Virtual Hopf phenomenon: A new precursor of period-doubling bifurcations

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A new type of noise-induced precursor of a period-doubling bifurcation is described. The usual noise rise in the power spectrum at half the fundamental frequency, typical just before the onset of period doubling, may be preceded by a pair of symmetrically located noise bumps characteristic of a Hopf bifurcation precursor. The continuous evolution from the Hopf precursor to the period-doubling precursor may be understood in terms of the behavior of the Floquet multipliers as a control parameter is varied. The necessary appearance of the virtual Hopf phenomenon may be proven for a large class of low-dimensional systems, including the driven damped pendulum and the Lorenz equations, *between each successive period doubling in the infinite cascade* leading to chaos.

I. INTRODUCTION

This paper concerns the effect of external noise on systems that undergo period-doubling bifurcations. The bulk of previous work on this topic focused on the dynamics of discrete mappings,[1-3] rather than differential equations,[4] subject to a random perturbative term. That body of work emphasized the scaling properties for systems displaying an *infinite cascade* of period doublings, which accumulate at a critical parameter value marking the onset of chaos. A particularly interesting feature of this topic is the interplay of truly random noise and underlying deterministic chaotic dynamics.

More recently, a theory was developed to understand the effect of noise near the onset of instabilities of periodic solutions of ordinary differential equations.[5] Rather than looking at an entire sequence of bifurcations (such as the period-doubling cascade), the focus was on *single* bifurcations. It was found that external noise induced new broadband lines in the observed power spectrum: The nearer the bifurcation point, the more prominent are the noise-induced lines. Furthermore, each general class of instability—in the usual sense of bifurcation theory for periodic orbits[6]—has its own characteristic noise-induced spectrum. These spectra have been computed for the five most common codimension-one bifurcations: saddle-node, transcritical, pitchfork, period-doubling, and Hopf bifurcations.[5] (The codimension-one bifurcations are the instabilities typically encountered as a single parameter is varied.)

It follows that, if external noise is present, the type of instability to be suffered by a system can be deduced *before* the onset of the bifurcation. The noise-induced power spectra have consequently been named "noisy precursors" of their respective dynamical instabilities.

The theoretical predictions for the height, width, and shape of the precursor peaks as the bifurcation point is approached have been tested by experiments on voltage-driven p-n junctions.[7] The experimental results are in excellent agreement with the theoretical predictions, for the two kinds of bifurcations tested, namely, period-doubling and Hopf bifurcations.

Having decided that the theory of noisy precursors gives a good basic understanding of the effects of noise near the onset of simple instabilities, one can look for new predictions based on the theory. One such prediction—the "virtual Hopf" phenomenon—is the main subject of this paper. Qualitatively, this phenomenon refers to a sequence of events (as some parameter varies) whereby the noisy precursor of a Hopf bifurcation continuously changes into the precursor characteristic of a period-doubling bifurcation. Although experimental observation of this phenomenon has not yet been reported in the literature, one may prove that it must occur in certain systems (see Sec. III).

An understanding of the virtual Hopf phenomenon allows one to make contact with a somewhat different problem, namely, the way in which noise makes unobservable the very high period bifurcations in the period-doubling route to chaos. As mentioned above, this is a problem which has been directly addressed in previous work on noisy discrete mappings. To describe this phenomenon, Crutchfield and Huberman introduced the notion of a "bifurcation gap."[4] One way of understanding the gap is to average the deterministic dynamical behavior over a window of parameter values[1]—this picture leads to good quantitative results. In the present paper the bifurcation gap is viewed in a new light by using the derived results for the power spectra associated with the virtual Hopf phenomenon. This is done by focusing on the driven, damped pendulum, which is known to undergo an infinite cascade of period doublings[8-10] and also to undergo the virtual Hopf phenomenon between each successive period doubling (see Sec. IV).

This paper is organized as follows. The main ideas and predictions of the theory of noisy precursors are summarized in Sec. II. The virtual Hopf phenomenon is introduced in Sec. III, where its existence is proven for a specific example, namely, the driven damped pendulum. Section IV gives a discussion of the bifurcation gap: First a brief review of the gap based on scaling ideas for discrete mappings is presented, and then the gap is reconsidered in terms of the virtual Hopf scenario. Finally, a discussion centered on the *generality* of the virtual Hopf phenomenon—and the likelihood of observing it in experiments—is presented in Sec. V.

II. REVIEW OF NOISY-PRECURSOR THEORY (REF. 5)

The theory of noisy precursors examines the effect of external random noise on the power spectrum of nonlinear systems having periodic output. In the absence of noise the power spectrum consists of sharp spikes at integer multiples of the fundamental frequency. Imagine that as a parameter λ is varied, the deterministic system can undergo a dynamical instability at some critical value λ_0. Then for λ near λ_0, external noise induces new broadband peaks in the power spectrum. These lines become more prominent as λ approaches λ_0.

