Dynamics of Fréedericksz transition in a fluctuating magnetic field

F. Sagués
Departamento de Física Teórica y Departamento de Química Física, Universidad de Barcelona, Diagonal 647, Barcelona 08028, Spain

M. San Miguel
Departamento de Física Teórica, Universidad de Barcelona, Diagonal 647, Barcelona 08028, Spain
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Fréedericksz transition under twist deformation in a nematic layer is discussed when the magnetic field has a random component. A dynamical model which includes the thermal fluctuations of the system is presented. The randomness of the field produces a shift of the instability point. Beyond this instability point the time constant characteristic of the approach to the stationary stable state decreases because of the field fluctuations. The opposite happens for fields smaller than the critical one. The decay time of an unstable state, calculated as a mean first-passage time, is also decreased by the field fluctuations.

I. INTRODUCTION

Dynamical instabilities occur in many different physical systems. It is known that mean-field theories have, in general, a wider domain of applicability in the study of these instabilities than in the analysis of equilibrium phase transitions.[1] This fact greatly simplifies, in particular, the analysis of dynamical properties. Global fluctuations are usually modeled in a mean-field treatment by a space-independent random term in a Langevin-type dynamical model.[1] These fluctuations play a crucial role in the description of several dynamical processes such as the decay of unstable states.[2–4] On the other hand, the instability occurs at some critical value of a control parameter of the system. The effect of random fluctuations of the control parameter in such transitions has also been considered.[5] The modeling of this "external noise" situation introduces an additional random term in the dynamical model.

There exist very interesting features common to classes of instabilities and nonequilibrium transitions. This is still true in the case of random control parameters. But there also exist peculiarities of each individual physical system under consideration. A proper understanding of these transitions, and especially of the effect of external noise, requires a detailed study of concrete systems. In this context an interesting possibility is the study of instabilities in nematic liquid crystals.[6–8] Experimental studies of their electrohydrodynamic instabilities, in which the electric field has a random component, have already been carried out.[9] The external noise in these experiments is the random part superposed to the electrical field. In this context it has been recently suggested by Horsthemke *et al.*[5,10] that the study of the Fréedericksz instability has certain advantages, namely, it admits a relatively simple theoretical description, and it is easily accessible to experimentation. In this transition an applied magnetic field larger than a critical one changes the preferred orientation of the molecules. The fluctuations of the control parameter (external noise) correspond here to some imposed random time dependence of the magnetic field. The purpose of this paper is to elucidate the changes induced by a random magnetic field in the Fréedericksz transition. We only consider here the simplest case of twist deformation.[6–8] Horsthemke *et al.*[5,10] have discussed the effect of the field fluctuations in the stationary state and in the location of the instability point. Here, on the contrary, we focus primarily on dynamical properties such as relaxation times and the decay of unstable states, which have not yet been considered. In the case of a nonrandom field, these properties have been experimentally studied some time ago.[11–13] Even in this case there is no stochastic dynamical theory reported so far which takes consistently into account the thermal fluctuations of the system. Such fluctuations have also been neglected in Refs. 5 and 10. Their consideration is necessary to study the decay of unstable states. In general it permits a more complete description of the system. In this paper we introduce a dynamical model which includes the internal (thermal) fluctuations and also the external noise associated with the field. This is a mean-field model represented by a stochastic equation for an averaged amplitude of the most unstable mode. This description is equivalent to supposing that the molecules rotate in planes parallel to the plates containing the sample and the layer remains spatially homogeneous in any of these planes. In the limit of a nonrandom field, and neglecting thermal fluctuations, we recover the statical and dynamical descriptions associated with the standard Fréedericksz transition.[6–8,11,13] From the point of view of the external-noise problem, the most significant feature is that the dynamical model is nonlinear (quadratic) in the magnetic field. This fact precludes the use of the simplest Gaussian white-noise assumption for the field fluctuations. In this paper we present a first study of dynamical properties in the presence of nonlinear external noise including a calculation of mean first-passage times associated with decay processes. We calculate the dependence of different relaxation times on the noise parameters.

A general analysis of a quadratic-noise situation shows that the main effect of the noise is a systematic (nonrandom) contribution in the dynamical equation.[14,15] Such a systematic main contribution can be understood in terms of a modified potential which in turn explains our main findings. These findings refer to the modifications produced by the field fluctuations in the characteristic times associated with the different relaxation processes: We find that beyond a shifted instability point, the time constant associated with the approach to the stationary stable state is decreased. The opposite happens in the relaxation process with fields smaller than the critical one. We also find that the decay time, needed by the system to leave an unstable state (produced by switching on the field to a value beyond the instability point), is reduced by the field fluctuations. Estimates for the different characteristic times and their modifications are given below. For typical values, these modifications are of the order of 10%. These predictions on the effect of external noise on a well-defined system should be very helpful to check the validity of the standard modeling of external-noise situations.[5] This modeling consists normally in introducing random parameters in phenomenological dynamical equations.

The outline of this paper is as follows. In Sec. II we introduce our dynamical model. Its stationary solution is briefly discussed and a shift of the instability point is found. Section III is devoted to the dynamical properties. We first calculate by a linearization procedure the time constants (differential relaxation times) associated with the late stages of any relaxation towards the stationary stable state. Secondly we make a mean first-passage time calculation[16,17] of the decay time of an unstable state. Section IV contains a summary of results and conclusions. Details of the derivation of the stochastic model are given in the Appendix.

II. DYNAMICAL MODEL AND STATIONARY SOLUTION

The excess free-energy density of a nematic liquid crystal in its nematic state due to local deformations, with respect to the state of uniform orientation, is given by a distortion-free energy[6–8,18]

$$f_d[\mathbf{n}(\mathbf{r})]=\tfrac{1}{2}K_{11}(\nabla\cdot\mathbf{n})^2+\tfrac{1}{2}K_{22}(\mathbf{n}\cdot\nabla\times\mathbf{n})^2+\tfrac{1}{2}K_{33}[\mathbf{n}\times(\nabla\times\mathbf{n})]^2\,, \tag{2.1}$$

where $\mathbf{n}(\mathbf{r})$ is the director giving the preferred orientation at point $\mathbf{r}$. The three elastic constants K_{11}, K_{22}, and K_{33} are respectively associated with splay, twist, and bend deformations. In the presence of a magnetic field $\mathbf{H}$, there is an additional free-energy density which, except for constant terms, is[6–8,18]

$$f_m[\mathbf{n}(\mathbf{r})]=-\tfrac{1}{2}\chi_a(\mathbf{n}\cdot\mathbf{H})^2\,, \tag{2.2}$$

where $\chi_a=\chi_{\parallel}-\chi_{\perp}$ is the anisotropic part of the magnetic susceptibility. Torques exerted by the magnetic field on the molecules are opposed by elastic torques which tend to maintain a preferred direction fixed by boundary conditions. Fréedericksz transition occurs at a critical value of the field H_c beyond which the molecules in the bulk turn in the direction of $\mathbf{H}$. The stationary configurations and the value of H_c can be determined, for a given geometry, by minimizing the free energy[6–8]

$$F[\mathbf{n}(\mathbf{r})]=\int d\mathbf{r}\{f_d[\mathbf{n}(\mathbf{r})]+f_m[\mathbf{n}(\mathbf{r})]\}\,. \tag{2.3}$$

