|n\rangle$ we find that the uncertainties are

$$(\Delta Q)_n^2=(n+\tfrac{1}{2})(\hbar/M_0\Omega)\ , \qquad (25a)$$

$$(\Delta P)_n^2=(n+\tfrac{1}{2})\hbar M_0\Omega[1+(\nu/\Omega)^2\tan^2(\nu t)]\ , \qquad (25b)$$

so that the product of physical uncertainties is

$$\Delta q_n\,\Delta p_n=(n+\tfrac{1}{2})\hbar[1+(\nu/\Omega)^2\tan^2(\nu t)]^{1/2}$$
$$\geq\Delta q_\alpha\,\Delta p_\alpha\ , \qquad (25c)$$

where we have used Eq. (9.3) of Ref. 14.

The expectation values of E and H in state $|\alpha\rangle$ are easily derived from Eqs. (16), (21), (22), and (23b). The expectation values in state $|n\rangle$ are given by

$$\langle n|E|n\rangle=\langle n|E_0|n\rangle+\tfrac{1}{2}M_0\omega_0^2R^2+\tfrac{1}{2}S^2/M_0$$
$$-FR\cos(\nu t)\ , \qquad (26a)$$

$$\langle n|H|n\rangle=\langle n|H_0|n\rangle$$
$$+\tfrac{1}{2}M_0\{[\dot{R}(t)-\dot{R}(0)]^2$$
$$+[\Omega^2-\nu^2\sec^2(\nu t)]R^2(t)\}$$
$$-FR\cos(\nu t)\ , \qquad (26b)$$

where, on using Eq. (3.16) of Ref. 14,

$$\langle n|E_0|n\rangle=(n+\tfrac{1}{2})\hbar(\omega_0^2/\Omega)[1+\tfrac{1}{2}(\nu/\omega_0)^2\sec^2(\nu t)]\ , \qquad (26c)$$

$$\langle n|H_0|n\rangle=(n+\tfrac{1}{2})\hbar\Omega[1-\tfrac{1}{2}(\nu/\Omega)^2\sec^2(\nu t)]\ . \qquad (26d)$$

V. THE GREEN FUNCTION $G(Q,Q_0,t)$

A formal integral of the evolution equation

$$HG=i\hbar\partial G/\partial t,\quad G(Q,Q_0,0)=\delta(Q-Q_0)\ , \qquad (27)$$

for the Green function is given by

$$G(Q,Q_0,t)=\exp\left[-\frac{i}{\hbar}\int_0^t H(\tau)d\tau\right]\delta(Q-Q_0)\ . \quad (28)$$

Hence $G(Q,Q_0,t)$ satisfies the equation

$$Q(-t)G(Q,Q_0,t)=Q_0G(Q,Q_0,t),\ \ t>0\ . \quad (29)$$

We put $P(0)=-i\hbar(\partial/\partial Q)$ in Eq. (13a) and solve the Schrödinger equation (29) to find, with $\bar{F}=F(-t)$ and $t>0$,

$$G(Q,Q_0,t)=G_0(Q_0,t)\exp\left[\frac{iM_0\Omega}{2\hbar\sin(\Omega t)}[Q^2\cos(\Omega t)+2\bar{F}Q-2Q_0Q]\right]\ . \quad (30)$$

In order to calculate the function $G_0(Q,t)$ we invert Eqs. (13a) and (13b) to give

$$Q(0)=[(\nu/\Omega)\tan(\nu t)\sin(\Omega t)+\cos(\Omega t)][Q(t)-R(t)]-[\sin(\Omega t)/M_0\Omega][P(t)-S(t)]\ . \quad (31)$$