To gain a qualitative understanding of the origin of the noisy-precursor lines, consider the situation depicted in Fig. 1. For definiteness, suppose that the system is close to the onset of a period-doubling instability, and focus on an initial condition $\mathbf{q}_0$ lying just off of the stable T-periodic orbit $\mathbf{x}_0$ [Fig. 1(a)]. As the orbit through $\mathbf{q}_0$ relaxes to the limit cycle $\mathbf{x}_0$, it repeatedly intersects the Poincaré section P, at points $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots$. Figure 1(b) illustrates that successive intersections approach the fixed point $\mathbf{q}_\infty$ in an *alternating fashion.* (This is why a period doubling is sometimes called a "flip bifurcation" in the theory of discrete mappings.[6]) Consequently, the transient part of the orbit is a damped, $2T$-periodic oscillation. The power spectrum of such an orbit will have a broad peak at circular frequency $\omega=\pi/T$—that is, at half the fundamental frequency of the stable orbit $\mathbf{x}_0$. In a purely deterministic system the trajectory would settle down to the attractor $\mathbf{x}_0$ for all times, and the transient would contribute only negligibly to the measured power spectrum. The effect of external noise is to continually kick the system off of the stable orbit, so that the transient behavior contributes significantly to the observed spectrum.

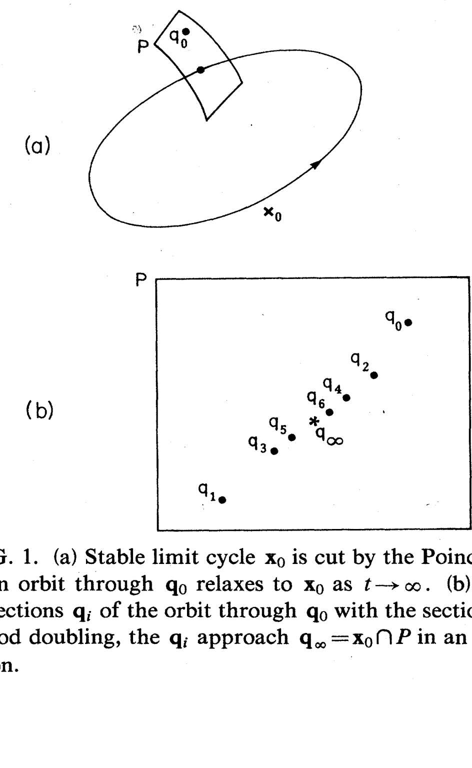

FIG. 1. (a) Stable limit cycle $\mathbf{x}_0$ is cut by the Poincaré section P: An orbit through $\mathbf{q}_0$ relaxes to $\mathbf{x}_0$ as $t\to\infty$. (b) Successive intersections $\mathbf{q}_i$ of the orbit through $\mathbf{q}_0$ with the section P. Near a period doubling, the $\mathbf{q}_i$ approach $\mathbf{q}_\infty=\mathbf{x}_0\cap P$ in an alternating fashion.

The quantitative analysis of the above model begins with the set of differential equations governing the deterministic system, augmented by a small stochastic term $\xi(t)$,

$$\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x};\lambda)+\xi, \quad \mathbf{x}\in\mathbb{R}^N \tag{2.1}$$

where λ is a tunable parameter, and ξ is white noise,

$$\langle\xi(t)\rangle=0, \quad \langle\xi_i(t)\xi_j(t+\tau)\rangle=\Gamma_{ij}\delta(\tau) , \tag{2.2}$$

so that the Γ_{ij} give a measure of the strength of the noise. Let $\mathbf{x}_0$ be a T-periodic solution of the noise-free system:

$$\mathbf{x}_0(t+T)=\mathbf{x}_0(t) . \tag{2.3}$$

For small deviations $\boldsymbol{\eta}=\mathbf{x}-\mathbf{x}_0$, one may linearize Eq. (2.1) about $\mathbf{x}_0$, with the result

$$\dot{\boldsymbol{\eta}}=[D\mathbf{F}(\mathbf{x}_0;\lambda)]\cdot\boldsymbol{\eta}+\xi , \tag{2.4}$$

where $D\mathbf{F}$ is the matrix of periodic functions

$$(D\mathbf{F})_{ij}=\left.\frac{\partial F_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_0} . \tag{2.5}$$

For convenience, assume time is rescaled so that $D\mathbf{F}$ has period 2π. Equation (2.4) is linear with periodic coefficients, and an exact solution can be constructed using the results from Floquet theory.[11] It is the *Floquet multipliers* μ_k that play the central role in such an analysis: They are defined in terms of special solutions $\boldsymbol{\psi}_k$ of the homogeneous equation associated with Eq. (2.4), with the property

$$\boldsymbol{\psi}_k(t+2\pi)=\mu_k\boldsymbol{\psi}_k(t) . \tag{2.6}$$

Since Eq. (2.1) is real, the multipliers μ_k are either real or come in complex-conjugate pairs. Moreover, stability of the basic oscillation $\mathbf{x}_0$ requires that all of the μ_k lie inside the unit circle:

$$|\mu_k|\leq 1 . \tag{2.7}$$

An instability occurs when one or more of the μ_k exits the unit circle—close to the onset of the instability, it is the near-critical multiplier(s) μ_0 that dominate the observed power spectrum. For codimension-one bifurcations, there exists only a single critical μ_0 (saddle-node, transcritical, pitchfork, and period-doubling cases) or a critical pair μ_0,μ_0^* (Hopf case). Figure 2 summarizes how the noisy-precursor lines depend on the value of μ_0. The size and shape of the precursors depend on the distance of μ_0 from the unit circle—writing $|\mu_0|=1-\epsilon$, the precursors sharpen and grow as the bifurcation point $\epsilon=0$ is approached. The position of the precursors depends on the argument θ of μ_0.

