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Stability of finger patterns in Hele-Shaw cells

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The finger of fluid formed during the displacement of a more viscous fluid is a well-known example of nonequilibrium pattern formation. Previous efforts claimed that the steady-state solution discovered by Saffman and Taylor was linearly unstable, in contradiction with experimental observations. In this work, we present a new stability analysis which demonstrates stability and resolves the aforementioned conflict. We describe how this lends support to a new paradigm governing interfacial pattern-forming systems, that of "microscopic solvability."

Recently, considerable progress has been made in understanding pattern selection in nonequilibrium dynamical processes such as crystallization and multiphase fluid flow. Specifically, there is a growing amount of evidence that these systems choose their spatial structure via a solvability condition that arises when one considers the effect of the (microscopic) capillary length on the family of macroscopically consistent steady-state solutions. This mechanism has been shown to apply to simplified models of dendritic growth,[1,2] and to fingering in a Hele-Shaw cell.[3,4] In the case of true diffusively controlled crystallization, there is suggestive evidence,[5] but as of yet no direct proof.

Under this new paradigm of "microscopic solvability," the stability characteristics of the steady-state solution do not determine the velocity, but are nonetheless important for such properties as the presence of sidebranches. In this regard, the stability of the Saffman-Taylor finger[3] in the Hele-Shaw cell has to date been a complete mystery. Experimentally[6] and computationally,[7] a fluid forced into the gap between two parallel plates saturated with more viscous fluid will eventually[8] form a single fingerlike interfacial pattern which then propagates with constant velocity. The pattern is a reproducible function of the external conditions and is apparently stable with respect to small perturbations for a wide range of velocities. However, the one attempt to compute the spectrum of the linear stability operator around the selected pattern concluded that the finger was *unstable* at all velocities. Clearly, this result is inconsistent with the scenario sketched previously.

The purpose of this work is to resolve this problem by presenting a new stability analysis of the Saffman-Taylor finger. We derive a singular integro-differential stability operator and study its properties by analytical and computational techniques. Our result is that a proper treatment of this operator and in particular of the effect of finite surface tension leads to stability for a wide range of velocities. The relevance of this result for some recent experiments will be discussed briefly.

The standard geometry for the Hele-Shaw cell is shown in Fig. 1. The equations of motion for the fluid-fluid interface are

$$\begin{aligned}&\nabla^2 T = 0 \ , \\ &-\hat{\mathbf{n}}\cdot\nabla T = \hat{\mathbf{n}}\cdot\frac{d\mathbf{x}}{dt} \quad \text{(at the interface)} \ , \\ &T(\text{interface}) = -\gamma\kappa \ ,\end{aligned} \tag{1}$$

together with the boundary conditions $T \sim -x$ as $x\to\infty$, $\partial T/\partial y = 0$ at $y=\pm 1$. Here, T is proportional to the pressure, κ is the interface curvature, and $\gamma = \sigma b^2/12\mu v a^2$, for surface tension σ, viscosity μ, gap thickness b, and cell width a. We have for simplicity neglected the pressure drop in the less viscous fluid and these equations are strictly applicable to the case of displacement by a fluid of almost vanishing viscosity, such as air.

Saffman and Taylor[3] solved these equations for steady-state interface shapes in the limit of zero surface tension. They found a family of possible patterns labeled by λ, the ratio of finger width to cell size. Using the fact that T satisfies Laplace's equation, we can construct a coordinate system (ϕ,ψ), $\phi=-T$, $\partial_\mu\psi = \epsilon_{\mu\nu}\partial_\nu\phi$. The complex coordinate $z = x+iy$ is then an analytic function of $\rho = \phi + i\psi$. The solution is given by the mapping

$$z = \rho + 2/\pi(1-\lambda)\ln(1+e^{-\pi\rho}) \ . \tag{2}$$

The interface is determined parametrically by setting $\phi = 0$ in (2).