Replacing t by $-t$, we let both sides of Eq. (31) operate on the Green function. Using $P(-t)=-(\hbar/i)(\partial/\partial Q_0)$ and Eq. (29) $[Q(-t)G]$, we find

$$\frac{1}{G}\frac{\partial G}{\partial Q_0}=-\frac{iM_0\Omega}{\hbar\sin(\Omega t)}\left[Q-(Q_0-\bar{R})\left[\frac{\nu}{\Omega}\tan(\nu t)\sin(\Omega t)+\cos(\Omega t)\right]+\frac{\bar{S}}{M_0\Omega}\sin(\Omega t)\right]\ , \quad (32)$$

where $\bar{S}=S(-t)$ and $\bar{R}=R(-t)$. By differentiating Eq. (30) with respect to Q_0, we obtain

$$\frac{\partial G}{\partial Q_0}=\left[\frac{1}{G_0}\frac{\partial G_0}{\partial Q_0}-\frac{iM_0\Omega}{\hbar\sin(\Omega t)}Q\right]G\ . \quad (33)$$

Comparing Eqs. (32) and (33)

$$\frac{1}{G_0}\frac{\partial G_0}{\partial Q_0}=\frac{iM_0\Omega}{\hbar\sin(\Omega t)}\left[(Q_0-\bar{R})\left[\frac{\nu}{\Omega}\tan(\nu t)\sin(\Omega t)+\cos(\Omega t)\right]-\frac{\bar{S}}{M_0\Omega}\sin(\Omega t)\right]\ , \quad (34)$$

and hence with $N_G(t)$ a normalization factor,

$$G_0(Q_0,t)=N_G(t)\exp\left\{\frac{iM_0\Omega}{2\hbar\sin(\Omega t)}\left[(Q_0^2-2\bar{R}Q_0)\left[\frac{\nu}{\Omega}\tan(\nu t)\sin(\Omega t)+\cos(\Omega t)\right]-\frac{2\bar{S}}{M_0\Omega}Q_0\sin(\Omega t)\right]\right\}\ . \quad (35)$$

Substituting Eq. (35) into Eq. (30) we obtain

$$G(Q,Q_0,t)=N_G(t)\exp\left\{\frac{iM_0\Omega}{2\hbar\sin(\Omega t)}\left[Q^2\cos(\Omega t)+Q_0(Q_0-2\bar{F})\left[\frac{\nu}{\Omega}\tan(\nu t)\sin(\Omega t)+\cos(\Omega t)\right]\right.\right.$$
$$\left.\left.-2QQ_0+2\bar{R}Q-\frac{2\bar{S}}{M_0\Omega}Q_0\sin(\Omega t)\right]\right\}\ . \quad (36)$$

To determine $N_G(t)$, we use the relation

$$\int_{-\infty}^{\infty}G^*(Q,Q_0,t)G(Q,Q_1,t)dQ=\delta(Q_0-Q_1)\ , \quad (37)$$

where, for convenience, we take normalization with respect to Q. Then we find

$$N_G(t)=\{M_0\Omega/[2\pi\hbar|\sin(\Omega t)|]\}^{1/2}\ . \quad (38)$$

When we set $\bar{R}=\bar{S}=\bar{F}=0$ in Eq. (36) we are left with the Green function (8.1) obtained in Ref. 14.

VI. THE WAVE FUNCTION FOR THE QUASICOHERENT STATE $|\alpha(t)\rangle$

At time $t=0$, the Dirac operator (19) reduces to

$$A(0)=(2M_0\hbar\Omega)^{-1/2}[M_0\Omega Q(0)+iP(0)]\ . \quad (39)$$

The Schrödinger equation corresponding to Eq. (23b) with $t=0$ is, with $\alpha=\alpha(0)$,

$$\left[\frac{\hbar}{2M_0\Omega}\right]^{1/2}\left[\frac{d}{dQ}+\frac{M_0\Omega}{\hbar}Q\right]\psi_\alpha(Q,0)=\alpha\psi_\alpha(Q,0)\ , \quad (40)$$

with solution

$$\psi_\alpha(Q,0)=N_\alpha\exp\left[-\frac{M_0\Omega}{2\hbar}Q^2+\left[\frac{2M_0\Omega}{\hbar}\right]^{1/2}\alpha Q\right]\ , \quad (41a)$$

$$N_\alpha=(M_0\Omega/\pi\hbar)^{1/4}\exp[-\tfrac{1}{2}(|\alpha|^2+\alpha^2)]\ . \quad (41b)$$