For example, a period-doubling bifurcation corresponds to a single multiplier exiting the unit circle at -1, so a noisy-precursor line occurs at $\omega=\frac{1}{2}=$ half the fundamental (and at odd-integer multiples of this frequency). A Hopf bifurcation corresponds to a pair of critical multipliers exiting at $e^{i\theta},e^{-i\theta}$, yielding pairs of precursor lines at $\omega=\theta$ and $1-\theta$ (and at $m\pm\theta$ for all positive integers m).

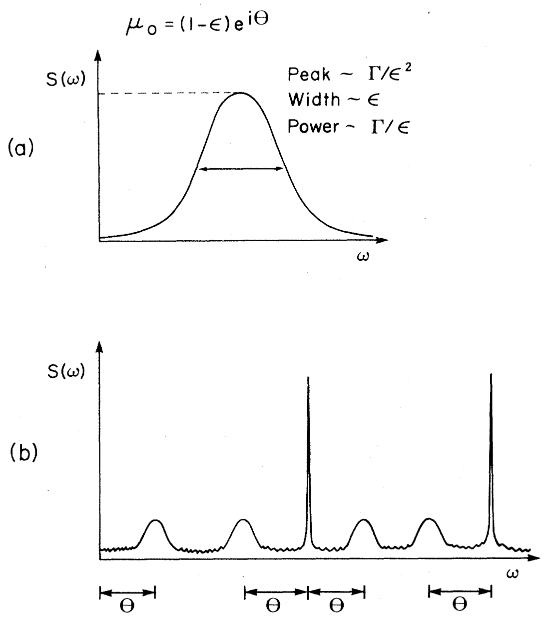

FIG. 2. (a) Scaling of the height, width, and area of the Lorentzian precursors as a function of Γ, the external noise strength, and ϵ, the distance of the near-critical multiplier μ_0 from the unit circle. (b) Position of the precursors depends on θ, the argument of μ_0.

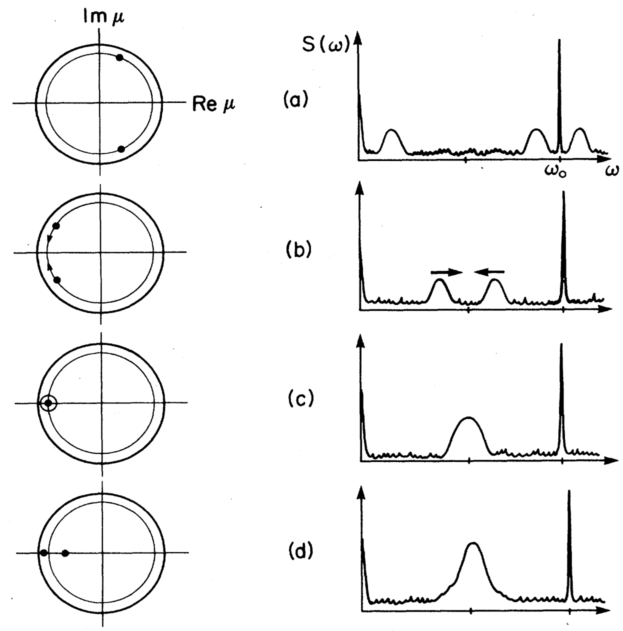

FIG. 3. Behavior of the Floquet multipliers (left-hand side) and the corresponding power spectra (right-hand side) for the virtual Hopf phenomenon. As a parameter is varied, the precursor characteristic of a Hopf bifurcation (a) changes into the precursor of a period-doubling instability (d).

The basic predictions of Fig. 2 are in excellent agreement with experiments performed on driven p-n junctions.[7] We therefore conclude the following: To understand the effect of noise on a dynamical system, understand first the behavior of the Floquet multipliers. It has been emphasized elsewhere that the effect of noise depends intimately on the Liapunov exponents of the underlying dynamical system.[1,12] In the present case, in which the system possesses a stable periodic orbit, there is a simple relationship between the Floquet multipliers μ_k and the corresponding Liapunov exponents λ_k:

$$\ln|\mu_k| = \lambda_k \ . \tag{2.8}$$

Physically, $-\lambda_0$ gives the relaxation rate (near a bifurcation) for the noise-free system after experiencing a small impulse perturbation.

III. THE VIRTUAL HOPF PHENOMENON

We are now ready to launch into a description of the main subject of this paper: the virtual Hopf phenomenon. Specifically, consider the sequence of events illustrated in Fig. 3. The left-hand side of Figs. 3(a)–3(d) shows the hypothetical behavior of a pair of near-critical Floquet multipliers as some parameter is varied, and the corresponding power spectra one would observe if noise was present are shown on the right-hand side.

Initially, the complex-conjugate pair lies very close to the unit circle [Fig. 3(a)], a situation that generally occurs just before the onset of a Hopf bifurcation. However, instead of exiting the unit circle, the multipliers move along the circle $|\mu| = 1-\epsilon$ [Fig. 3(b)], until they meet on the negative real axis [Fig. 3(c)]. The corresponding power spectra show the precursor bumps shifting and then merging at half the fundamental frequency $\omega = \omega_0/2$. Upon changing the parameter further, the two multipliers split up along the negative real axis [Fig. 3(d)] until one of them crosses the unit circle at $\mu = -1$, heralding the onset of a period-doubling bifurcation. During this last phase, the power spectrum shows the usual noisy precursor characteristic of a period-doubling instability.

