RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 32, NUMBER 3 SEPTEMBER 1985

Evidence for a new period-doubling sequence in four-dimensional symplectic maps

Jian-Min Mao* and Indubala I. Satija†
Center for Nonlinear Studies, MS B258, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

Bambi Hu
Department of Physics, University of Houston–University Park, Houston, Texas 77004
(Received 30 April 1985)

We have numerically investigated period-doubling bifurcations in four-dimensional symplectic maps. Our study indicates the existence of a universally self-similar period-doubling sequence. Unilke the two-dimensional case, the fixed-point map has two unstable directions under the period-doubling operator with two relevant eigenvalues 8.721 097 2 and −15.0786. The four orbital scaling factors along and across the dominant symmetry surface are, respectively, 16.1449, −4.018 07, 16.36, and −7.5393.

The discovery of universally self-similar period-doubling bifurcations in one-dimensional maps[1] aroused a great deal of interest in the study of period doubling in conservative maps.[2–6] One-parameter family of two-dimensional area-preserving maps is found to exhibit an infinite sequence of period-doubling bifurcations analogous to one-dimensional dissipative maps. However, the critical exponents δ and α for two-dimensional area-preserving maps are different from their corresponding values in the dissipative case. Also, the universality results for period doubling in one-dimensional maps extend to higher-dimensional dissipative systems. An interesting, but still open question, is whether the self-similar period-doubling patterns of area-preserving maps carry over to higher-dimensional symplectic maps. Symplectic maps are of interest because they serve to describe the time evolution of Hamiltonian systems through a Poincaré surface of section.

In this paper, we report a numerical study of period-doubling bifurcations in two-parameter families of four-dimensional symplectic maps. These maps may be viewed as Poincaré sections of Hamiltonian systems with three degrees of freedom. Four-dimensional symplectic maps are also relevant to the study of crystal physics.[5] We have found evidence for the existence of a complete period-doubling sequence with two universal δ's: 8.721 097 2 and −15.0786. This implies that the unstable manifold of the fixed-point function is two dimensional. We have also obtained four orbital scaling exponents: −4.018 07, 16.1449, 16.36, and −7.5393, describing scaling along and across the dominant symmetry surface, respectively.

Consider the following class of conservative maps:

$$X_{i+1}=-Y_i+F(X_i),\quad Y_{i+1}=X \ , \tag{1}$$

where X, Y, and F denote

$$X=(x,z) \ ,$$
$$Y=(y,t) \ , \tag{2}$$
$$F=(f_1(x,z),f_2(z,x)) \ .$$

This map is symplectic if the functions f_1 and f_2 satisfy the condition

$$\partial_z f_1=\partial_x f_2 \ . \tag{3}$$

For our study, f_1 and f_2 are chosen to have the following form:

$$f_1(x,z)=f_a(x)-b(x+z) \ ,$$
$$f_2(z,x)=f_a(z)-b(z-z) \ , \tag{4}$$

where the function $f_a(x)$ depends on a parameter a. When b is equal to zero, these equations decouple into two area-preserving maps. This matrix equation is an obvious higher-dimensional generalization of the two-dimensional Hénon map. Also, this class of maps is reversible, i.e., they can be factorized into a product of two involutions I_1 and I_2 ($I_i^2=1$):

$$I_1:\ X_{i+1}=X_i,\quad Y_{i+1}=-Y_i+F(X_i) \ ;$$
$$I_2:\ X_{i+1}=Y_i,\quad Y_{i+1}=X_i \ . \tag{5}$$

Fixed points of I_1 and I_2 lie on a surface called the dominant symmetry surface. A 2^k-periodic point on the dominant symmetry surface will also have its 2^{k-1} iterate on it. This greatly simplifies the process of locating the periodic orbits.

The stability of a periodic orbit in a given dimension is determined by the Floquet multipliers which are the eigenvalues of the linearized map T. The matrix T is real and has determinant equal to unity. In two dimensions, this condition restricts the eigenvalues of T to be conjugate points $e^{i\theta}$ and $e^{-i\theta}$ on the unit circle, or reciprocal points ρ and ρ^{-1} on the real axis. The former case corresponds to stable orbits; the latter case, unstable orbits. Hence, the parameter that determines the stability of a two-dimensional periodic orbit is the trace of T. The generalization of this condition to four dimensions has been worked out.[7] In this case, the eigenvalues of the matrix T are of the form $\rho e^{i\theta}$, $\rho e^{-i\theta}$, $\rho^{-1}e^{i\theta}$, and $\rho^{-1}e^{-i\theta}$. The stability regime is bounded by the following curve in the ($\mathrm{Tr}\,T$, $\mathrm{Tr}\,T_{12}$) plane:

$$B\geqslant -2+2A,\quad B\geqslant -2-2A,\quad B\leqslant \frac{A^2}{4}+2 \ , \tag{6}$$

where

$$A=\mathrm{Tr}\,T=\sum_i \lambda_i,\quad B=\mathrm{Tr}\,T_{12}=\sum_{i<j}\lambda_i\lambda_j \ . \tag{7}$$