We are mainly interested here in the dynamics of this transition in the case in which the magnetic field takes random values in time. We first consider the case of a nonrandom field $\mathbf{H}$. In a general situation the dynamics are quite complex because the evolution of $\mathbf{n}(\mathbf{r},t)$ is coupled to a velocity field $\mathbf{v}(\mathbf{r},t)$, and hydrodynamic effects are important.[6–8] Nevertheless, for a pure twist deformation there is no hydrodynamic flow: There is no translational motion of molecules caused by the magnetic field and so there is no coupling of $\mathbf{n}(\mathbf{r},t)$ with the velocity field.[6,8,19] In this case and neglecting director inertia, a simple dynamical relaxational model of the time-dependent Ginzburg-Landau (TDGL) type seems appropriate. We choose a geometry in which the plates containing the sample are perpendicular to the z axis and are separated a distance d. The molecules are initially aligned along the x axis, and a magnetic field is applied along the y axis. We make the ordinary assumption that the director remains in the x-y plane.[6,8,13] We also assume that the system remains homogeneous in the x-y plane. (See the Appendix for a discussion of this last assumption.) The director is then written as

$$n_x=\cos\phi(z,t'),\quad n_y=\sin\phi(z,t') \tag{2.4}$$

and the free energy (2.3) becomes[6–8,11–13]

$$F[\phi(z)]=(S/2)\int_{-d/2}^{d/2}dz[K_{22}(\partial_z\phi)^2-\chi_aH^2\sin^2\phi]\,, \tag{2.5}$$

where S is the area of the sample in the plane x-y. In equilibrium at a given temperature, $e^{-F[\phi(z)]/k_BT}$ gives the probability density of a configuration $\phi(z)$. We introduce a TDGL-type model as

$$\partial_{t'}\phi(z,t')=-\frac{1}{S\gamma}\frac{\delta F[\phi(z)]}{\delta\phi(z,t')}+\hat{\xi}(z,t')\,, \tag{2.6}$$

where γ is the twist viscosity and $\hat{\xi}(z,t')$ is a Gaussian white noise of zero mean which accounts for the internal fluctuations of the system. It satisfies the following fluctuation-dissipation relation:

$$\langle\hat{\xi}(z,t_1')\hat{\xi}(z',t_2')\rangle=2\hat{\epsilon}\delta(t_1'-t_2')\delta(z-z')\,, \tag{2.7}$$

$$\hat{\epsilon}=\frac{k_BT}{\gamma S}\,, \tag{2.8}$$

where k_B is the Boltzmann constant and T the absolute temperature. Equation (2.8) guarantees that the stationary distribution of the configurations $\phi(z)$ associated with (2.6) is the canonical distribution

$$P_{st}[\phi]=\mathcal{N}\exp\left[-\frac{F[\phi(z)]}{k_BT}\right] \tag{2.9}$$

with $\mathcal{N}$ a normalization constant.

The dynamical model (2.6) reduces, in the absence of fluctuations ($\hat{\epsilon}=0$), to the one considered in Refs. 6, 8, 11, and 13. The stochastic model gives a pure relaxational dynamics to the equilibrium solution (2.9). This model is of the standard form used with great success in the study of critical dynamics[20] and nonequilibrium dynamics associated with phase transitions.[21] Models of this sort have been used in the context of nematodynamics, for instance, in Ref. 22. The rationale behind these semiphenomenological equations is that in a dynamical description in terms of the macroscopic variable ϕ, many microscopic degrees of freedom have been averaged out. The effect of these microscopic variables is taken into account by a heat bath modeled by the random force $\hat{\xi}$. The assumptions on $\hat{\xi}$ are justified by the large number of averaged microscopic variables (Gaussian) and their evolution in a much faster time scale than $\phi(z)$ (white noise). The stochastic model (2.6) can also be considered as an example of hydrodynamic (i.e., semimacroscopic) equations with fluctuating forces.[23]

According to (2.5) the explicit form of (2.6) is

$$\partial_{t'}\phi=\frac{K_{22}}{\gamma}\partial_z^2\phi+\frac{\chi_a H^2}{\gamma}\sin\phi\cos\phi+\hat{\xi}\ . \tag{2.10}$$

$\phi(z,t')$ can be analyzed in its Fourier modes. For a sample with boundary conditions $\phi(z=\pm d/2,t')=0$, only the lowest mode $\theta'(t')\cos(\pi z/d)$ is unstable for fields $3H_c>H>H_c$ where $H_c=[(K_{22}\pi^2)/(\chi_a d^2)]^{1/2}$ (Ref. 5). In these circumstances only the amplitude of the lowest mode shows a macroscopic variation with time, while fluctuations of the other modes are damped out. We can then neglect the coupling of the unstable modes to the stable one, and an equation for the amplitude $\theta'(t')$ follows from (2.10). Following the usual approximation[6,8,10,11,13] we expand $\sin\phi\cos\phi$ for small fluctuations in powers of ϕ. We obtain

$$\partial_{t'}\theta'(t')=-\tau_0^{-1}\theta'(t')+\tau_0^{-1}\frac{H^2}{H_c^2}\theta'(t')\left[1-\frac{\theta'^2(t')}{2}\right]+\xi'(t')\ , \tag{2.11}$$

where

$$\tau_0=\frac{\gamma d^2}{K_{22}\pi^2}=\frac{\gamma}{\chi_a H_c^2} \tag{2.12}$$

is the natural time scale and $\xi'(t')$ represents the internal fluctuations acting on the amplitude variable $\theta'(t')$. It is a Guassian white noise, with zero mean and correlation

$$\langle \xi'(t_1')\xi'(t_2')\rangle=2\epsilon'\delta(t_1'-t_2')\ , \tag{2.13}$$

$$\epsilon'=\frac{\hat{\epsilon}}{d/2}=2\frac{k_B T}{\gamma V}\ . \tag{2.14}$$

A more detailed derivation of (2.11) can be given allowing first for a dependence of the angle ϕ on the spatial coordinates x and y. Equation (2.11) is then obtained as a strict mean-field theory for the most unstable mode of a quantity spatially averaged in the x-y plane (see the Appendix). Introducing a dimensionless time $t=\tau_0^{-1}t'$, (2.11) can be written as

$$\partial_t\theta(t)=-U'(\theta)+\xi(t)=f(\theta)+h^2 g(\theta)+\xi(t)\ , \tag{2.15}$$

$$U(\theta)=\frac{(1-h^2)}{2}\theta^2+\frac{h^2}{8}\theta^4\ , \tag{2.16}$$

$$f(\theta)=-\theta,\quad g(\theta)=\theta(1-\theta^2/2)\ , \tag{2.17}$$

where $\theta(t)=\theta'(\tau_0 t)$, $h=H/H_c$, and $\xi(t)=\tau_0\xi'(t')$ has a dimensionless noise intensity $\epsilon=\epsilon'\tau_0=2\tau_0 k_B T/\gamma V=2k_B T/\chi_a H_c^2 V$. For MBBA [4-methoxybenzylidene-4'-(n-butyl)aniline], at room temperature ($\gamma\sim 1$P, $K_{22}\sim 10^{-6}$ dyn),[24] and for a sample of width $d\sim 100\ \mu$m, the order of magnitude of the time scale τ_0 is $\tau_0\sim 10$ sec, and the internal noise intensity $\epsilon\sim 8\times 10^{-11}$.