The sequence of power spectra in Fig. 3 is referred to as the virtual Hopf phenomenon, since the noisy precursor of a Hopf bifurcation—instead of giving way to the actual Hopf instability—changes continuously into the precursor of a period-doubling bifurcation.

The following question now arises: Is the sequence of events of Fig. 3 a common precursor of period doubling, or does it represent a very rare phenomenon? It is now shown that the virtual Hopf phenomenon is not rare.

To show that it happens at all, consider the equation

$$\beta_L\beta_C\dddot{\delta} + \beta_C\ddot{\delta} + \dot{\delta}(1+\beta_L\cos\delta) + \sin\delta = I \tag{3.1}$$

which was first studied in connection with a dc-biased Josephson junction circuit. The solutions of Eq. (3.1) show a wide variety of dynamical behavior as the three dimensionless parameters β_L, β_C, and I are varied. Of relevance here is that, for most values of β_L and β_C studied, a simple oscillation $\delta_0(t)$ undergoes a period-doubling bifurcation as I is lowered from a high value.[13–16] This system is particularly noteworthy because it has proven possible to accurately determine, using *analytic calculations,* both the basic oscillation $\delta_0(t)$ and the value of I at which the onset of period doubling occurs.[15,16] The suc-

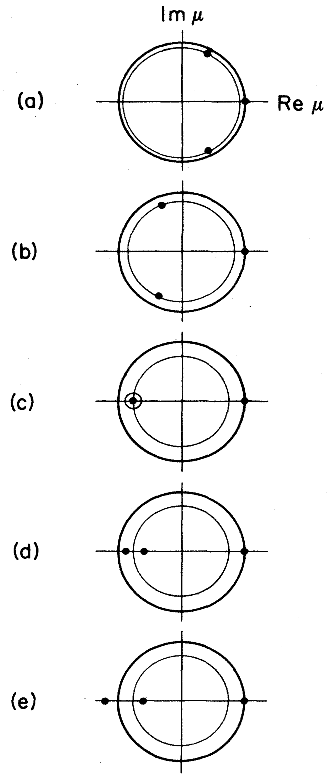

FIG. 4. Computed behavior of the Floquet multipliers for the system (3.1), with $\beta_L=3.0$, $\beta_c=0.1$, and (a) $I=10.0$, (b) $I=7.5$, (c) $I=5.1$, (d) $I=5.0$, (e) $I=4.9$.

cess of the analytic approximations rests on the fact that $\delta_0(t)$ has very little higher harmonic content. The reliable representation for $\delta_0(t)$ means that one may reliably compute the Floquet multipliers μ_k for this system. As was shown in an earlier paper,[5] for $\beta_L=3.0$, $\beta_C=0.1$, the μ_k behave as in Figs. 4(a)–4(e) as I is lowered from 10.0. This is essentially the same sequence as Fig. 3 and represents a directly calculated example of the virtual Hopf phenomenon.

[A technical point should be explained here. In systems governed by autonomous equations, such as (3.1), there is always a single multiplier at $\mu=+1$. Geometrically, this corresponds to the fact that the orbit is neutrally stable to perturbations along the limit cycle. The chief effect of this multiplier is to broaden the sharp lines in the power spectrum arising from the basic oscillation $\mathbf{x}_0$.[5] The remaining features of the noisy-precursor spectrum are governed by the behavior of the other, unconstrained multipliers, as summarized in Sec. II.]

In general, the direct analytic computation of μ_k is impractical, since one must first compute explicitly a stable periodic solution of the governing nonlinear differential equation. Nevertheless, it will now be shown that the behavior illustrated in Fig. 3 must occur in one of the most pedestrian of dynamical systems, namely, the driven damped pendulum

$$\ddot{\theta}+\gamma\dot{\theta}+\omega_0^2\sin\theta=A+B\cos t\ . \tag{3.2}$$

This may be written in the standard form

$$\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x},t),\quad \mathbf{x}\in\mathbb{R}^2 \tag{3.3}$$

with

$$\mathbf{x}=\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix}\dot{\theta}\\ -\gamma\dot{\theta}-\omega_0^2\sin\theta+A+B\cos t\end{bmatrix}. \tag{3.4}$$

The dynamics of this system has been documented extensively in the physics literature.[8–10] As the four parameters γ, ω_0, A, and B are varied, the system exhibits a variety of instabilities. Of particular interest here is the observed cascade of period doublings leading to chaos as, say, the parameter B is increased from zero.

The main point is this: *Between any two period doublings in the infinite cascade, the Floquet multipliers behave as in Fig. 5.* Consequently, in the presence of external noise, the power spectrum will show the virtual Hopf sequence over and over again.