We have followed Louisell[22] in taking a convenient choice of phase for N_α, which accounts for the difference between Eq. (4.1b) and Eq. (9.5a) of Ref. 14. The quasicoherent state at any time $t>0$ may be calculated using the Green function (35)

$$\psi_\alpha(Q,t)=\int_{-\infty}^{\infty}dQ_0G(Q,Q_0,t)\psi_\alpha(Q_0,0)\ , \quad (42)$$

which yields the result

$$\psi_\alpha(Q,t)=\left[\frac{M_0\Omega}{\hbar}\right]^{1/4}\exp\left[\frac{iM_0}{2\hbar\sin\Omega t}\left[Q(Q-2\bar{R})[\Omega\cos(\Omega t)+\nu\tan(\nu t)\sin(\Omega t)]-(Q-\bar{R})^2\Omega e^{-i\Omega t}-\frac{2\bar{S}}{M_0}Q\sin(\Omega t)\right]\right]$$
$$\times\exp\left[\left[\frac{2M_0\Omega}{\hbar}\right]^{1/2}(Q-\bar{R})\tilde{\alpha}-\tfrac{1}{2}\tilde{\alpha}^2(t)-\tfrac{1}{2}|\tilde{\alpha}|^2\right], \quad (43a)$$

where

$$\tilde{\alpha}=\alpha e^{-i\Omega t}, \quad (43b)$$

and we have dropped a phase factor $\exp(i\pi/4)$.

VII. THE WAVE FUNCTION FOR DISPLACED NUMBER STATES $|n(t)\rangle$

The Schrödinger equation

$$H\psi=i\hbar\partial\psi/\partial t, \quad (44)$$

with H given by Eq. (7), may be separated by introducing

$$y=Q-\xi(t), \quad (45)$$

where $\xi(t)$ is to be determined, cf. Ref. 23. Let us write

$$\psi(Q,t)=\chi(y,t). \quad (46)$$

Then

$$\partial\psi/\partial Q=\partial\chi/\partial y,\quad \partial^2\psi/\partial Q^2=\partial^2\chi/\partial y^2, \quad (47a)$$

$$\partial\psi/\partial t=\partial\chi/\partial t-\dot{\xi}\partial\chi/\partial y, \quad (47b)$$

and Eq. (44) transforms to

$$\frac{\partial^2\chi}{\partial y^2}-\left[\frac{M_0\omega_0}{\hbar}\right]^2(y+\xi)^2\chi-\frac{iM_0}{\hbar}\left[2[\nu\tan(\nu t)(y+\xi)+\dot{\xi}]\frac{\partial\chi}{\partial y}+\nu\tan(\nu t)\chi\right]+\frac{2M_0}{\hbar^2}\cos(\nu t)F(t)(y+\xi)\chi=-\frac{2iM_0}{\hbar}\frac{\partial\chi}{\partial t}. \quad (48)$$

It should be noted that the term in $\dot{\xi}$ arising from Eq. (47b) has been transposed to the left-hand side of Eq. (48), so that the equation ceases to be of the form $H\chi=i\hbar\partial\chi/\partial t$. This point is important in any attempt to generalize the concept of energy eigenvalues.

We seek a separation of the form

$$\chi(y,t)=Y(y)T(t)\exp\left[\frac{iM_0}{\hbar}[(\tfrac{1}{2}y+\xi)\nu\tan(\nu t)+\dot{\xi}]y\right], \quad (49)$$

which leads to the following partially separated equation

$$(-\hbar^2/2M_0)(Y''/Y)+\tfrac{1}{2}M_0\Omega^2y^2+[M_0(\ddot{\xi}+\Omega^2\xi)-F(t)\cos(\nu t)]y$$
$$=i\hbar(\dot{T}/T)-\tfrac{1}{2}M_0\{\omega_0^2\xi^2-[\nu\tan(\nu t)\xi+\dot{\xi}]^2\}+\cos(\nu t)F(t)\xi. \quad (50)$$