Here, λ's are the eigenvalues of the matrix T. Typically, period doubling takes place when a pair of eigenvalues passes through -1. This condition is satisfied along a line $2\,\mathrm{Tr}+\mathrm{Tr}\,T_{12}+2=0$ in the stability region.

We have carried out a two-parameter search for period-doubling sequences by simultaneously varying a and b. Hence, both pairs of eigenvalues can be moved independently on the unit circle. The period-doubling parameters are determined when the two pairs of eigenvalues pass through -1 and 1, corresponding to $A=0$. Results for the map corresponding to $f_a(x)=1-ax^2$ are shown in Table I. At first sight the period-doubling parameter values did not seem to exhibit any scaling property in an obvious fashion. To unravel a possible underlining self-similar structure, we will use a general method of testing for scaling developed by Guckenheimer, Hu, and Rudnick.[8]

Let $\Delta a_1 = a_i - a_{i-1}$, $\Delta b_i = b_i - b_{i-1}$, and define

$$(\Delta a_i, \Delta b_i) = \tilde{\delta}(\Delta a_{i+1}, \Delta b_{i+1}) \ . \tag{8}$$

The matrix $\tilde{\delta}$ was found to converge rapidly; moreover, its eigenvalues δ_1 and δ_2 were seen to obey very nice scaling relations (see Table II). The exponent $\delta_1 = 8.721\,097\,2$ is the same as the two-dimensional case, whereas $\delta_2 = -15.0786$ is new and signifies the existence of another unstable direction. [As a matter of fact, if we apply this scaling-matrix method to analyze the data presented in Table III(a) of Ref. 9(b), we obtain the same δ_1 and δ_2, despite the fact that the form of the maps is completely different.]

To calculate the orbital scaling factors, we make the following coordinate change:

$$X \rightarrow 2X, \quad Y \rightarrow 2Y - F(X) \ , \tag{9}$$

so that Eq. (1) is transformed to the generalized DeVogelare map

$$X_{i+1} = -Y_i + F(X_i), \quad Y_{i+1} = X_i - F(X_{i+1}) \ . \tag{10}$$

The usefulness of this form of the map comes from the fact that $Y=0$ is the dominant symmetry surface. Hence, the orbital scaling matrix $\tilde{S}$ is decoupled:

$$\tilde{S} = \begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & \tilde{\beta} \end{pmatrix} \ , \tag{11}$$

TABLE I. Convergence of the period-doubling parameters a and b for a four-dimensional symplectic map with $f_a(x)=1-ax^2$.

Period $=2^n$ n	a	b
1	1.551 052 531 382 147	0.884 566 136 193 24
2	3.154 929 859 908 943	0.286 068 854 598 49
3	3.310 106 308 833 480	0.244 813 629 780 80
4	3.302 711 501 079 377	0.249 186 673 247 67
5	3.303 272 425 207 032	0.249 158 055 154 25
6	3.303 252 894 478 886	0.249 185 866 130 81
7	3.303 256 133 362 466	0.249 187 022 559 87
8	3.303 256 140 697 924	0.249 187 290 167 15
9	3.303 256 165 659 491	0.249 187 311 907 23
10	3.303 256 166 921 683	0.249 187 314 993 39
11	3.303 256 167 172 515	0.249 187 315 307 92
12	3.303 256 167 194 420	0.249 187 315 346 59
13	3.303 256 167 197 197	0.249 187 315 350 86
14	3.303 256 167 197 506	0.249 187 315 351 36
15	3.303 256 167 197 543	0.249 197 315 351 41
16	3.303 256 167 197 547	0.249 187 315 351 42

where $\tilde{\alpha}$ and $\tilde{\beta}$ are 2×2 matrices describing, respectively, the self-similarity along and across the dominant symmetry surface. They are defined as

$$(X_0^i - X_{1/2}^i) = \tilde{\alpha}(X_0^{i+1} - X_{1/2}^{i+1}), \quad Y_{1/4}^i = \tilde{\beta}\, Y_{1/4}^{i+1} \ . \tag{12}$$

Here (X_0, Y_0), $(X_{1/2}, Y_{1/2})$, and $(X_{1/4}, Y_{1/4})$ are, respectively, the initial, halfway, and quarterway points of a limit cycle. This definition is the same as that used for two-dimensional area-preserving maps.[2-6] The scaling exponents α_1, α_2 and β_1, β_2 are the eigenvalues of the matrices $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. Again we notice that the exponents α_1 and β_1 are the same as those for the two-dimensional area-preserving case, whereas α_2 and β_2 are new. We have computed the exponents for various functions f_a, and universality is seen to be well obeyed.

