Model for Taylor-Couette flow

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Model equations for circular symmetric Taylor-Couette flow are derived in the narrow-gap limit by a mode truncation of the full Navier-Stokes equations. For static Taylor numbers these model equations can be transformed into the Lorenz equations for Rayleigh-Bénard convection. Linear growth rates of the most unstable modes and the torque on the cylinders evaluated within this model reproduce well the known results. The response to periodic modulation of the forcing is different from that of the convective Lorenz model.

I. INTRODUCTION

Taylor-Couette flow and Rayleigh-Bénard convection display hydrodynamic instabilities, which have been studied quite intensively. Although the origin of the instabilities as well as the geometries differ, there are similarities. It has been pointed out,[1] that the Taylor-Couette problem for narrow gap is analogous to that of Rayleigh-Bénard convection. Thus one might hope that approximations of the flow in the latter system are also applicable in the former. Such an approximation was suggested by Lorenz[2] for convection in a heated fluid layer close to threshold by truncating the full hydrodynamic field equations to a few modes. Here we shall derive by a similar procedure the analogous model equations for Taylor vortex flow between concentric cylinders.

To that end we consider the narrow-gap case explained in Sec. II. In analogy to the convective Lorenz model we assume a free-slip boundary condition in axial direction, i.e., in the turning direction of the Taylor vortex rolls. Our mixed boundary conditions—the tangential ones are rigid—and the mode truncation of the Navier-Stokes equations are described in Sec. III. Linear and nonlinear properties of the resulting set of model equations are discussed in Sec. IV. In Sec. V we investigate within our model the stability behavior of the basic flow under periodic modulation of the inner cylinder's rotation rate. The validity of the performed approximations is summarized in Sec. VI, and the Appendix deals with the linear stability problem.

II. NARROW-GAP APPROXIMATION

Consider an incompressible fluid of density ρ and kinematic viscosity ν between two infinite, concentric cylinders of radii R_1 and R_2, where the inner one is rotating with angular velocity Ω_1 and the outer one is at rest. The problem is governed by the Navier-Stokes equations and the incompressibility condition

$$\partial_t \mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{U}=-\frac{1}{\rho}\nabla P+\nu\Delta\mathbf{U}\ ,\qquad \nabla\cdot\mathbf{U}=0\ . \tag{2.1}$$

Here we only consider axisymmetric fields in the narrow-gap case, where the gap width $d=R_2-R_1$ is small compared to the radii. In lowest-order δ^0 of the small parameter $\delta=d/R_1\ll 1$, one finds the deviations $\mathbf{u}=(u,v,w)$ and p from the basic Couette flow having only azimuthal component $V_0(x)=1-x$ to satisfy the equations

$$(\partial_t-\partial_x^2-\partial_z^2)u+\partial_x p=-(u\,\partial_x+w\,\partial_z)u+Tv^2+2TV_0 v\ , \tag{2.2a}$$

$$(\partial_t-\partial_x^2-\partial_z^2)v=-(u\,\partial_x+w\,\partial_z)v-u\partial_x V_0\ , \tag{2.2b}$$

$$(\partial_t-\partial_x^2-\partial_z^2)w+\partial_z p=-(u\,\partial_x+w\,\partial_z)w\ , \tag{2.2c}$$

$$\partial_x u+\partial_z w=0\ . \tag{2.2d}$$

We have introduced the Taylor number $T=(\Omega_1 R_1 d/\nu)^2\delta$ and $x=(r-R_1)/d$. Length, time, pressure, azimuthal, radial, and axial velocity are measured in units of d, d^2/ν, $\rho\nu^2/d^2$, $R_1\Omega_1$, and ν/d, respectively. Equations (2.2) together with appropriate boundary conditions are the starting point for further approximations.