The proof of this assertion relies on a theorem[11] constraining the product of all the Floquet multipliers μ_k:

$$\prod_{k=1}^{N}\mu_k=\exp\left[\int_0^T \operatorname{div}\mathbf{F}(\mathbf{x}_0)dt\right], \tag{3.5}$$

where $\mathbf{x}_0(t)$ is a stable T-periodic solution, and $\mathbf{F}$ is the vector field of the governing differential equation [see, e.g., Eq. (3.4)]. The constraint Eq. (3.5) has a simple physical interpretation.[17,18] The left-hand side gives the factor by which volumes of phase space expand or contract under one iteration of the Poincaré return map (see

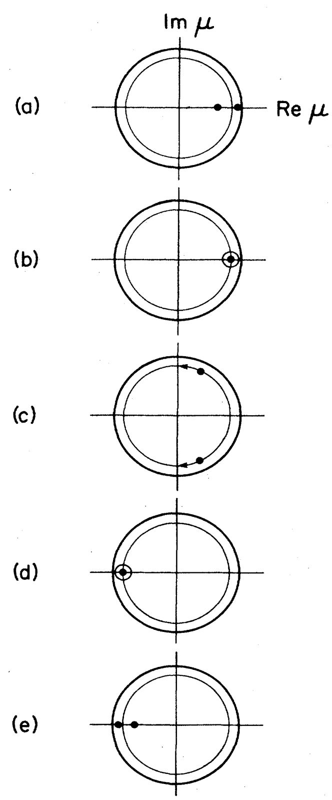

FIG. 5. Behavior of the two Floquet multipliers for the driven, damped pendulum (3.2), between any two successive period-doubling bifurcations.

Fig. 1). The right-hand side is the change in the infinitesimal comoving volume V,

$$\frac{1}{V}\frac{dV}{dt}=\mathrm{div}\,\mathbf{F}(\mathbf{x}_0(t)) ,$$

integrated over one period of the orbit $\mathbf{x}_0$.

For Eq. (3.3), div **F** is a constant,

$$\mathrm{div}\,\mathbf{F}=\mathrm{Tr}\,(D\mathbf{F})=-\gamma , \tag{3.6}$$

and since the system is two dimensional, there are only two Floquet multipliers, so Eq. (3.5) becomes

$$\mu_1\mu_2=e^{-\gamma T}<1 \tag{3.7}$$

since we assume that the pendulum is truly damped, and thus $\gamma>0$. From Eq. (3.7), one may conclude two important facts regarding the Floquet multipliers for this system.

(i) If μ_1 and μ_2 are complex (and therefore $\mu_1=\mu_2^*$) then they lie on the circle

$$|\mu|^2=e^{-\gamma T} \tag{3.8}$$

which lies inside the unit circle.

(ii) If μ_1 and μ_2 are real, then stability of the basic solution $\mathbf{x}_0$ requires $|\mu_k|<1$, so

$$e^{-\gamma T/2}<|\mu_k|<1, \quad k=1,2 . \tag{3.9}$$

Together, Eqs. (3.8) and (3.9) demand the occurrence of the virtual Hopf sequence between any two successive period doublings. Consider the parameter window $B_1<B<B_2$, where B_1 and B_2 are the bifurcation points for consecutive period doublings. For B slightly greater than B_1, one of the multipliers must lie near $+1$ (since, by *decreasing* B, the system would undergo a pitchfork bifurcation). If ϵ denotes a small, positive number, then one may write

$$\mu_1=1-\epsilon, \quad \mu_2=\frac{e^{-T\gamma}}{1-\epsilon} . \tag{3.10}$$

How can one of these multipliers move continuously from the positive real axis [Eq. (3.10)] to the point $\mu=-1$ and still satisfy the necessary constraint conditions (3.8) and (3.9)? The only way is [see Figs. 5(a)—5(e)] for them to meet at $\mu=e^{-\gamma T/2}$, move along the circle (3.8) as a complex-conjugate pair until they meet again at $\mu=-e^{-\gamma T/2}$, and then split up along the negative real axis. Figures 5(c)—5(e) are precisely the virtual Hopf sequence depicted in Fig. 3, as claimed.

Note that the above proof rests on only three facts: the existence of consecutive period doublings, the two-dimensional governing equation (so that there are only two multipliers), and that div **F** is strictly negative. As a result, the argument holds for other driven oscillators as well (such as the driven Duffing oscillator[19,20]). Moreover, since autonomous systems necessarily have one multiplier constrained to be $\mu=+1$, the proof is easily extended to include *third-order autonomous* systems with div $\mathbf{F}<0$ that display successive period doublings, such as the Lorenz equations[21,22] and the Koch-Miracky equation (3.1).

The question as to whether the virtual Hopf phenomenon might occur in higher-dimensional systems is addressed in Sec. V.

Note that it has been assumed implicitly that the damping is finite, $\gamma=-\mathrm{div}\,\mathbf{F}<\infty$. Otherwise, a multiplier could pass continuously from $+1$ to -1 along the real axis without violating constraint (3.7). In terms of the Poincaré return map, this implies that phase-space volumes contract by a finite amount per iteration. This is clearly a necessary physical property for flows, since one must be able to follow trajectories uniquely both forward and backward in time. In the more general study of discrete dynamical systems, however, it is possible to have infinite contraction, allowing eigenvalues to pass through the origin.

IV. BIFURCATION GAP REVISITED

Aside from being an interesting phenomenon in its own right, the virtual Hopf sequence provides a new way of viewing the so-called bifurcation gap first described by Crutchfield and Huberman.[4] This section begins with a brief review of the basic features of the bifurcation gap. Then, the gap is reevaluated in terms of the noisy-precursor picture, and an explicit expression for the power spectrum during the virtual Hopf sequence is produced. The two views are compared briefly in Sec. V.