To effect a complete separation, we choose $\xi(t)$ to satisfy

$$\ddot{\xi}+\Omega^2\xi=F(t)\cos(\nu t)/M_0. \quad (51)$$

We take $F(t)$ as in Eq. (8), then the solution of Eq. (51) is

$$\xi(t)=C\cos(\Omega t+\theta)+\tfrac{1}{2}f_0\left[\frac{\cos[(\lambda+\nu)t+\phi]}{\Omega^2-(\lambda+\nu)^2}+\frac{\cos[(\lambda-\nu)t+\phi]}{\Omega^2-(\lambda-\nu)^2}\right], \quad (52)$$

where C and θ are arbitrary. It is possible, and obviously convenient, to choose C and θ so that $\xi=R(t)$, as defined in Eq. (13c). Then, to meet the requirement of finite $Y(y)$, we have from Eq. (50)

$$(-\hbar^2/2M_0)(Y_n''/Y_n)+\tfrac{1}{2}M_0\Omega^2y^2=\hbar\Omega(n+\tfrac{1}{2}),\quad n=0,1,2,\ldots \quad (53)$$

and solutions of the form (49) in terms of the original variable Q are

$$\psi_n(Q,t)=\left[\frac{M_0\Omega}{\pi\hbar}\right]^{1/4}2^{-n/2}(n!)^{-1/2}H_n\left[\left[\frac{M_0\Omega}{\hbar}\right]^{1/2}(Q-R)\right]$$
$$\times\exp\left[-\frac{M_0}{2\hbar}\{[\Omega-i\nu\tan(\nu t)](Q^2+R^2)-2\Omega QR-2i(Q-R)\dot{R}\}\right]$$
$$\times\exp\left[-i\Omega(n+\tfrac{1}{2})t+i\left[\omega_0^2\int_0^t R^2(t')dt'-\int_0^t[\dot{R}+\nu\tan(\nu t')R]^2dt'-2f_0\int_0^t R(t')\cos(\lambda t'+\phi)\cos(\nu t')dt'\right]\right]. \quad (54)$$

In the absence of a driving force, so that $R \rightarrow 0$, Eq. (54) agrees with Eq. (3.8) of Ref. 14. As we noted above in connection with Eq. (48), these fundamental solutions ψ_n are *not* time-dependent eigenfunctions of H. In the case $R \rightarrow 0$, ψ_n is easily seen to be an eigenfunction of the "augmented" Hamiltonian

$$H' = \tfrac{1}{2}P^2/M_0 + \tfrac{1}{2}M_0[\omega_0^2 + \nu^2\sec^2(\nu t)]Q^2 - \tfrac{1}{2}\nu\tan(\nu t)(QP + PQ) , \tag{55}$$

with eigenvalues as in Eq. (53). The success of the separation in Ref. 14 depends on the cancellation of a term $\frac{1}{2}M_0\nu^2\sec^2(\nu t)Q^2\psi$. With $F(t)$, as in Eq. (8), an "augmented" Hamiltonian, which is a complicated extension of Eq. (55), can be written down, but any significance of the extra terms is unclear. A better approach is to ask under what conditions Eq. (54) may be regarded as a quasiperiodic state[24] with a corresponding quasienergy.[25] This requires

$$H(t+\tau) = H(t), \quad \tau = \pi/\nu \tag{56a}$$

$$\psi_n(t+\tau) = \exp(-i\mathscr{E}_n\tau/\hbar)\psi_n(t) . \tag{56b}$$

Ω is obviously incommensurate with ν and ω_0, but in the case $\lambda = \omega_0$, $R(t)$ does not depend on Ωt. Thus ψ_n is quasiperiodic if, and only if

$$\lambda = \omega_0, \quad r\lambda = s\nu \quad (r,s \text{ integers}) . \tag{56c}$$

It should be noted that Eq. (56c) cannot hold in the resonance cases described in Eq. (15). Also we see from Eq. (14a) that Eq. (56b) cannot be satisfied in the presence of a constant driving force.