III. MIXED BOUNDARY CONDITIONS AND MODE EXPANSION

The natural boundary conditions are $u,v,w,\partial_x u=0$ at $x=0,1$. An orthogonal set of functions satisfying these conditions is, for instance, the system of hyperbolic trigonometric functions commonly used in the Bénard problem[1] with rigid-rigid boundaries. Here we impose mixed boundary conditions at the cylinder surfaces: No slip in azimuthal direction, i.e., $v=0$, but free slip in z-direction, i.e., $\partial_x w=0$. Hence there are no axial friction forces acting between the fluid and the cylinder walls. The effect of free slip is to enhance the onset of Taylor vortices. For $\delta\to 0$ the critical Taylor number and wave number for the onset of Taylor vortex flow are $T_c=1695$ and $k_c=3.12$,[1] respectively, if rigid boundaries are imposed, whereas for mixed boundary conditions we have $T_c=654$ and $k_c=2.23$ (cf. Appendix). That is in close analogy to the lowering of the convective threshold in the Bénard problem by use of free-free boundaries. With the above conditions the axisymmetric Taylor vortex field can be decomposed into sine and cosine normal modes

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$$u(x,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}2\sqrt{2}\hat{u}(n,m,t)\sin(n\pi x)\cos(mkz)\,,$$
$$v(x,z,t)=\sum_{n=1}^{\infty}\sqrt{2}\left[\sum_{m=1}^{\infty}2\hat{v}(n,m,t)\sin(n\pi x)\cos(mkz)+\hat{v}(n,0,t)\sin(n\pi x)\right],\qquad(3.1)$$
$$w(x,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}-2\sqrt{2}\hat{w}(n,m,t)\cos(n\pi x)\sin(mkz)\,.$$

Here the axial periodicity length of the Taylor vortex flow state is $2\pi/k$. In analogy to the derivation of the Lorenz model for Rayleigh-Bénard convection, we shall truncate the mode expansion (3.1) of the velocity field and keep only a minimal set. Linear stability analysis of (2.2) (cf. Appendix) shows that to lowest approximation the critical velocity field is given by the $n=m=1$ terms of the expansion (3.1). The corresponding neutral stability curve is $T_s=(\pi^2+k^2)^3/k^2$, which is the same as for Rayleigh-Bénard convection. Thus the minimum $T_c=\frac{27}{4}\pi^4$ and the associated critical wave number $k_c=\pi/\sqrt{2}$ takes the same value in both cases. A first approximation would be to keep only the fundamental modes $\hat{u}(1,1,t)$, $\hat{v}(1,1,t)$, $\hat{w}(1,1,t)$ in (3.1). However, since Eq. (2.2) involves a quadratic nonlinearity, these fundamentals are not able to interact. To allow for nonlinear effects, we also keep $\hat{v}(1,0,t)$ and $\hat{v}(2,0,t)$. Projecting the Navier-Stokes equation (2.2) onto the above-mentioned modes we obtain

$$\partial_t\hat{v}(1,1,t)=-(\pi^2+k^2)\hat{v}(1,1,t)+\hat{u}(1,1,t)[1+\sqrt{2}\pi\hat{v}(2,0,t)]\,,\qquad(3.2a)$$

$$\partial_t\hat{u}(1,1,t)=-(\pi^2+k^2)\hat{u}(1,1,t)+T\frac{k^2}{\pi^2+k^2}\hat{v}(1,1,t)+T\frac{16\sqrt{2}k^2}{3\pi(\pi^2+k^2)}\hat{v}(1,1,t)\hat{v}(1,0,t)\,,\qquad(3.2b)$$

$$\partial_t\hat{v}(1,0,t)=-\pi^2\hat{v}(1,0,t)\,,\qquad(3.2c)$$

$$\partial_t\hat{v}(2,0,t)=-4\pi^2\hat{v}(2,0,t)-2\sqrt{2}\pi\hat{v}(1,1,t)\hat{u}(1,1,t)\,,\qquad(3.2d)$$

where we have used, that $\hat{w}(1,1,t)=(\pi/k)\hat{u}(1,1,t)$. Note that $\hat{v}(1,0,t)$ is damped away exponentially. Therefore, with the scaling

$$X=\frac{2\pi}{\pi^2+k_c^2}\hat{u}(1,1,t),\quad \tau=\frac{1}{\pi^2+k_c^2}\,,$$
$$Y=2\pi\hat{v}(1,1,t)\,,\qquad(3.3)$$
$$Z=-\sqrt{2}\pi\hat{v}(2,0,t),\quad b=4\pi^2\tau\,,$$

our model equations for the critical modes read

$$\tau\dot{X}=-X+Y(1+\epsilon)\,,$$
$$\tau\dot{Y}=-Y+X(1-Z)\,,\qquad(3.4)$$
$$\tau\dot{Z}=-bZ+XY\,.$$