The Langevin equation (2.15) exhibits, in the limit of vanishing fluctuations, a dynamical instability (Fréedericksz transition) at $h=1$. For $h<1$, $\theta=0$ is the stationary stable state. It corresponds to the single minimum of the potential U. For $h>1$, $\theta=0$ is a stationary unstable state and two new stable stationary states $\theta_\pm=\pm[2(1-1/h^2)]^{1/2}$ appear. These two states are the two-equivalent minima of $U(\theta)$. They correspond to the two possible alignments of the molecules for $H>H_c$. When the fluctuations of the system are taken into account, the instability is described by the behavior of the stationary distribution $P_{st}(\theta)$ associated with (2.15). The Fokker-Planck equation for the probability distribution $P_{st}(\theta)$ equivalent to (2.15) is

$$\partial_t P(\theta,t)=\partial_\theta[U'(\theta)P(\theta,t)]+\epsilon\partial_\theta^2 P(\theta,t)\ . \tag{2.18}$$

The stationary solution of (2.18) is

$$P_{st}(\theta)=\mathcal{N}\exp\left[-\frac{U(\theta)}{\epsilon}\right]. \tag{2.19}$$

It has a single peak at $\theta=0$ for $h<1$. For $h>1$, $P_{st}(\theta)$ is a symmetric distribution with two maxima at $\theta=\theta_\pm$.

We now consider the effect of external fluctuations in the magnetic field. The experimental situation that we envisage is similar to the one considered by Kai *et al.* and Kawakubo *et al.*[9] for an electrohydrodynamic transition: The magnetic field has a constant mean value and a random part $w'(t')$ produced by an appropriate noise generator. The modeling of the random part $w'(t')$ cannot be made by a Gaussian white noise. On the one hand, it is not always realistic. On the other hand, it is mathematically ill-defined. Indeed, replacing H by $\overline{H}+w'(t')$ in (2.11) we obtain a stochastic differential equation involving $w'^2(t')$ which makes no sense if $w'(t')$ is a white noise. Here we model $w'(t')$ by an Ornstein-Uhlenbeck process, that is, a Gaussian process with zero mean and correlation

$$\langle w'(t_1')w'(t_2')\rangle=(D'/\tau')\exp(-|t_1'-t_2'|/\tau')\ . \tag{2.20}$$

The process $w'(t')$ is characterized by two independent parameters: noise intensity D' and correlation time τ'. These two parameters can be controlled experimentally (see, for example, Refs. 9 and 25). The correlation time τ' is given by the inverse of the cutoff frequency of the power spectrum of the noise. For a typical value of 5 kHz, $\tau'\sim 2\times 10^{-4}$ sec. In our case the value of D'/τ' is restricted by the requirement that only the lowest mode in (2.10) becomes unstable. It has been estimated[5] that for $D'/\tau'\leq 5\times 10^{-2}H_c^2$ this requirement is fulfilled. For an applied magnetic field $H \sim H_c$ this allows fluctuations in H^2 of the order of 5%. In the dimensionless time of (2.15), the stochastic differential equation for $\theta(t)$ is

$$\partial_t \theta(t) = f(\theta) + [\bar{h} + w(t)]^2 g(\theta) + \xi(t) , \quad (2.21)$$

where $\bar{h} = \bar{H}/H_c$, and $w(t) = H_c^{-1} w'(\tau_0 t)$ is an Ornstein-Uhlenbeck process characterized by a dimensionless noise intensity $D = D'/(\tau_0 H_c^2)$, and a dimensionless correlation time $\tau = \tau_0^{-1}\tau'$. With the above estimates for D' and τ', we have $\tau \sim 2\times 10^{-5}$ and $D \sim 10^{-6}$.

In order to discuss the effect of the stochastic fluctuations of the magnetic field, a first problem is to find the equation for the probability distribution $P(\theta,t)$ associated with (2.21). The difficulty comes from the nonlinearity of (2.21) in $w(t)$. An approximate Fokker-Planck equation which accounts for the main effects of the nonlinear noise can be obtained in the limit $D \ll 1$, $\tau \ll 1$ with D/τ finite. This approximation corresponds to the ordinary situation in which the noise evolves in a fast time scale $\tau \ll 1$, but has a finite strength measured by the integral of the spectral density $S(w)$: $D/\tau = \int_0^\infty dw\, S(w)$.[26] In this approximation we only retain contributions of zeroth $(D/\tau)\tau^0$, and first $[(D/\tau)\tau, (D/\tau)^2\tau]$ order in τ.[14,15] In our situation this means keeping terms of an estimated order of magnitude of 5×10^{-2}, 10^{-6}, and 5×10^{-8}. The first term neglected is of order $(D/\tau)\tau^2 \sim 2\times 10^{-11}$. This procedure corresponds to a consistent Markovian limit in which $\eta(t)$ defined by

$$\eta(t) = w^2(t) + 2\bar{h} w(t) - D/\tau \quad (2.22)$$

is replaced by a Gaussian white noise of zero mean value and with the same noise intensity $\bar{D}$ (Refs. 14 and 15)

$$\bar{D} = \int_0^\infty dt' \langle \eta(t)\eta(t') \rangle = D(4\bar{h}^2 + D/\tau) . \quad (2.23)$$