A. The bifurcation gap

In the absence of noise the period-doubling route to chaos consists of an infinite number of bifurcations which accumulate geometrically at some critical parameter value.[23,24] Crutchfield and Huberman[4] first studied the effect of random noise on a differential equation known to possess a period-doubling cascade: a driven, damped oscillator in a quartic potential. That work, carried out on an analog computer, showed that the noise truncates the observable sequence after a finite number of bifurcations. The authors introduced the notion of a bifurcation gap to describe the manner in which noise washed out the fine structure of the deterministic bifurcation diagram. The simulations showed evidence that the size of the bifurcation gap scaled with the input noise level.

One way of quantifying the effect of the noise, and the one of interest for this paper, was based on extensive digital computations performed for the discrete logistic map.[1] First, one defines κ_p as the maximum input noise strength for which one can resolve an orbit of (at most) period p. That is, the slightest increase of noise above κ_p will wash out the period p orbit, making the largest observable periodic orbit $\frac{1}{2}p$ periodic. Scaling arguments lead to the result[1]

$$\frac{\kappa_p}{\kappa_{2p}}=\delta^{2/\gamma} , \tag{4.1}$$

where δ is the familiar Feigenbaum constant ($\delta=4.699\,20\ldots$), and γ is a new scaling exponent introduced to describe the scaling between the noise strength and the parameter value for which the Lyapunov characteristic exponent first becomes positive. This exponent was determined from numerical simulations to be $\gamma=0.82\pm0.01$, so that Eq. (4.1) becomes

$$\frac{\kappa_p}{\kappa_{2p}}=\kappa^2, \quad \kappa=6.62 . \tag{4.2}$$

Thus, raising the noise level by 16.4 dB should obscure the highest-order period-doubling bifurcation.

A similar understanding of the bifurcation gap based on the effect of noise on a one-dimensional finite-difference equation has also been reached using nonequilibrium field-theoretic techniques[3] and via a renormalization-group approach,[2] the latter approach yielding κ to several significant figures: $\kappa=6.619\,03$.

Strictly speaking, Eq. (4.1) holds only asymptotically as $p\rightarrow\infty$, since it is based on scaling arguments valid in this limit. Nevertheless, such scaling arguments often give quantitatively good results away from the accumulation point for the period-doubling cascade. Indeed, the prediction Eq. (4.1) agrees with measurements made on a periodically driven p-n junction.[25]

B. Power spectra for the virtual Hopf sequence

Now consider the effect of noise on the period-doubling cascade in terms of noisy precursors. In particular, consider the driven pendulum, Eq. (3.2), immediately after a period-doubling bifurcation to an orbit of period T. The multipliers thus sit as illustrated in Fig. 5(a). Consequently, there is a noise bump centered at the new fundamental frequency $\omega_0=2\pi/T$. Since the deterministic frequency spike at ω_0 grows continuously (as a function of parameter λ) from zero height at the bifurcation point λ_0, one must sweep the parameter some finite amount past the bifurcation point in order to unambiguously view the new frequency spike above the background noise.

As λ is increased further, two things happen to help resolve the new spike above the noise: the spike at ω_0 continuous to grow, while the precursor bump diminishes. The key point here is that the growth of the spike and the diminution of the noise bump *may be insufficient* to enable the spike to be unambiguously observable above the noise background. This would effectively wash out the bifurcation.

To determine the condition for obscuring the spike at ω_0, consider again the virtual Hopf sequence of Fig. 5 and see what happens to the noise level at $\omega=\omega_0$. Between Figs. 5(a) and 5(b), the precursor bump, centered at ω_0, diminishes. Between Figs. 5(b) and 5(d), this bump splits in half, and the two daughters shift their centers continuously, without changing their size and shape. During this phase, again, the amount of noise at ω_0 steadily decreases. Just at Fig. 5(d) the daughters merge into a single bump at $\omega=\omega_0/2$, and this bump grows [Fig. 5(e)] as the next period doubling is approached. During this final phase, the noise level at $\omega=\omega_0$ essentially remains constant (in fact, it increases slightly) even though the precursor bump becomes narrower.

Consequently, the minimum noise level at ω_0 during the virtual Hopf sequence occurs when the two multipliers meet on the negative real axis. The noise spectrum at this parameter value is now computed explicitly.

Assume, first of all, that the stable T-periodic solution $\theta_0(t)$ of Eq. (3.2) has been found. For convenience, imagine that time is rescaled so that $T=2\pi$—the task then is to find the value of the power spectrum $S(\omega)$ at $\omega=1$. The deviation η from the periodic solution satisfies the linearized equation

$$\ddot{\eta}+\gamma\dot{\eta}+[\omega_0^2\cos\theta_0(t)]\eta=\xi(t) , \tag{4.3}$$

where the noise $\xi(t)$ is taken to be zero mean and δ correlated with variance Γ:

$$\langle\xi(t)\rangle=0, \quad \langle\xi(t)\xi(t')\rangle=\Gamma\delta(t-t') . \tag{4.4}$$

The homogeneous equation associated with Eq. (4.3) has the special Floquet solutions

$$\chi_k(t+2\pi)=\mu_k\chi_k(t), \quad k=1,2 . \tag{4.5}$$

In general, the μ_k satisfy the constraint [see Eq. (3.7)]

$$\mu_1\mu_2=e^{-2\pi\gamma} . \tag{4.6}$$

There is a technical point to consider: The existence of two linearly independent solutions of the form Eq. (4.5) requires that the two multipliers be distinct, in which case the results of Ref. 5 are immediately relevant. However, the case of interest here corresponds to

$$\mu_1=\mu_2=-e^{-\pi\gamma} . \tag{4.7}$$

To rectify the situation, the calculation will be carried out for the μ_k sitting on the negative real axis, with $\mu_1\neq\mu_2$ [as in Fig. 5(e)], and taking the limit as $\mu_1\rightarrow\mu_2$.