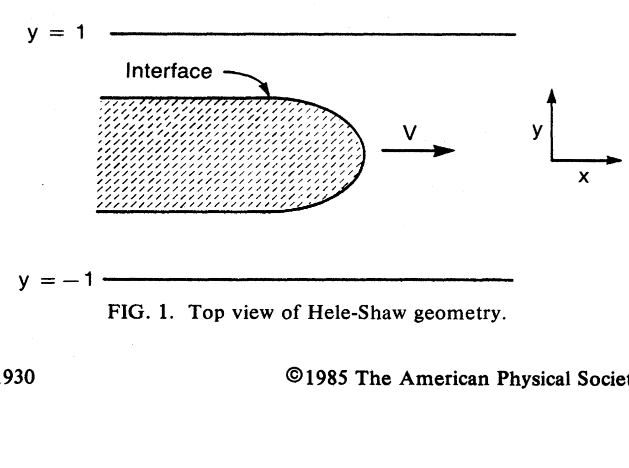

FIG. 1. Top view of Hele-Shaw geometry.

It was shown by McLean and Saffman and Vanden-Broeck[4] that inclusion of finite γ selects a discrete set of values for λ, and at the same time slightly modifies the shape of the interface. In the small γ limit, λ approaches $\frac{1}{2}$, and the shape correction becomes very small. Our method will be to derive a stability operator by neglecting the shape correction entirely but still including the surface tension. This method is justified by the fact that the effect of nonzero γ is crucial even when the shape correction is infinitesimal. This idea parallels an approach due to Muller-Krumbhaar and Langer[9] in their study of dendritic crystal growth, and the reader is referred there for more details. This approximation enables us to derive an explicit expression for the stability operator, something which considerably eases the subsequent analysis.

Assume that the perturbed interface is parametrized by $\phi=\delta(\psi)$, for δ small and ϕ,ψ the coordinate system given by (2). The field T can now be expanded

$$T=-\phi+\sum_{n=0} a_n \cos\pi n\psi e^{-\pi n\phi} \ , \tag{3}$$

which satisfies the sidewall condition, and where we have assumed a symmetric perturbation. We now impose the two boundary conditions at the interface and keep terms only to $O(\delta)$. After some algebra, this yields

$$-\delta(\psi)+\sum a_n \cos\pi n\psi=-\gamma\kappa^{(1)}(\delta) \ , \tag{4a}$$

$$\sum \pi n a_n \cos\pi n\psi=(\tfrac{1}{2}\dot{\delta}+\sin\pi\psi\delta'+\pi\delta)/(1+\cos\pi\psi) \ , \tag{4b}$$

where the linearized curvature is given by

$$\kappa^{(1)}=-2\delta''\cos\frac{\pi\psi}{2}-\pi\sin\frac{\pi\psi}{2}\delta' -\frac{\pi^2}{2}\delta\left[\cos\frac{\pi\psi}{2}+\frac{1}{\cos\pi\psi/2}\right] \ .$$

To go to the final form of the stability operator, we eliminate the unknown coefficients a_n. The function defined by

$$f(\psi)\equiv\sum_n a_n e^{-in\pi\psi}$$

is analytic and vanishes for large negative imaginary ψ. Therefore,

$$\int_{-\infty}^{\infty} d\psi' \frac{f(\psi')}{\psi'-\psi-i\epsilon}=0$$

or

$$\mathrm{Im}f(\psi)=\frac{-1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d\psi' \frac{\mathrm{Re}f(\psi')}{\psi'-\psi} \ . \tag{5}$$

Combining this with the definition of f and with Eq. (4a), we can rewrite the left-hand side of Eq. (4b) as

$$-\frac{1}{\pi}\frac{\partial}{\partial\psi}\mathrm{P}\int_{-\infty}^{\infty} d\psi' \frac{\delta(\psi')-\gamma\kappa^{(1)}[\delta(\psi')]}{\psi'-\psi} \ .$$

Making use of the periodicity of δ under $\psi\to\psi+2$, we can arrive at the spectral equation

$$\tfrac{1}{2}\dot{\delta}+\sin\pi\psi\delta'+\pi\delta =\tfrac{1}{2}(1+\cos\pi\psi)\mathrm{P}\int_{-1}^{1} d\psi' (\delta-\gamma\kappa^{(1)})'\cot\frac{\pi}{2}(\psi-\psi') \ . \tag{6}$$