In this approximation (2.21) becomes

$$\begin{aligned}\partial_t \theta(t) &= -\bar{U}'(\theta) + g(\theta)\eta(t) + \xi(t) \\ &= f(\theta) + \alpha^2 g(\theta) + g(\theta)\eta(t) + \xi(t) , \end{aligned} \quad (2.24)$$

$$\bar{U}(\theta) = U(\theta) + \frac{1}{2}\frac{D}{\tau}\theta^2 \left[\frac{\theta^2}{4} - 1 \right] = \frac{(1-\alpha^2)}{2}\theta^2 + \frac{\alpha^2}{8}\theta^4 , \quad (2.25)$$

$$\alpha^2 \equiv \bar{h}^2 + D/\tau , \quad (2.26)$$

where h is now replaced by $\bar{h}$ in $U(\theta)$. According to the Gaussian white-noise assumption for $\eta(t)$, the Fokker-Planck equation, in the Stratonovich interpretation, associated with (2.24) is

$$\begin{aligned}\partial_t P(\theta,t) &= \partial_\theta [\bar{U}'(\theta) - \bar{D} g(\theta) g'(\theta)] P(\theta,t) \\ &\quad + \partial_\theta^2 [\bar{D} g^2(\theta) + \epsilon] P(\theta,t) . \end{aligned} \quad (2.27)$$

A comparison of (2.15), (2.16), and (2.18) with (2.24), (2.25), and (2.27) shows that at this formal level there are two different effects of the fluctuations of the magnetic field. First, there is a systematic effect due to the nonvanishing value of $\langle w^2(t) \rangle$. This effect is taken into account by a modification of the potential U which is now replaced by $\bar{U}$, or equivalently by the substitution of h^2 by $\alpha^2 = \bar{h}^2 + D/\tau$ in (2.25). We remark that such a systematic effect is a peculiarity of the nonlinearity of the noise.[14] A second effect is the introduction of a multiplicative (state-dependent) noise in (2.24).

The instability point is shifted by the randomness of the magnetic field. We first consider the strict limit $D \to 0$, $\tau \to 0$, D/τ finite. Equation (2.24) exhibits in this limit an instability at $\alpha^2 = 1$, that is,

$$\bar{h}_F^2 = 1 - D/\tau . \quad (2.28)$$

For $\bar{h} > \bar{h}_F$, the stable stationary states are given by the two equivalent minima of $\bar{U}(\theta)$ located at

$$\bar{\theta}_\pm = \pm [2(1 - 1/\alpha^2)]^{1/2} . \quad (2.29)$$

The shift in the critical value of the square of the field in the amount D/τ is a consequence of the systematic effect of the nonlinear noise. The location of the instability point is further modified when taking into account the multiplicative noise in (2.24). The instability can then be described in terms of the stationary solution of (2.27):

$$P_{st}(\theta) = \mathcal{N} [\bar{D} g^2(\theta) + \epsilon]^{-1/2} \times \exp\left[\int d\theta \frac{f(\theta) + \alpha^2 g(\theta)}{\bar{D} g^2(\theta) + \epsilon} \right] . \quad (2.30)$$

According to (2.30) the position of the extrema of $P_{st}(\theta)$ is given by

$$f(\theta) + \bar{h}^2 g(\theta) = -(D/\tau) g(\theta) + \bar{D} g(\theta) g'(\theta) . \quad (2.31)$$

The right-hand side of (2.31) vanishes when the magnetic field does not fluctuate. It shows the effect of the field fluctuations in the position of the extrema. The first term in the rhs originates in the systematic effect of the nonlinear noise. The second one is due to the multiplicative character of $\eta(t)$. Substituting the explicit expressions of $f(\theta)$ and $g(\theta)$, (2.17), in (2.31) we have

$$(\alpha^2 - 1)\theta - \tfrac{1}{2}\alpha^2\theta^3 - \bar{D}\theta(1 - \tfrac{1}{2}\theta^2)(1 - \tfrac{3}{2}\theta^2) = 0 . \quad (2.32)$$

This implies that $\theta = 0$ is always an extremum of $P_{st}(\theta)$. When $\alpha^2 > 1 + \bar{D}$ a pair of additional symmetric extrema appear at

$$\theta_{\max}^\pm = \pm \left[\frac{(4 - \alpha^2/\bar{D}) + [(2 + \alpha^2/\bar{D})^2 - 12/\bar{D}]^{1/2}}{2} \right]^{1/2} . \quad (2.33)$$

The single extremum displayed at $\theta = 0$ when $\alpha^2 < 1 + \bar{D}$ is a maximum, while the pair (2.33) are maxima when $\alpha^2 > 1 + \bar{D}$. In the latter regime $\theta = 0$ corresponds to a minimum of $P_{st}(\theta)$. The extrema $\theta_{\max}^\pm$ approach $\bar{\theta}_\pm$ as $\bar{D} \to 0$ with α fixed. If the position of the extrema are taken as indicators of the nonequilibrium transition,[5] the above discussion implies that the threshold value $\bar{h}_F$ for the Fréedericksz transition, including the effect of the multiplicative noise, is given by $\alpha_F^2 = 1 + \bar{D}$, that is,

$$\bar{h}_F^2 = \frac{1 - D/\tau + D^2/\tau}{1 - 4D} . \quad (2.34)$$

Consistently with the approximation leading to (2.24) we expand (2.34) in powers of τ. Keeping first-order contributions in D or τ with D/τ finite,

$$\bar{h}_F^2 = 1 - D/\tau + D(4 - 3D/\tau) \ . \tag{2.35}$$

As $D \to 0$ with D/τ finite we recover the previous result (2.28).

The dominant contribution, $-D/\tau$ in (2.35) implies that rapid ($\tau \ll 1$) external quadratic noise destabilizes the system lowering the threshold value of the Fréedericksz instability. This instability shift is directly given by the fluctuation of H^2. In our calculation it is of the order of 5%. The second contribution in (2.35) is common to any problem with multiplicative noise. In our case it is estimated to be only of the order of 10^{-4}%. On these bases we will only consider the systematic effect of the noise on the dynamical properties discussed in the next section.

The result (2.34) has been independently obtained by Horsthemke and Lefever in a model in which internal fluctuations are neglected.[5,10] Although internal fluctuations do not modify the threshold value (2.34), they are crucial to describe the dynamical properties discussed in Sec. III. Horsthemke, Lefever and collaborators[10] have also considered the case in which internal fluctuations are neglected and $w(t)$ is a dichotomic Markov noise with correlation function $\langle w(t) w(t') \rangle = \Delta^2 \exp[-\lambda(t-t')]$. In the limit $\lambda \to \infty$, Δ^2 fixed, they have found $\bar{h}_F^2 = 1 - \Delta^2$. This coincides with (2.28) when the natural identifications of parameters $D/\tau = \Delta^2$, $\tau^{-1} = \lambda$ is made. Beyond this limit the results obtained with the dichotomic Markov model and with an Ornstein-Uhlenbeck one do not coincide. This fact is in agreement with the general discussion of Ref. 15, where the two models for a noise entering quadratically in a stochastic differential equation have been compared.