We are now in the position to compute the correlation function $C(\tau)$:

$$C(\tau)=\langle\langle\eta(t)\eta(t+\tau)\rangle_{\text{ens}}\rangle_t , \tag{4.8}$$

where the $\langle\ \rangle_{\text{ens}}$ denotes an ensemble average and $\langle\ \rangle_t$ denotes a time average. The general derivation of Ref. 5 may be immediately applied to the special case of two multipliers, yielding

$$\langle\eta(t)\eta(t+\tau)\rangle_{\text{ens}}=\sum_{r,k=1}^{2}\Gamma h_r(t)h_k(t+\tau)e^{-\gamma/2\tau} \times\int_0^t e^{-\gamma(t-t')}g_r(t')g_k(t')dt' , \tag{4.9}$$

where the limit Eq. (4.7) has been taken, and where the functions $h_n(t)$, $g_n(t')$ are 4π-periodic functions of time, such that

$$h_n(t+2\pi)=-h_n(t) ,$$
$$g_n(t'+2\pi)=-g_n(t') . \tag{4.10}$$

A detailed derivation of Eq. (4.9) can be found in Ref. 5: As will be seen, an explicit representation for the functions h_n, g_n is unimportant for the present purposes.

In any measurement of a power spectrum one needs to take a time series of much longer duration than any relevant characteristic time of the system being studied. Consequently, the following ordering is appropriate:

$$t\gg\gamma^{-1}\gg T . \tag{4.11}$$

In the case of light damping ($\gamma\ll1$), the integral appearing in Eq. (4.9) has a simple form. The integrand is the product of a fast oscillating factor and a slow decaying exponential. In view of the ordering (4.11), one has

$$\int_0^t e^{-\gamma(t-t')} g_r(t') g_k(t') dt' \simeq \frac{Q_{rk}}{\gamma} , \quad (4.12)$$

where Q_{rk} is the average value of the oscillating function $g_r(t')g_k(t')$.

Equation (4.9) thus becomes

$$\langle \eta(t)\eta(t+\tau) \rangle_{\text{ens}} = \sum_{r,k=1}^{2} \Gamma \frac{Q_{rk}}{\gamma} h_r(t) h_k(t+\tau) e^{-\gamma/2\tau} , \quad (4.13)$$

and the correlation function $C(\tau)$ is the time average of this,

$$C(\tau) = \frac{\Gamma}{\gamma} e^{-\gamma/2\tau} \Lambda(\tau) , \quad (4.14)$$

where

$$\Lambda(\tau) = \sum_{r,k=1}^{2} \langle Q_{rk} h_r(t) h_k(t+\tau) \rangle_t \quad (4.15)$$

is a 4π-periodic function of τ, such that

$$\Lambda(\tau + 2\pi) = -\Lambda(\tau) , \quad (4.16)$$

this last fact following from Eq. (4.10). Expanding $\Lambda(\tau)$ in a Fourier series yields

$$\Lambda(\tau) = \sum_{n \text{ odd}} \alpha_n e^{in\tau}, \quad \alpha_{-n} = \alpha_n^* , \quad (4.17)$$

the even-n terms in the summation being absent owing to Eq. (4.16).

Finally, the power spectrum $S(\omega)$ is given by the Fourier transform of Eq. (4.14),

$$S(\omega) = \Gamma \sum_{n \text{ odd}} \frac{\alpha_n}{(\gamma/2)^2 + (\omega - n/2)^2} , \quad (4.18)$$

so that the broadband part of the power spectrum is a sum of Lorentzians centered at $\omega = n/2$, where n is an odd integer.

From Eq. (4.18) come two important facts. First, the output noise level at any fixed value of ω is proportional to the input noise intensity. Second, for small damping ($\gamma \ll 1$), the noise level away from the frequencies $\omega = n/2$ is essentially independent of γ.

Equation (4.18) gives the *smallest* value for the noise spectrum (at $\omega = 1$) during the virtual Hopf sequence. The next question is as follows: How large does the spike at $\omega = 1$ grow, and will it become larger than the noise level at $\omega = 1$? The answer comes from a result originally due to Feigenbaum,[26] which gives a scaling relation for the maximum amplitude attained by successive subharmonics in a period-doubling cascade. The result states that the maximum power level drops by a factor of $(6.56)^2$ at each bifurcation.[26]

Thus, suppose the noise level given by Eq. (4.15) is sufficiently large that the frequency spike at $\omega = 1$ cannot be resolved. Then, since Eq. (4.18) varies linearly with Γ, increasing Γ by a factor of $(6.56)^2$ will surely obscure the previous period doubling as well. In terms of the notation of Eq. (4.2),

$$\frac{\kappa_p}{\kappa_{2p}} = (6.56)^2 \quad (4.19)$$

which should be compared with Eq. (4.2).