For the $\gamma=0$ case, this equation can be solved simply by assuming a Fourier series for δ. It is easy to show that the spectrum is given by $\omega_n\equiv\dot{\delta}_n/\delta_n=2\pi n$, with corresponding eigenfunctions

$$\delta_n=\sum_{m=0}^{n+1} b_{m;n}\cos\pi m\psi \ , \tag{7a}$$

$$b_{m;n}(n-m+1)=(m+1)b_{m+1;n} \ . \tag{7b}$$

These modes all vanish at $\psi=1$ and are all unstable. This agrees with results obtained previously.[3,4] The first question to answer is why the introduction of nonzero γ can completely alter this result. This can be addressed by recasting the stability matrix at $\gamma=0$ into a basis given by pure Fourier modes. From (7a), this matrix has nonzero elements on the diagonal equal to $(n-1)$ and in positions $(n,n+1)$ equal to n. This matrix is supersensitive to perturbations (for finite n, eigenvalues change by $n!\epsilon$) and is, in fact, very similar to a matrix used as a textbook illustation of an ill-conditioned matrix.[10] This analysis demonstrates that arbitrarily small surface tension can in principle completely alter the spectrum. Note that this ill conditioning invalidates any approach based on expansion in the $\gamma=0$ eigenfunctions, or in Fourier modes.

We proceed by recognizing that nonzero surface tension allows for the existence of continuum modes which vary as $e^{\pi ks}$, where s is the arclength along the interface. s is related to ψ via

$$\frac{d\psi}{ds}=2\cos\frac{\pi\psi}{2} \text{ or } \cos\frac{\pi\psi}{2}=\frac{1}{\cosh\pi s} \ . \tag{8}$$

Any mode with finite wavelength will be extremely singular in ψ. We reexpress the stability equation (6) as

$$\frac{-1}{2}\mathrm{P}\int_{-\infty}^{\infty} ds' \frac{1+\sinh\pi s\sinh\pi s'}{\sinh\pi s'-\sinh\pi s}\{\delta(s')-\gamma\hat{\kappa}^{(1)}[\delta(s')]\}' =\tfrac{1}{2}\cosh^2\pi s[(\tfrac{1}{2}\omega+\pi)\delta(s)+\tanh\pi s\,\delta'(s)] \ , \tag{9}$$

with

$$\hat{\kappa}^{(1)}=-\tfrac{1}{2}\cosh\pi s\,\delta''-\pi\sinh\pi s\,\delta' -\tfrac{1}{2}\pi^2\delta(\cosh\pi s+1/\cosh\pi s) \ .$$

We can now substitute the form $\delta\sim e^{\pi ks}$ and identify the dispersion relationship for these modes. Clearly, the real part of k must be negative to ensure convergence of the principal-value integral at large s'. After some algebra, we find for $q\equiv k+1$

$$\frac{\omega}{2\pi}=-q-\tfrac{1}{2}\gamma\pi^2q^3\tan\frac{\pi q}{2} \ . \tag{10}$$

Before turning to our numerical results, it is worthwhile to comment on the form of Eq. (10). This dispersion relation has two pieces—the first can be associated with having transformed to the moving frame of reference and the second is due to surface tension. The usual *positive* contribution on the right-hand side representing the Mullins-Sekerka[11] instability (which gives rise to the nontrivial pattern by making a horizontal planar front unstable) is absent. This is due to the walls preventing fluid motion asymptotically far from the tip region. Now, the normal displacement of the interface is related to δ via a factor of $\cosh\pi s$. As we shall see, all modes have $\mathrm{Re}q>0$ and hence cause large displacements at large distances. The tip stabilization, which will be demonstrated numerically, is directly traceable to the