III. RELAXATION TIMES

Dynamical properties of interest associated with the Fréedericksz instability are the relaxation times.[6,8,11,13] In principle one has to distinguish between the relaxation of small fluctuations around the stationary stable states (differential relaxation time), and the relaxation of the system after a sudden change of the parameters in which the system is driven through the instability point. In the first case there is an exponential relaxation characterized by a single well-defined time constant. In the second case the dynamical evolution involves, in general, different stages. In any case there is always a last stage with an exponential relaxation which coincides with the relaxation of a small fluctuation around the stationary stable state approached by the system. This last stage is the only significant one when a field larger than the critical field (and not too large) is switched off: $\theta(t)$ relaxes exponentially to $\theta = 0$. We first discuss such exponential relaxations. The discussion of the dynamical evolution of the system from the unstable state created when a field larger than the critical is switched on is postponed to Sec. III B. Earlier studies of these dynamical properties[6,8,11–13] have been based on a pure deterministic analysis. A more consistent presentation is given here considering the internal fluctuations of the system. As a new problem which has not been discussed so far we also consider random fluctuations of the field. We have seen in the previous section that the dominant effect of such fluctuations is taken into account replacing $U(\theta)$ in (2.15) by a modified potential $\bar{U}(\theta)$ given in (2.25). This is the only effect which survives in the limit $D \to 0$, $\tau \to 0$, D/τ finite. Here we study dynamical properties only in this limit. Internal fluctuations must be kept in our model because the decay of an unstable state does not occur in their absence. Our stochastic description is thus given by (2.24) neglecting the multiplicative noise of external origin

$$\partial_t \theta(t) = -\bar{U}'(\theta) + \xi(t) \ . \tag{3.1}$$

A. Differential relaxation times

The late stage of the relaxation of θ towards the stable stationary states $\theta = 0$ for $\bar{h}^2 < 1 - D/\tau$ and to $\theta = \bar{\theta}_\pm$ for $\bar{h}^2 > 1 - D/\tau$ can be studied linearizing (3.1) around these points. In fact the quantity which is experimentally measured is a time-dependent heat conductivity which is proportional to $\theta^2(t)$.[13] The equation for $\theta^2(t)$ follows from (3.1)

$$\partial_t \theta^2(t) = -2(1-\alpha^2)\theta^2 - \alpha^2\theta^4 + 2\theta\xi(t) \ . \tag{3.2}$$

Linearizing around $\theta^2 = 0$, the mean value $\langle \theta^2 \rangle_t$ satisifies

$$\partial_t \langle \theta^2 \rangle_t = -2(1-\alpha^2)\langle \theta^2 \rangle_t + 2\epsilon \tag{3.3}$$

so that it reaches the equilibrium value $\epsilon/(1-\alpha^2)$ with a time constant $T_1(\bar{h}, D/\tau)$,

$$T_1(\bar{h}, D/\tau) = 2[(1-\alpha^2)]^{-1} = [2(1-\bar{h}^2 - D/\tau)]^{-1} \ . \tag{3.4}$$

Setting $D/\tau = 0$ in (3.4) we recover the relaxation time for the case of a nonrandom field h.[6,8,11,13] The effect of the field fluctuations is to slow down the relaxation process subtracting a quantity $2D/\tau$ to T_1^{-1}. This linearization procedure implies a critical slowing down at the shifted instability point $\bar{h}^2 = \bar{h}_F^2 = 1 - D/\tau$. For a field $h = (\frac{1}{2})^{1/2} (H^2 = H_c^2/2)$, and in the absence of external noise $T_1 = \tau_0 \simeq 10$ sec. For the estimated value of $D/\tau \simeq 5 \times 10^{-2}$, the external noise increases the relaxation time T_1 in an amount of the order of 10%.

In the case $\bar{h}^2 > 1 - D/\tau$, we linearize (3.2) around $\bar{\theta}_\pm^2$, and $\delta\theta^2 = \theta^2 - \bar{\theta}_\pm^2$ satisfies

$$\partial_t \langle \delta\theta^2 \rangle_t = -2(\alpha^2 - 1)\langle \delta\theta^2 \rangle + 2\epsilon \tag{3.5}$$

so that the time constant $T_2(\bar{h}, D/\tau)$ for this process is

$$T_2(\bar{h}, D/\tau) = [2(\alpha^2 - 1)]^{-1} = [2(\bar{h}^2 + D/\tau - 1)]^{-1} \ . \tag{3.6}$$

The effect of the field fluctuations is now the opposite one. The relaxation time is faster than in the case of a nonrandom field. Now, for a field $h = (\frac{3}{2})^{1/2} (H^2 = \frac{3}{2} H_c^2)$, $T_2(D/\tau = 0) = \tau_0 \simeq 10$ sec, and the external noise diminishes T_2 also by 10%.

If one studies the relaxation of θ instead of θ^2, in principle one should consider the fact that the internal fluctuations $\xi(t)$ connect the two states $\bar{\theta}_+$ and $\bar{\theta}_-$ for $\bar{h}^2 > 1 - D/\tau$. The relaxation towards one of these states occurs with the same time constant T_2. This is obtained linearizing (3.1) around $\bar{\theta}_+$ or $\bar{\theta}_-$. The linearization makes sense because the equilibration between the two states (which would yield $\langle\theta\rangle_t \to 0$ as $t\to\infty$) only occurs on a much longer time scale, except very close to the instability point where $\bar\theta_+$ and $\bar\theta_-$ join each other. On the other hand, the relaxation of θ towards $\theta=0$ for $\bar h^2<1-D/\tau$ occurs with a time constant $2T_1$.

B. Decay of an unstable state

We now turn our attention to the relaxation process induced by switching on the field and its fluctuations from $\bar h=0$, $D/\tau=0$, to a value beyond the instability point $\alpha_F^2=1$. The system relaxes from the original stable steady state at $\theta=0$ towards $\theta=\bar\theta_\pm$. Immediately after the change of parameters, the system finds itself in an unstable steady state. The decay is initiated by the internal fluctuations of the system. Previous studies[13] of this relaxation process for a nonrandom field use a deterministic description [(2.15) with $\xi(t)=0$]. The decay of the unstable state is then made possible replacing the initial condition $\theta^2=0$ by a mean value $\langle\theta^2\rangle\neq 0$. It is argued that the distribution of initial conditions around $\theta=0$ implied by considering $\langle\theta^2\rangle$ is caused by thermal fluctuations obeying an equipartition theorem. In our formulation of the problem such fluctuations are modeled by the random term $\xi(t)$ in (2.15). The distribution of initial conditions is given by the stationary distribution (2.19). Several stochastic theories have been recently proposed to describe the decay of an unstable state of a general system.[2-4] From the point of view of the calculation of the relevant time scales, an important quantity is the time that the system takes to leave the unstable state. This is given by the mean first-passage time (MFPT) to leave the immediate vicinity of the unstable state.[16,17] After this time lag the process is essentially described by the exponential relaxation described previously.[27] Experimental evidence of such time lags has been reported for the Fréedericksz transition, in other geometries, in Refs. 11 and 13. Here we present a calculation of the MFPT based on the stochastic model (3.1). In the limit $D/\tau=0$ we obtain the corresponding result for the nonrandom-field case.