In summary, the result (4.19) rests on three facts. First, an understanding of how the precursor peaks vary between each bifurcation in the infinite cascade tells one to focus on the calculation of the power spectrum at the point where Eq. (4.7) holds. Second, this calculation shows that the minimum noise level at the fundamental frequency is just proportional to Γ. Finally, the universal scaling of successive spikes in the power spectrum leads to Eq. (4.19).

It should be emphasized that Shraiman *et al.*[3] first pointed out that the scaling argument leading to Eqs. (4.1) and (4.2) agrees numerically very closely with the ratio of successive subharmonics in the power spectrum. In one sense, the derivation leading to Eq. (4.19) simply exploits this "coincidence." On the other hand, the derivation of Eq. (4.19) shows that this is not merely a numerical coincidence: Feigenbaum's subharmonic scaling relation, together with the theory of noisy precursors, leads to Eq. (4.19) without any additional scaling hypothesis.

V. DISCUSSION AND CONCLUSIONS

The theory of noisy precursors studies the effect of noise on ordinary differential equations (ODE's). One finds that the central role in the analysis is played by the Floquet multipliers. The virtual Hopf phenomenon corresponds to a specific behavior of the multipliers preceding a period-doubling bifurcation, in which the power spectrum first shows precursors characteristic of a Hopf bifurcation, but then evolves into the period-doubling precursor. Not only is the corresponding behavior of the multipliers hypothetically possible, but (as demonstrated in Sec. III) it *must* occur between each successive period doubling in the driven, damped pendulum. The proof of this was possible because of the low dimensionality of that system's governing equation—in low-dimensional systems, Eq. (3.5) is a very severe constraint. The proof also required that the dissipation was finite. While this might strike one at first as a somewhat technical point, a moment's reflection shows that it is related to the practical issue of the *likelihood* of experimentally resolving the virtual Hopf phenomenon. The magnitude of the dissipation [γ in Eq. (3.2)] determines how close the complex-conjugate multipliers are to the unit circle during the virtual Hopf sequence. Thus, the size of the precursor bumps depends on the dissipation [Fig. 2(a)]—the smaller the dissipation, the more prominent the precursors. Consequently, while the dissipation for an ODE will naturally be finite, the probability of detecting the virtual Hopf phenomenon above the background will be enhanced if the dissipation can be lowered as much as possible. It should be clear that the proof of Sec. III is easily extended to *all* second-order nonautonomous systems, with positive dissipation, which are known to undergo successive period doublings. Furthermore, the argument extends to all third-order autonomous systems with positive dissipation [such as Eq. (3.1) and the Lorenz equations] which might undergo successive period doublings. This is because there is an additional constraint in the autonomous case that one multiplier always be $+1$.

Is the virtual Hopf phenomenon thus limited to second-order nonautonomous and third-order autonomous systems? (It cannot occur in still-lower-dimensional systems, since it requires the existence of a pair of complex-conjugate multipliers.) This remains an open question; however, it is tempting to think that, since systems undergoing period-doubling cascades often behave like very-low-dimensional systems, the virtual Hopf phenomenon might be fairly typical.

A theory analogous to that developed for ODE's is possible for discrete mappings subject to noise, the central role being played by the eigenvalues of the mapping linearized about the bifurcating fixed point. Such a theory will have all of the same codimension-one precursors derived in Ref. 5. A linear-response calculation, representing an important start in this direction, has been carried out for period doubling of a one-dimensional discrete mapping.[27] There is one interesting difference between the noisy precursors for the two cases, however. While period doubling can occur in one-dimensional mappings, the virtual Hopf phenomenon requires at least a two-dimensional (real) mapping. This is because the existence of a pair of complex-conjugate eigenvalues requires at least two eigenvalues, which in turn requires that the mapping be (at least) two dimensional. In contrast, the lowest-dimensional ODE allowing period-doubling bifurcations (second order, driven) can also show the virtual Hopf phenomenon.

This last point suggests a "test" for our understanding of the bifurcation gap. Suppose we have a high-dimensional system of ODE's that is known to undergo a period-doubling cascade. It is often pointed out that these systems behave very much like a corresponding low-dimensional system, for example, a one-dimensional noninvertible mapping. But it could be, in order to properly capture the effect of external noise, that the system behaves more like (say) a two-dimensional mapping that shows a virtual Hopf sequence. Perhaps this point should not be emphasized too greatly, for even if it is true, the much simpler one-dimensional map might prove perfectly adequate so far as deterministic effects are concerned.

Finally, the virtual Hopf sequence provides one with a new way to view the period-doubling bifurcation gap. The two viewpoints described in Sec. IV—one based on discrete mappings, the other on the noisy precursors for ODE's—share some things in common. In particular, they both rely on the deterministic scaling laws of systems. On the other hand, the former understanding rests either on the notion of averaging the deterministic dynamics over a range of parameter values to model the effect of noise[1] or on additional scaling laws for the noisy mapping, whereas the latter is based on the role of *transient* orbits and their contribution to the observed power spectrum in the presence of noise.

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