In sum, we have succeeded in finding a complete period-doubling sequence in four-dimensional symplectic maps.

TABLE II. Period-doubling bifurcation rates δ_1 and δ_2, and orbital scaling factors α_1, β_1 and α_2, β_2 along and across the dominant symmetry surface, respectively.

Period $=2^n$ n	δ_1	δ_2	α_1	α_2	β_1	β_2
3	6.335 730 0	−13.872 058	−0.016 689	−0.003 317 2	0.000 005	−0.000 004 3
4	8.455 412 9	8.455 412	0.006 529	0.006 529 3	0.012 851	0.012 851 8
5	9.394 969 0	−17.738 353	−19.326 037	−4.727 903 4	6.206 889	6.206 889 8
6	8.847 625 6	−16.842 358	16.965 790	−4.030 998 8	13.169 810	13.169 810 9
7	8.723 543 0	−15.303 906	15.955 490	−4.017 347 4	15.579 237	−8.743 703 7
8	8.722 287 7	−15.049 664	16.183 370	−4.018 263 7	15.876 244	−7.693 843 1
9	8.721 100 0	−15.092 972	16.135 221	−4.018 066 8	16.214 081	−7.539 500 3
10	8.721 109 7	−15.075 200	16.148 114	−4.018 079 2	16.261 098	−7.545 752 7
11	8.721 096 6	−15.079 738	16.143 809	−4.018 076 5	16.314 274	−7.538 013 1
12	8.721 097 2	−15.078 419	16.145 397	−4.018 076 7	16.335 769	−7.539 838 0
13	8.721 097 2	−15.078 773	16.144 747	−4.018 076 7	16.349 052	−7.539 211 2
14	8.721 097 2	−15.078 672	16.145 025	−4.018 076 7	16.355 799	−7.539 390 0
15	8.721 097 2	−15.078 641	16.144 902	−4.018 076 7	16.359 544	−7.539 336 8
16	8.721 097 2	−15.078 659	16.144 957	−4.018 076 7	16.361 541	−7.539 352 0

There exist two universal bifurcation rates δ_1 and δ_2, which imply that the fixed-point function has two relevant eigenvalues under the renormalization transformation. Within numerical precision, we also found $\alpha_2 = \alpha_1^2$ and $\delta_2 = 2\beta_2$. The relation $\alpha_1^2 = \alpha_2$ is similar to the two-dimensional cases,[2] where α^2 rescaling emerges due to the curvature of the symmetry line. However, we have no clue as to why $\delta_2 = 2\beta_2$.

In the future, we would like to understand the universality of period doubling in four-dimensional symplectic maps by studying the fixed points of the renormalization transformation. Perhaps this study would shed more light on such new features as the existence of a new unstable direction, the commonality of exponents between two and four dimensions, and the relations among the exponents.

We are indebted to R. S. MacKay for his critical comments and suggestions. We would like to thank R. Helleman and K. Kaneko for useful discussions. One of us (J.M.M.) would like to thank the Center for Nonlinear Studies at Los Alamos for its hospitality.

*Permanent address: Department of Physics, University of Houston, Houston, TX 77004.

†Permanent address: Bartol Research Foundation, University of Delaware, Newark, DE 19716.

[1]M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); **21**, 669 (1979).

[2]G. Benettin, C. Cercignani, L. Galgani, and A. Giorgilli, Lett. Nuovo Cimento **28**, 1 (1980).

[3]T. C. Bountis, Physica D **2**, 577 (1981).

[4]J. Greene, R. S. MacKay, F. Vivaldi, and M. J. Feigenbaum, Physica D **3**, 468 (1981).

[5]P. Bak, T. Janssen, and B. A. Tjon, Phys. Lett. **87A**, 139 (1982).

[6]R. S. MacKay, Ph.D. dissertation, Princeton University, 1982 (unpublished).

[7]R. Broucke, AIAA J. **7**, 6 (1969).

[8]J. Guckenheimer, B. Hu, and J. Rudnick (unpublished).

[9](a) T. Janssen and T. A. Tjon, J. Phys. A **16**, 673 (1983); (b) **16**, 697 (1983).