The explicit form of (3.1) is

$$\partial_t\theta(t)=(\alpha^2-1)\theta-\tfrac{1}{2}\alpha^2\theta^3+\xi(t)\ . \tag{3.7}$$

We wish to calculate the MFPT, T, of $\theta^2(t)$ through $\theta^2=\Omega^2$. In this calculation we follow the method developed by Haake *et al.*[17] This method is expected to give good results if for the final value of the control parameter α, $\theta=0$ and $\theta=\bar\theta_\pm$ are well separated in comparison with the strength ϵ of the fluctuations. In this case Ω is chosen as a point neither close to $\theta=0$ nor to $\theta=\bar\theta_\pm$. For small enough values of ϵ, T turns out to be independent of the precise value of Ω. The requirement of a small value of ϵ is safely satisified with our estimate of $\epsilon\sim 8\times10^{-11}$. The calculation is based on an asymptotic solution of (3.7) given by its deterministic solution with a random initial condition θ_0. The randomness of θ_0 accounts for the main dynamical effect of the thermal fluctuations of the system

$$\theta_{as}^2(t,\theta_0)=\frac{2(\alpha^2-1)}{\alpha^2}e^{2(\alpha^2-1)t}\times\theta_0^2\left[\frac{2(\alpha^2-1)}{\alpha^2}+(e^{2(\alpha^2-1)t}-1)\theta_0^2\right]^{-1}, \tag{3.8}$$

$$\theta_0=\theta(0)+\int_0^\infty dt\, e^{-(\alpha^2-1)t}\xi(t)\ . \tag{3.9}$$

The same basic idea involving a deterministic mapping of an initial random process is used in the quasideterministic theory (QDT) of de Pasquale *et al.*[4] for the decay of an unstable state. The MFPT is given by the average over an asymptotic passage time distribution $W(t)$,

$$T=\int_0^\infty dt\, tW(t)\ . \tag{3.10}$$

$W(t)$ is defined by ($\mathscr{H}$ is the Heaviside's function)

$$W(t)=\left\langle\delta[\theta_{as}^2(t,\theta_0)-\Omega^2]\mathscr{H}\left[\frac{d}{dt}\theta_{as}^2(t,\theta_0)\right]\times\frac{d}{dt}\theta_{as}^2(t,\theta_0)\right\rangle_{Q(\theta_0)}, \tag{3.11}$$

where the average is over the distribution $Q(\theta_0)$ of the random variable θ_0. This distribution includes the average over the different stochastic trajectories induced by $\xi(t)$ with a fixed initial condition and also the average over the initial conditions $\theta(0)$. In the physical situation under consideration the magnetic field and its fluctuations are switched on at $t=0$, so that the distribution of $\theta(0)$ is a Gaussian distribution given by (2.19) with $h=0$. The distribution of the random variable $\int_0^\infty dt\, e^{-(\alpha^2-1)t}\xi(t)$ is also a Gaussian of zero mean value and variance $\epsilon/(\alpha^2-1)$. Therefore it follows from (3.9) that $Q(\theta_0)$ is also a Gaussian distribution given by

$$Q(\theta_0)=\left[\frac{2\pi\epsilon}{a}\right]^{-1/2}e^{-a\theta_0^2/2\epsilon}, \tag{3.12}$$

where

$$a\equiv(\alpha^2-1)/\alpha^2\ . \tag{3.13}$$

From (3.11) and (3.12), and following the same procedure as in Ref. 17, we obtain

$$W(t)=\frac{2(\alpha^2-1)}{\pi^{1/2}}\left[\frac{a}{2\epsilon}\frac{\Omega^2e^{-2(\alpha^2-1)t}}{1-[\alpha^2/2(\alpha^2-1)]\Omega^2}\right]^{1/2}\times\exp\left[-\frac{a}{2\epsilon}\frac{\Omega^2e^{-2(\alpha^2-1)t}}{1-[\alpha^2/2(\alpha^2-1)]\Omega^2}\right], \tag{3.14}$$

and

$$T=\frac{1}{2(\alpha^2-1)}\left[\ln\left[\frac{a}{2\epsilon}\frac{\Omega^2}{1-[\alpha^2/2(\alpha^2-1)]\Omega^2}\right]-\psi(\tfrac{1}{2})\right], \tag{3.15}$$

where ψ is the digamma function.[28]

In the particular case of a nonrandom field, the MFPT is given by (3.15) with α^2 replaced by h^2. The effect of the distribution of initial conditions $\theta(0)$ is contained in the parameter a in (3.15). Setting $\theta(0)=0$ for all trajectories $\theta_{as}(t)$, we obtain (3.15) with a replaced by α^2-1. The consideration of a distribution of initial conditions obviously decreases the value of T with respect to the case of fixed initial condition $\theta(0)=0$. In practice the smallness of ϵ dominates the actual value of T. In fact, for asymptotically small values of ϵ we obtain from (3.15)

$$T \simeq [2(\alpha^2-1)]^{-1}\ln(1/2\epsilon) = [2(\bar{h}^2+D/\tau-1)]^{-1}\ln(1/2\epsilon) \quad (3.16)$$

so that, in this limit, T is independent of the values of Ω and a. This asymptotic result is also valid if at $t=0$ there is an applied nonvanishing field below its threshold value.

The expression for T in (3.16) contains two well-differentiated factors. The $\ln(1/2\epsilon)$ factor takes into account the fluctuations of the system, so that when $\epsilon \to 0$, T becomes infinite. The existence of this factor makes clear that T is much larger than the differential relaxation times T_1 and T_2 given in (3.4) and (3.6). This fact implies that when the field is switched on, the system spends in the vicinity of the unstable state a time $T \gg T_2$ before approaching the stable state $\bar{\theta}^2_{\pm}$ in an exponential relaxation with time constant T_2. This is true both for random and nonrandom fields. The prefactor $[2(\alpha^2-1)]^{-1}$ in (3.16) indicates the dependence of T on the curvature of the potential $\bar{U}(\theta)$ around $\theta=0$. In the approximation leading to (3.16) the randomness of the magnetic field only modifies this factor. The estimate which follows from (3.16) for the MFPT T_0 in the case of a nonrandom field $h=(\frac{3}{2})^{1/2}$ is $T_0 \simeq 23\tau_0 \simeq 200$ sec. Identifying h with $\bar{h}$, the relative variation caused by the fluctuations of the field is

$$\frac{T-T_0}{T_0} = \frac{-D/\tau}{\bar{h}^2+D/\tau-1}, \quad T_0 \equiv T(D/\tau=0) \ . \quad (3.17)$$

Therefore the field fluctuations diminish the time that the system needs to leave the unstable state. For $\bar{h}=(\frac{3}{2})^{1/2}$ and $D/\tau \simeq 5\times 10^{-2}$ this gives a variation of 10%.