When Eq. (56c) is satisfied, Eq. (54) gives the quasienergy

$$\mathscr{E}_n = \hbar\Omega(n + \tfrac{1}{2}) + Mf_0A + \tfrac{1}{2}M_0(B - \omega_0^2 C) , \tag{57}$$

where

$$A = \frac{1}{\tau}\int_t^{t+\tau} R(t')\cos(\lambda t' + \phi)\cos(\nu t')dt' , \tag{58a}$$

$$B = \frac{1}{\tau}\int_t^{t+\tau} [\dot{R} + \nu\tan(\nu t')R]^2 dt' , \tag{58b}$$

$$C = \frac{1}{\tau}\int_t^{t+\tau} R^2(t')dt' . \tag{58c}$$

As an example, let us consider the case $\lambda = \nu = \omega_0$. Unless $\phi = \pm\pi/2$ the integrand in Eq. (58b) has a double pole in the range of integration and, consequently, $B \rightarrow \infty$. This is a consequence of the periodic vanishing of $M(t)$. Taking $\phi = \pi/2$ a simple calculation shows that the quasienergies are

$$\mathscr{E}_n = \hbar\Omega(n + \tfrac{1}{2}) - M_0 f_0^2/(32\omega_0^2) . \tag{59}$$

When $\phi = 0$, the Hamiltonian given by Eqs. (7) and (8) is invariant under $t \rightarrow -t$. Then $\overline{R} = R$ and, using Eqs. (43), (54), and the expansion in Hermite polynomials

$$\exp(2z\theta - \theta^2) = \sum_{n=2}^{\infty} H_n(z)\theta^n/n! ,$$

$$z = (M\Omega/\hbar)^2(Q - R), \quad \theta = 2^{-1/2}\tilde{\alpha} , \tag{60}$$

one may check that Eq. (23c) holds, where $\psi_n(Q,t) = \langle Q \mid n(t) \rangle$. The case $\phi \neq 0$ is more difficult, owing to the occurrence of $\overline{R}$ in Eq. (43a).

VIII. DISCUSSION

The exact solution of the motion of an oscillator with mass pulsating according to $M = M_0\cos^2(\nu t)$ and driven by a force $F = M_0 f_0\cos(\lambda t + \phi)$ has been obtained in both the Heisenberg and Schrödinger pictures. In the case $\phi = 0$, the system is invariant with respect to time-reversal and then, as discussed in Sec. VII, the solutions in the Heisenberg and Schrödinger pictures may be connected via the Green function calculated in Sec. V. This forms a useful check. Although the periodic vanishing of the mass gives rise to some undesirable divergences in the solution, the system is an important one to study. We believe that, as a time-periodic system, it is unique in possessing an exact solution. Our reasons are set out in Ref. 14.

From a physical point of view, as discussed in Refs. 16 and 26, a pulsating mass given by Eq. (1), or, alternatively, the more tractable form $M_0\exp[2\mu\cos(2\nu t)]$, which excludes $M = 0$ and has a variable strength parameter α or μ, is preferable. The weak case $\mu \ll 1$ is considered in Refs. 20 and 26. It would be interesting to compare the strong case $\mu \gg 1$ with the present model. This would require numerical computation and could form the basis of some future work.

The Heisenberg picture shows very clearly that if the mass and driving force are not exactly in phase ($\phi \neq 0$), then a catastrophic resonance occurs if $\lambda = \Omega \pm \nu$, and $\Omega = (\omega_0^2 + \nu^2)^{1/2}$. If $\phi = 0$ the resonance weakens to secular type. In the variable-strength model with $\mu \ll 1$ and the same driving force and any phase difference ϕ, we find secular-type resonances at $\lambda = \omega_0$, $\omega_0 + \nu/2$, and $|\omega_0 - \nu/2|$, as discussed in Ref. 20.

As Eberly[27] remarks, a solution in the Schrödinger picture is more difficult than in the Heisenberg picture. Furthermore, the Schrödinger solution is more powerful. Although in principle it is possible to pass from one picture to the other via ψ_α, using Eq. (60), we did not find this practicable rather we had to seek the correct separation via Eq. (49). We regard the fundamental "number state" wave function given in Eq. (54) as the most important outcome of our work.

Although we are convinced that no further exactly solvable *periodic* mass laws exist, there might be some further physically interesting *oscillatory* masses (hopefully that exclude $M = 0$) that are amenable to exact solution, for instance, an oscillatory modification of the law in Ref. 17. However, we must always remember that exactly solvable models in quantum mechanics are extremely rare.

*Present address: Department of Mathematics, King's College, Strand, London WC2R 2LS, England.

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