Here we have introduced

$$\epsilon=(T-T_c)/T_c\,.\qquad(3.5)$$

For static driving with constant ϵ one can obtain the standard Lorenz model with Prandtl number $\sigma=1$ by the transformation $X\to x$, $Y\to y/(1+\epsilon)$, $Z\to z/(1+\epsilon)$

$$\tau\dot{x}=-x+y\,,$$
$$\tau\dot{y}=-y+x(1+\epsilon-z)\,,$$
$$\tau\dot{z}=-bz+xy\,.$$

Recently, Hsieh and Chen[3,4] derived similar equations for Taylor-Couette flow with rigid boundaries. For a narrow gap and constant ϵ their system can be written as

$$\dot{x}=-c_7x+c_6y\,,$$
$$\dot{y}=-c_1y+\frac{c_1c_7}{c_6}x\left[1+\epsilon-\frac{c_3c_6}{c_1c_7}z\right],\qquad(3.6)$$
$$\dot{z}=-c_4z+c_5xy\,,$$

with constants c_i explained in Ref. 4. Because of the rigid boundary conditions one would expect that (3.6) yields more realistic results than (3.4). This is true for the critical parameters, $k_c=3.1$ and $T_c=2002$ ($\delta=0.\bar{1}$), but not for the torque on the cylinders to be discussed in the next section.

IV. PROPERTIES OF THE MODEL

A. Growth rates

To investigate the growth rates of the linearly unstable modes for slightly supercritical driving, $\epsilon=(T-T_c)/T_c>0$, one may approximate the time evolution of X and Y through exponential functions. Then one finds that the wave number k_m for the mode with maximum growth rate grows according to

$$\frac{k_m}{k_c}=(1+\epsilon)^{1/4}\,,\qquad(4.1)$$

with an initial slope of $\alpha=\frac{1}{4}$. This result is in excellent agreement with a high-precision numerical calculation of Dominguez-Lerma[5] based on the linearized version of (2.1) with the natural boundary conditions; in the range $0<\delta\leq 0.3$ the difference is less than 0.5%. Even for δ as large as 0.9 the difference in α is only about 6%.

B. Torque

The torque on the cylinders shows a characteristic change, when the driving becomes supercritical. Hence comparison between numerical data and predictions of the model are another test for its validity. The only contribution to the torque in infinite systems comes from nonoscillatory (in z-direction) modes. In reduced units the torque on the inner cylinder reads

$$G-G_0=-(1+2Z)\,. \tag{4.2}$$

It is useful to define $g=(G-G_0)/G_c$, where G_0 and G_c are the torques due to the laminar Couette profile and its critical value, respectively. Here we have

$$g=2\frac{\epsilon}{(1+\epsilon)^{1/2}}\cong 2\epsilon\,. \tag{4.3}$$

In the limit $\delta\to 0$ Davey[6] and Stuart[7] found $g/\epsilon=1.528$ and 1.4472, respectively, using rigid boundaries. Thus the transport of angular momentum from the inner to the outer cylinder, measured by g, is favored in the presence of free slip in z direction. This result is in analogy to the Rayleigh-Bénard problem, where the corresponding quantity, the Nusselt number, measuring the amount of convective heat transport, is enhanced by free boundaries.[8,9] In the limit $\delta\to 0$ the model proposed by Chen and Hsieh[4] yields $g/\epsilon\cong 6.5$, which is much bigger than all previous results. This value for g/ϵ seems to be caused by too high amplitudes: Calculations of the radial velocity in the middle of the gap show that the value obtained from (3.6) is about six times bigger than ours.

In general the model (3.4) is affected by mode truncation and the choice of boundary conditions. Linear stability analysis (cf. Appendix) suggests, that for static driving modes with $n>1$, play a minor role for small ϵ, and the main difference to the rigid system arises from the mixed boundary conditions.