IV. SUMMARY AND CONCLUSIONS

In this paper we have analyzed the Fréedericksz transition in a twist geometry in the presence of a fluctuating magnetic field and have taken into account thermal fluctuations. We imagine an experimental situation in which controlled random perturbations are superposed to the magnetic field in analogy with experimental studies of the electrohydrodynamic instability.[9] Our description is also useful to analyze the effect of natural fluctuations of the magnetic field.

In agreement with recent studies[10] we have found that the magnetic field fluctuations produce a shift of the instability point. This shift is directly given by the strength of the magnetic fluctuations. For fluctuations of H^2 of 5%, the shift in H_c^2 is of 5%. Our new results refer to dynamical properties. The characteristic relaxation times are modified in a larger relative amount than the shift in the instability point. Again, for fluctuations of H^2 of 5%, the differential relaxation times T_1, T_2, and the time needed to leave an unstable state are estimated to be modified by 10%. T_1 becomes larger while T_2 and the decay time diminish. Our formulas (3.4) and (3.6) for T_1 and T_2 can be applied to a number of physical situations that we can envisage. Small fluctuations of the director angle can be produced by small changes in the mean value of the applied field H. These fluctuations decay with time constants T_1 or T_2 depending on the final value of the field. If the fluctuations of the field are controlled by a noise generator, we have the additional possibility of changing the stationary state of the system by modifying the parameters of the external noise acting on the field. Finally, we note that T_1 with $\bar{h}=0$ is the time constant for the exponential relaxation of the system when the mean value of the field is set to zero starting with a situation in which $\bar{h}\neq 0$ is either larger or smaller than $\bar{h}_F$.

Equation (3.16) for T gives the time lag between the switching on of a field larger than the critical one and the time in which the director turns following the field. This formula is also of interest in the absence of field fluctuations. In fact, we do not know of any other consistent calculation of this time lag for the Fréedericksz transition. Our estimate of T_0 is compatible with experimental results in other geometries.[13]

Our conclusions are based on a stochastic dynamical model which incorporates the thermal fluctuations of the system. In the absence of field fluctuations this model gives a relaxational dynamics driven by the equilibrium free energy. The main assumptions of our final model equation (2.15) are two. First, we assume that the system is spatially homogeneous in the plane of the sample. This assumption corresponds to a mean-field treatment in which only the most unstable Fourier mode in the plane of the sample is considered. Spatial fluctuations in that plane are not essential to describe global relaxational processes. The random term associated with the thermal fluctuations models the global fluctuations of the order parameter. Secondly, we neglect the coupling of the unstable mode (in the direction perpendicular to the sample) to the stable modes. For $H < 3H_c$ only an unstable mode exists. In the limit of vanishing thermal fluctuations our model reduces to the standard one in the literature.[6,8,13] Fluctuations of the magnetic field are modeled as in other experimental studies[9,25] by a Gaussian broad-band noise with two adjustable parameters. A requirement is imposed in these parameters so that the condition of a single unstable mode is respected.

Although our results follow from a mathematical development of the dynamical model, they admit a clear physical interpretation. Fréedericksz instability is controlled by the square of the magnetic field. The dominant contribution of the field fluctuations is then a systematic contribution given by the noise strength D/τ. Essentially, the effect of fluctuations is to change the value of the square of the (reduced) field h^2 by $\bar{h}^2+D/\tau$. The time constants have then to be roughly given by the formulas for the case without field fluctuations with $\bar{h}^2$ replaced by $\bar{h}^2+D/\tau$. This replacement only modifies the potential $U(\theta)$ which describes the instability. The situation can then be described in terms of the modified potential $\bar{U}(\theta)$. For $\bar{h}^2+D/\tau<1$, $\bar{U}(\theta)$ is flatter at $\theta=0$ than the potential $U(\theta)$. This obviously implies a slowing down of the relaxation towards $\theta=0$. T_1 increases with field fluctuations. For $\bar{h}^2+D/\tau>1$, $\bar{U}(\theta)$ has deeper wells at $\theta=\bar{\theta}_\pm$ than those of $U(\theta)$ at $\theta_\pm$. In turn, this implies a smaller value of T_2. Likewise, the maximum of $\bar{U}(\theta)$ at $\theta=0$ for $\bar{h}^2+D/\tau>1$ is more pronounced than the corresponding one of $U(\theta)$. As a consequence, the time spent by the system before leaving the unstable state when a field is switched on from $\bar{h}=0$ is smaller than in the absence of field fluctuations.

APPENDIX

In this Appendix we give a more detailed derivation of our final model equation (2.11) in the absence of magnetic field fluctuations. Neglecting thermal fluctuations, the dynamical equation for the director $\mathbf{n}(\mathbf{r},t')$ can be written in terms of the free-energy functional (2.3) as[6]

$$\partial_{t'}\mathbf{n}(\mathbf{r},t')=-\frac{1}{\gamma}\frac{\delta F[\mathbf{n}(\mathbf{r},t')]}{\delta\mathbf{n}(\mathbf{r},t')} . \quad (A1)$$

We now consider the same geometry as that in (2.4) but we admit a dependence of the angle ϕ in the three spatial directions

$$n_x=\cos\phi(\mathbf{r},t'),\ \ n_y=\sin\phi(\mathbf{r},t') . \quad (A2)$$

In this situation (A1) reduces to an equation for the angle $\phi(\mathbf{r},t')$,

$$\partial_{t'}\phi(\mathbf{r},t')=-\frac{1}{\gamma}\frac{\delta F[\phi(\mathbf{r},t')]}{\delta\phi(\mathbf{r},t')} . \quad (A3)$$

The explicit form of the free-energy functional $F[\phi(\mathbf{r},t')]$ is given by (2.3) with

$$\begin{aligned} f_d[\phi(\mathbf{r},t')]=&\tfrac{1}{2}K_{11}[\cos^2\phi(\partial_y\phi)^2+\sin^2\phi(\partial_x\phi)^2\\ &-2\sin\phi\cos\phi\,\partial_x\phi\partial_y\phi]+\tfrac{1}{2}K_{22}(\partial_z\phi)^2\\ &+\tfrac{1}{2}K_{33}[\cos^2\phi(\partial_x\phi)^2+\sin^2\phi(\partial_y\phi)^2\\ &+2\sin\phi\cos\phi\,\partial_x\phi\partial_y\phi] , \end{aligned} \quad (A4)$$

$$f_m[\phi(\mathbf{r},t')]=-\tfrac{1}{2}\chi_a H^2\sin^2\phi . \quad (A5)$$

In the same way as in (2.6) we can now introduce a stochastic TDGL-type model

$$\partial_{t'}\phi(\mathbf{r},t')=-\frac{1}{\gamma}\frac{\delta F[\phi(\mathbf{r},t')]}{\delta\phi(\mathbf{r},t')}+\eta(\mathbf{r},t') , \quad (A6)$$

where $\eta(\mathbf{r},t')$ is a Gaussian white noise with correlation

$$\langle\eta(\mathbf{r}_1,t_1')\eta(\mathbf{r}_2,t_2')\rangle=2\frac{k_BT}{\gamma}\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(t_1'-t_2') . \quad (A7)$$

The physical ideas underlying (A6) are the same as in (2.6). In particular it gives a relaxation dynamics to an equilibrium solution e^{-F/k_BT}. This distribution gives the equilibrium probability of a configuration $\phi(\mathbf{r})$.