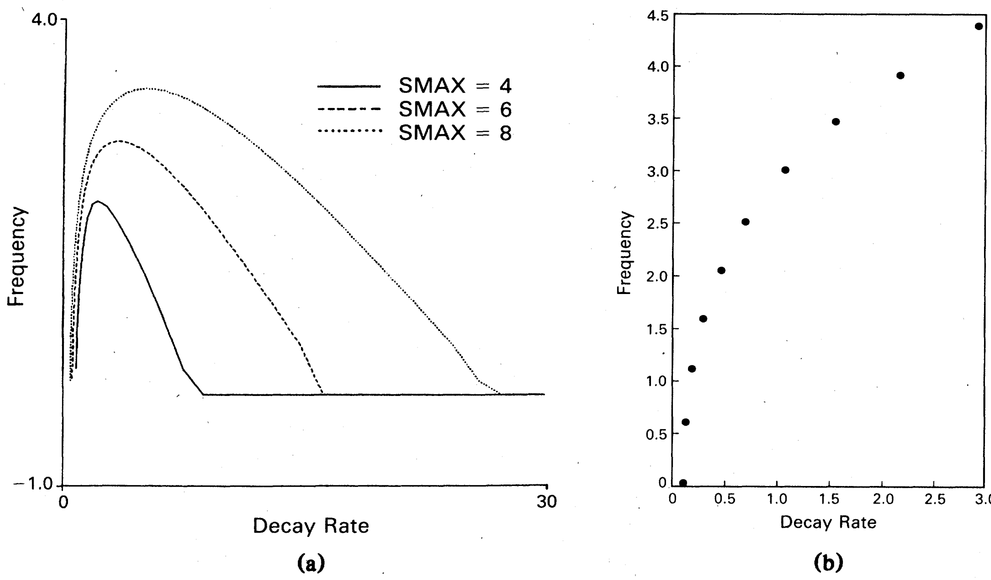

FIG. 2. Reω vs Imω for $\gamma = 0.005$. (a) For all curves, $N = 50S_{\max}$. (b) Bottom of the continuum, extrapolated to infinite $S_{\max}$.

coupling of displacements in the vicinity of the tip to exponentially growing displacements down the sides of the pattern and their suppression by the walls.

To study Eq. (9) numerically, we discretize the equation on a one-dimensional grid of N points with an arclength cutoff $S_{\max}$. Note that the singularity in the integral kernel must be treated carefully to ensure that the resulting matrix is accurate to $O(1/N^2)$. We then solve the resulting nonsymmetric eigensystem by using EISPACK library routines. For all $\gamma \geq 0.001$, the spectrum consists of exactly one discrete mode which has purely real ω, as well as a continuum with ω generally complex. The discrete mode is due to the translation invariance of the steady-state solution, and always has Re$\omega \sim 0$. We have checked that by approximately including the shape correction, one recovers $\omega = 0$ for this mode without affecting the continuum. This will be discussed at length elsewhere.

The remainder of the modes always have Re$\omega < 0$, giving rise to stability. They all satisfy the dispersion equation (10) with Req varying between 0 and 1. A typical set of data is presented in Fig. 2, where we have extrapolated to $S_{\max}$ infinite using the results of computations for values between 2.0 and 8.0 at the same value of the spatial resolution. More details will be presented in a longer paper in preparation, but we would like to note that the stability we see is completely independent of any assumptions made in the extrapolation; it is only the higher frequency modes which are all extremely stable that are sensitive to the details of our procedure. We have studied the spectrum at many differing values of the surface tension, again with the same conclusion. In fact, the modes most likely to be unstable have small Imq and *positive* Req, *guaranteeing* stability via the above dispersion relation. Finally, we have repeated this calculation with antisymmetric perturbations with similar results; the continuum is stable, and there is a single discrete mode now with negative Reω. For γ greater than 1.0×10^{-3}, Saffman-Taylor fingers are stable.

Howe does this compare with experiment? The stability of the single finger for this range of the parameters is clearly borne out by both real and computer experiments. This agreement is the main result of our paper. Now, at still smaller surface tension, there appears to be an antisymmetric instability which causes the finger to break apart.[12] Given our findings, there are two consistent possibilities. The discrete antisymmetric mode discussed above does tend to become less stable as γ is decreased, and it may cross the Re$\omega = 0$ axis at finite surface tension. However, it is also possible that the instability is only present at finite amplitude and that the finger is *linearly* stable even for infinitesimal surface tension. Some evidence for a finite amplitude instability has recently been provided by Schwartz and DeGregoria.[13] Resolution of this point awaits additional study, although we favor the latter possibility.

To summarize, our work has shown how the inclusion of surface tension can stabilize the Saffman-Taylor finger. The single finger pattern can indeed be understood by velocity selection and *subsequent* linear stability analysis. It falls neatly into the emerging scenario for pattern formation by interface propagation in nonequilibrium systems. This strengthens our belief in the universality of the idea of microscopic solvability.

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