V. PERIODIC MODULATION OF THE INNER CYLINDER'S ROTATION RATE

A. The stability problem

Of great interest is the behavior of the system under external modulation of the driving frequency Ω_1^0. In particular the stability boundary of Couette flow has attracted much research activities.[10–14] Let us consider a periodic modulation of the inner cylinder's rotation frequency with period $2\pi/\omega$,

$$\Omega_1(t)=\Omega_1^0\left[1+\Delta\,\mathrm{Re}\left[\sum_{n=1}^{\infty}a_n e^{-in\omega t}\right]\right]\,, \tag{5.1}$$

with relative amplitudes Δa_n. Then the time-dependent Couette solution is

$$V_0(x,t)=\mathrm{Re}\left[V_0(x)\left[1+\Delta\sum_{n=1}^{\infty}a_n e^{-in\omega t}\right]+\Delta\sum_{n=1}^{\infty}a_n e^{-in\omega t}\left[\frac{\sin[\gamma(1-x)]}{\sin\gamma}-V_0(x)\right]\right]\,, \tag{5.2}$$

where $\gamma=(in\omega)^{1/2}$, leading to Eq. (2.2) with $V_0(x,t)$ instead of $V_0(x)$. The following projection procedure is analogous to Sec. III, i.e., we truncate the full Navier-Stokes equations in the presence of modulation to the same set of modes kept in the static driving case. We obtain

$$\begin{aligned}\tau\dot X&=-X+pY(1+\epsilon)\,,\\ \tau\dot Y&=-Y+qX-XZ\,,\\ \tau\dot Z&=-bZ+XY\,,\end{aligned} \tag{5.3}$$

with abbreviations

$$\begin{aligned}\epsilon&=\frac{T}{T_c^{(0)}}-1\,,\\ p&=\mathrm{Re}\left[1+\Delta\sum_{n=1}^{\infty}\frac{a_n}{1-\gamma^2/4\pi^2}\frac{\tan(\gamma/2)}{\gamma/2}e^{-in\omega t}\right]\,,\\ q&=\mathrm{Re}\left[1+\Delta\sum_{n=1}^{\infty}\frac{a_n}{1-\gamma^2/4\pi^2}e^{-in\omega t}\right]\,.\end{aligned} \tag{5.4}$$

$T_c^{(0)}$ is the static threshold Taylor number and $T=(\Omega_1^0R_1d/\nu)^2\delta$. The stability of the basic flow $V_0(x,t)$ is then governed by a second-order differential equation for X,

$$\tau\ddot X+\left[2-\tau\frac{\dot p}{p}\right]\dot X+\frac{1}{\tau}\left[1-\tau\frac{\dot p}{p}-pq(1+\epsilon)\right]X=0\,. \tag{5.5}$$

This equation describes a damped harmonic oscillator, where the strength of the potential and damping are explicitly time dependent. The corresponding equation for the convective Lorenz model has a constant damping term and only the potential is modulated.

B. Stability boundary for small Δ

To investigate the stability boundary of the $X=0$ state, we expand the reduced threshold Taylor number or ϵ_c and the solution X in terms of the small parameter Δ,

$$\begin{aligned}X&=X^{(0)}+\Delta X^{(1)}+\Delta^2X^{(2)}+\cdots\,,\\ \epsilon_c&=\epsilon_c^{(0)}+\Delta\epsilon_c^{(1)}+\Delta^2\epsilon_c^{(2)}+\cdots\,,\end{aligned} \tag{5.6}$$

requiring that to each order in Δ the "coefficients" $X^{(n)}$ are marginal, i.e., periodic. That yields $\epsilon_c^{(0)}=0$. Furthermore, with the Fredholm alternative one finds $\epsilon_c^{(1)}=0$ as well as

$$\epsilon_c^{(2)}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{|q_n|^2}{1+\left[\frac{n\omega\tau}{2}\right]^2}\left[1+\left|\frac{p_n}{q_n}\right|^2-2\,\mathrm{Re}\left[\frac{p_n}{q_n}\right]\right]\geq 0\,, \tag{5.7}$$

where

$$\begin{aligned}p_n&=\frac{a_n}{2}\left[\frac{1}{1-\gamma^2/4\pi^2}\right]\frac{\tan(\gamma/2)}{\gamma/2}\,,\\ q_n&=\frac{a_n}{2}\left[\frac{1}{1-\gamma^2/4\pi^2}\right]\,.\end{aligned}$$