Expanding (A6) for small values of $\phi(\mathbf{r},t')$, its explicit form becomes

$$\partial_{t'}\phi(\mathbf{r},t')=\frac{1}{\gamma}[K_{11}\partial_y^2\phi+K_{22}\partial_z^2\phi+K_{33}\partial_x^2\phi+\chi_aH^2(\phi-\tfrac{2}{3}\phi^3)]+\eta(\mathbf{r},t') . \quad (A8)$$

In the rhs of (A8) we have neglected a term $(1/\gamma)(K_{33}-K_{11})\partial_x\phi\partial_y\phi$. This term can be neglected in an ordinary Landau expansion[18] in which only lowest-order terms in ϕ containing spatial derivatives are kept. A different reason for neglecting this term is that for ordinary MBBA, $K_{11}\simeq K_{33}$ (Ref. 24). In any case, if this term is kept, it does not contribute to our final equation (2.11).

A more detailed analysis of the dynamics specified by (A8) can be given by Fourier transforming $\phi(\mathbf{r},t')$ and $\eta(\mathbf{r},t')$. Proceeding in this way, we will be able to identify the more unstable modes involved in the relaxation processes that we study. The Fourier expansion is given by

$$\phi(\boldsymbol{\rho},z,t')=\sum_n\sum_{\mathbf{q}}\cos(2n+1)\frac{\pi z}{d}e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\theta'_{n,\mathbf{q}}(t') \quad (A9)$$

and analogously for $\eta(\mathbf{r},t')$. In (A9), $\boldsymbol{\rho}$ stands for a two-component position vector in a plane perpendicular to the relevant z direction. By inverse transforming the internal noise $\eta(\mathbf{r},t')$ it is easily seen that its Fourier modes are correlated through

$$\langle\eta_{m,\mathbf{q}_1}(t_1')\eta_{n,\mathbf{q}_2}(t_2')\rangle=2\left[\frac{2k_BT}{\gamma V}\right]\delta_{mn}\delta_{\mathbf{q}_1,-\mathbf{q}_2}\delta(t_1'-t_2') . \quad (A10)$$

Substituting (A9) and (A10) in (A8),

$$\begin{aligned}\lambda\dot\theta_{m,\mathbf{q}}(t')=&\{1-[(2m+1)^2(H_c/H)^2+\xi_2^2\mathbf{Q}^2]\}\theta'_{m,\mathbf{q}}(t')\\ &-\tfrac{2}{3}\sum_{\mathbf{q}_1,\mathbf{q}_2}\sum_{n,l,p}\theta'_{n,\mathbf{q}_1}(t')\theta'_{l,\mathbf{q}_2}(t')\theta'_{p,\mathbf{q}-(\mathbf{q}_1+\mathbf{q}_2)}(t')\left[\frac{2}{d}\int_{-d/2}^{d/2}dz\cos[(2m+1)\pi z/d]\cos[(2n+1)\pi z/d]\right.\\ &\left.\times\cos[(2l+1)\pi z/d]\cos[(2p+1)\pi z/d]\right]+\lambda\eta_{m,\mathbf{q}}(t') ,\end{aligned} \quad (A11)$$

where the following identifications have been made:

$$\lambda\equiv\frac{\gamma}{\chi_aH^2},\ \ \xi_2^2\equiv\frac{K_{22}}{\chi_aH^2} ,$$
$$H_c^2\equiv\frac{K_{22}\pi^2}{\chi_ad^2},\ \ \mathbf{Q}^2=\frac{K_{11}}{K_{22}}q_y^2+\frac{K_{33}}{K_{22}}q_x^2 . \quad (A12)$$

Linearizing (A11) we can do a linear stability analysis,

$$\lambda\dot\theta'_{m,\mathbf{q}}(t')=\xi_2^2(\mathbf{q}_c^2-\mathbf{Q}^2)\theta'_{m,\mathbf{q}}(t')+\lambda\eta_{m,\mathbf{q}}(t') , \quad (A13)$$

where $q_c(m)$ stands for the inverse of the characteristic length for the fluctuations of the mth mode in a plane perpendicular to the z axis

$$\mathbf{q}_c^2(m) \equiv \frac{1}{\xi_2^2}[1-(2m+1)^2(H_c/H)^2] . \quad \text{(A14)}$$

For a given m and $\mathbf{q}$, the mode $\theta'_{m,\mathbf{q}}(t')$ becomes unstable if the pair of inequalities

$$1-(2m+1)^2(H_c/H)^2>0, \quad \mathbf{q}_c^2>\mathbf{Q}^2 \quad \text{(A15)}$$

are both satisfied. If we are interested in relaxation processes not extremely close to the critical point $H=H_c$ we must focus our analysis on the most unstable mode, i.e., the fastest one. For fields $H_c<H<3H_c$, this corresponds to taking $m=0$, $q_x=2\pi/L_x$, $q_y=2\pi/L_y$, L_x,L_y being the dimensions of the sample in the x,y directions. It should be noted that for critical equilibrium dynamics the relevant modes would be the slowest ones, but here in what concerns the dynamics of relaxation processes, our interest lies in the fastes modes. Specializing the general equation (A11) for the mode $m=0,\mathbf{q}=0$ (the lower cutoff in the $\mathbf{q}$-space can be neglected in the limit $L\rightarrow\infty$), we obtain Eq. (2.11).

Equation (2.11) gives a model in which spatial fluctuations in the plane x-y are averaged out. It represents a strict mean-field theory for the most unstable mode $m=0$ of the spatially averaged quantity $\theta'_m(t')\equiv\theta'_{m,\mathbf{q}\rightarrow 0}(t')$,

$$\theta'_m(t')=\frac{1}{S}\int d\rho \left[\frac{2}{d}\right]\int_{-d/2}^{d/2} dz \cos(2m+1)\frac{nz}{d}\phi(\rho,z,t')$$

$$\equiv\frac{1}{S}\int d\rho\,\theta'_m(\rho,t') . \quad \text{(A16)}$$

The correlation length q_c^{-1} measures the spatial equilibrium fluctuations of the system in the x-y plane. In a mean-field study of a relaxation process the relevant fluctuations are the global fluctuations of the order parameter. In particular these are the fluctuations needed to analyze the nonequilibrium decay of an unstable state. They are well modeled by the random term $\xi'(t')$ in (2.11).

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