Thus the shift of the threshold for onset of vortex flow due to modulation is

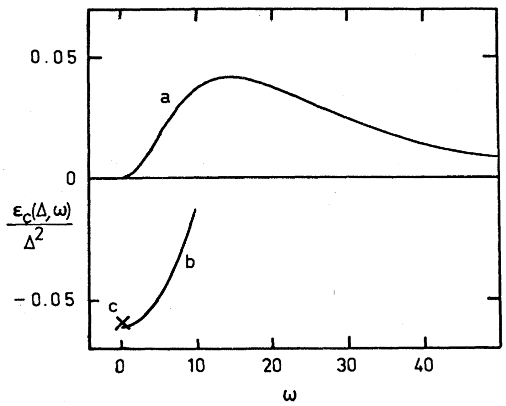

FIG. 1. Reduced threshold shift $\epsilon_c(\Delta,\omega)/\Delta^2$ as function of the modulation frequency of the inner cylinder's rotation rate ω for small modulation amplitude: a, present result, $a_1=1$; b, Hall (Ref. 12); and c, Riley and Laurence (Ref. 10).

$$\epsilon_c(\Delta,\omega)=\Delta^2\epsilon_c^{(2)}(\omega)+O(\Delta^4) \ . \tag{5.8}$$

Odd terms in Δ vanish because $\Delta \to -\Delta$ only adds a constant phase factor in (5.1). For an application of this technique to the convective Lorenz model see Ref. 15. For negative $\epsilon_c(\Delta,\omega)$ the basic flow is destabilized, whereas positive $\epsilon_c(\Delta,\omega)$ means a stabilization of the basic solution. The first nonvanishing correction to the static threshold $\epsilon_c(\Delta,\omega)=0$ is at order Δ^2 with $\epsilon_c^{(2)}$ giving the relative strength of the threshold shift. Here the threshold shift is always positive with a maximum for small ω and vanishes for $\omega \to 0,\infty$ [in Fig. 1 $\epsilon_c(\Delta,\omega)/\Delta^2$ is plotted for the case of harmonical modulation with strength $a_1=1$ (curve a)]. Therefore, within our model equations, modulation has a stabilizing effect. The convective Lorenz model for $\sigma=1$ shows a stabilization, that is similar in the high-frequency range. However, its low-frequency stabilization is much larger with a maximum $\epsilon_c^{(2)}(\omega\to 0)=0.125$.

Hall[12] and Riley and Laurence[10] investigated the stability of Couette flow using the narrow-gap approximation, too and found $\epsilon_c(\Delta,\omega)<0$. The destabilization was maximal for $\omega \to 0$ and vanishes for $\omega\to\infty$. Carmi and Tustaniwskyj[11] verified this behavior for finite gap width. Figure 1 shows the low-frequency modulation expansion of Hall (curve b) and a value for $\epsilon_c^{(2)}$ given by Riley and Laurence (curve c).

Both results contradict ours. To decide whether this discrepancy stems from mode truncation or changed boundary conditions, we carried out a linear stability analysis for the narrow-gap Eqs. (2.2a)–(2.2d) and modulated angular velocity with the full set of modes (3.1).[16] Technically we truncated the expansion at increasing order. Already the inclusion of the second harmonic ($n=2$) in radial direction leads to a destabilization $\epsilon_c^{(2)}(\omega)<0$ rather than stabilization of the Couette flow, and furthermore, $\epsilon_c^{(2)}(\omega\to 0)$ remains finite. Including higher modes, $n>2$, leads to a well-converging series of stability boundaries that do not differ qualitatively from the truncation at order $n=2$.

Therefore, the stabilizing effect of the model (5.3) is due to our truncating the radial modes at order $n=1$. The mixed boundary conditions, on the other hand, do not qualitatively alter the stability properties, except that $|\epsilon_c^{(2)}(\omega\to 0)|$ is smaller by a factor of ~ 17 from the results of Hall, and Riley and Laurence for rigid boundaries.

VI. CONCLUSION

The presented treatment of the transition from Couette flow to Taylor vortex flow in the narrow-gap limit starts from two simplifications, i.e., mixed boundary conditions and mode truncation. The resulting set of model equations confirm previous results in the case of static driving. Similar to the convective Lorenz model, the difference of k_c, T_c, and g to the rigid system are caused by the changed boundary conditions. The truncation approximation, on the other hand, influences only slightly the critical values. However, in the case of time-dependent driving, the threshold shift is more sensitive to mode restrictions and the second harmonic has to be taken into account. The discrepancies in $\epsilon_c^{(2)}$ between mixed and rigid cases that remain for $N\to\infty$, are then due to boundary effects. A natural consequence would be to investigate a model with more than three modes, which is under present consideration.

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APPENDIX

Here we present the linear stability analysis of (2.2) with the mixed boundary conditions $u=v=\partial_x^2 u=0$ at $x=0,1$. After linearizing we get

$$(\partial_t-\partial_x^2-\partial_z^2)(\partial_x^2+\partial_z^2)u=2TV_0\partial_z^2 v \ , \tag{A1a}$$

$$(\partial_t-\partial_x^2-\partial_z^2)v=u \ . \tag{A1b}$$

Since we have identical boundary conditions for u and v, it is convenient to write Eq. (A1) as one equation for v. (For the rigid boundary case see Ref. 1.) Using the expansion (3.1) with $m=1$, k variable, and $\hat{v}(n,1,t)=\hat{v}(n,1)e^{\sigma t}$, we are left with

$$a(\sigma,n,k)\hat{v}(n,1)=TB_{n',n}\hat{v}(n',1) \ , \tag{A2}$$

where

$$a(\sigma,n,k)=\frac{(\sigma+n^2\pi^2+k^2)^2(n^2\pi^2+k^2)}{4k^2}$$

and the symmetric matrix $B_{n',n}$ satisfies

$$B_{n',n}=\begin{cases}\frac{1}{4}, & n=n'\\ 0, & n-n' \text{ even, nonzero}\\ \dfrac{4n'n}{\pi^2(n^2-n'^2)^2}, & n-n' \text{ odd} \ .\end{cases} \tag{A3}$$

The solvability condition

TABLE I. Numerical values for the critical Taylor number T_c, wave number k_c, and relative amplitudes $x_n = \hat{v}(n,1)/\hat{v}(1,1)$ of harmonics within the marginally stable solution of (A2). N is the number of harmonics kept in (A2).

N	k_c	T_c	$10^2 x_2$	$10^5 x_3$	$10^5 x_4$	$10^7 x_5$
1	2.2214	657.511				
2	2.2272	654.257	1.383			
3	2.2272	654.256	1.383	2.126		
4	2.2272	654.256	1.383	2.129	2.167	
5	2.2272	654.256	1.383	2.129	2.167	1.053

$$\det[a(\sigma=0, n,k)\delta_{n',n} - TB_{n',n}] = 0 \quad \text{(A4)}$$

yields the curves of marginal stability for different eigenmodes of (A1) in the (k,T) plane. Because our basic functions (3.1) are not eigenfunctions for the problem (A1), $B_{n',n}$ contains off-diagonal elements. However, they decay like $|n-n'|^{-2}$. Thus we approximately solve (A4) by truncating the infinite matrices to $N \times N$ square matrices, where N is the number of harmonics retained. This approximation converges quite rapidly, as can be seen in Table I. For $N=1$ we get

$$k_c = \frac{\pi}{\sqrt{2}} \quad \text{and} \quad T_c = \tfrac{27}{4}\pi^4 .$$

For $N > 1$ the marginally stable solution obtained by solving (A2) contains higher harmonics of small amplitude, the relative size being less than 0.015 (see Table I). They cause corrections to the first approximation for T_c, which are of order 10^{-2}. Thus it seems reasonable for a first approximation to restrict oneself on the fundamental mode, if model equations for T immediately above T_c are investigated.

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