Two-state problems involving arbitrary amplitude and frequency modulations

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Analytic solutions for a class of pulses that includes an infinite variety of amplitude and frequency modulations which we previously obtained are applied specifically to the square, Gaussian, Lorentzian, hyperbolic-secant, and the exponential pulses, among others. For all the different types of amplitude- and its corresponding frequency-modulated functions, the outcome of the atomic excitation is given by the same simple analytic formulas. Our analytic solutions apply equally to two-state collision problems.

I. INTRODUCTION

The time evolution of a two-state problem in which the two states are coupled by a possibly time-dependent interaction is governed by the following coupled equations obtained from the time-dependent Schrödinger equation for the probability amplitudes $c_1(t)$ and $c_2(t)$ for the two states $|1\rangle$ and $|2\rangle$:

$$i\hbar\frac{dc_1(t)}{dt}=H_{11}(t)c_1(t)+H_{12}(t)c_2(t) , \tag{1a}$$

$$i\hbar\frac{dc_2(t)}{dt}=H_{21}(t)c_1(t)+H_{22}(t)c_2(t) , \tag{1b}$$

where $H_{ij}=\langle i\,|\,H\,|\,j\rangle$ represents the matrix element of the Hamiltonian H. Equations (1) have been studied in connection with the two-state collision problem[1,2] in which the matrix elements H_{ij} depend on the internuclear distance of the colliding nuclei and thus on time. Equations (1) also arise naturally in the studies of nuclear magnetic resonance[3] and coherent excitation of atomic and molecular systems by lasers,[4] where the matrix elements depend on the external magnetic or electric field.

Beginning with the pioneering work of Rosen and Zener,[5] Landau,[6] Zener,[7] and Rabi[8] some 50 years ago, the problem of finding analytic solutions of Eqs. (1) for various forms of time-dependent H_{ij} has again received considerable attention in recent years.[1,9–13] We have been able to achieve a considerable unification as well as extension of various analytic results recently with an analytic solution[13] of Eqs. (1) that includes an infinite variety of the time-dependent forms of the matrix elements. We shall use the language of laser physics, which relates the off-diagonal elements H_{12} and H_{21} to the amplitude modulation of the incident laser pulse, and the diagonal elements H_{11} and H_{22}, or H_{22}-H_{11} to the frequency modulation. Our analysis is applicable, of course, to other two-state problems that we have mentioned.

In this paper we develop our previous solution somewhat further, emphasizing the application to finite pulses whose amplitude modulations are of the types most commonly used: the square, Gaussian, Lorentzian, hyperbolic secant, and the exponential pulses. For all the different types of amplitude- and its corresponding frequency-modulation functions, the outcome of the atomic excitation is given by the same simple analytic formulas.

Section II describes some of the well-known transformations of Eqs. (1) that lead to equations and terminology more familiar to the laser physicists. Sections III—V describe various specific and general types of pulse functions in what we call the $(\alpha\beta\gamma)$ and $(\alpha\beta)$ models, respectively. Section VI and Table I summarize most of our results.

II. TRANSFORMATIONS OF EQS. (1) TO SOME OTHER FORMS

A transformation

$$c_j(t)=a_j(t)\exp\left[-\frac{i}{\hbar}\int_{-\infty}^{t}H_{jj}(t')dt'\right],\quad j=1,2 \tag{2}$$

applied to Eqs. (1) gives the following coupled differential equations of the more "standard" form:

$$i\hbar\frac{da_1(t)}{dt}=a_2(t)H_{12}(t)\times\exp\left[-\frac{i}{\hbar}\int_{-\infty}^{t}[H_{22}(t')-H_{11}(t')]dt'\right], \tag{3a}$$

$$i\hbar\frac{da_2(t)}{dt}=a_1(t)H_{21}(t)\times\exp\left[\frac{i}{\hbar}\int_{-\infty}^{t}[H_{22}(t')-H_{11}(t')]dt\right]. \tag{3b}$$

In the study of coherent excitation of atomic systems by lasers, another step is usually taken between Eqs. (1) and (3) as we shall outline below. The Hamiltonian of the system is

$$\hat{\mathscr{H}}=\hat{H}_a-\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} ,$$

where $\hat{H}_a$ is the unperturbed Hamiltonian of the atomic system, $\hat{\mathbf{d}}$ the atom's dipole moment operator, and $\hat{\mathbf{E}}$ the electric field operator evaluated at the position of the dipole. The frequency ω of the incident laser field $E(t)$ given by

$$E(t)=\mathscr{E}(t)\exp\left[i\int_{-\infty}^{t}\omega(t')dt'\right]+\text{c.c.} \quad (4)$$

is usually assumed to be nearly resonant with the transition frequency ω_0 between two particular levels of the atomic system that are dipole connected. The following substitutions are made into Eqs. (1):

$$c_1(t)=b_1(t)\exp\left[\tfrac{1}{2}i\int_{-\infty}^{t}\omega(t')dt'\right], \quad (5a)$$

$$c_2(t)=b_2(t)\exp\left[-\tfrac{1}{2}i\int_{-\infty}^{t}\omega(t')dt'\right]; \quad (5b)$$

and then the so-called rotating wave approximation[4] is made in which terms containing the rapidly oscillating factors $\exp\pm 2i\int_{-\infty}^{t}\omega(t)dt$ are ignored. The following coupled differential equations are then obtained that are still of the form given by Eqs. (1) but the matrix elements of which vary slowly with time compared to the optical frequencies:

$$i\hbar\frac{d}{dt}\begin{bmatrix}b_1\\b_2\end{bmatrix}=\tfrac{1}{2}\hbar\begin{bmatrix}-\Delta & -\Omega\\-\Omega & +\Delta\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}, \quad (6)$$

where

$$\Omega\equiv 2\hbar^{-1}d\mathscr{E}, \quad (7)$$

$$\Delta\equiv\omega_0-\omega \quad (8)$$

are referred to as the Rabi frequency and the atom-field detuning, respectively, d being the atom's dipole moment. We now apply a transformation of the type given by Eq. (2) to Eq. (6), obtaining again Eq. (3) in which the H_{ij} denote the following:

$$H_{12}=H_{21}=-\tfrac{1}{2}\hbar\Omega, \quad (9a)$$

$$H_{22}-H_{11}=\hbar\Delta. \quad (9b)$$

The Rabi frequency Ω is related through Eqs. (7) and (4) to the generally time-varying amplitude $\mathscr{E}(t)$ of the incident laser field, and the atom-field detuning Δ may depend on time, either because the frequency ω of the laser field is modulated or because the frequency separation ω_0 of the two levels is made time-dependent through an application of a direct magnetic or electric field.

To cover various types of two-state problems, we shall write Eqs. (3) in the following standard form:

$$i\begin{bmatrix}\dot a_1\\ \dot a_2\end{bmatrix}=\begin{bmatrix}0 & -\tfrac{1}{2}\dot A e^{-iB}\\ -\tfrac{1}{2}\dot A e^{iB} & 0\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}, \quad (10)$$

where $\dot A$ and $\dot B$ have the dimensions of angular frequencies, representing the Rabi frequency Ω and detuning Δ, respectively, for problems in quantum optics and magnetic resonance, while representing the intranuclear matrix elements $-2\hbar^{-1}H_{12}$ $(=-2\hbar^{-1}H_{21})$ and $\hbar^{-1}(H_{22}-H_{11})$, respectively, for collision problems. The notation $\dot A$ is in line with the notation of pulse envelope area A used in laser physics defined by[4,14]

$$A=\int_{-\infty}^{\infty}\dot A(t)dt. \quad (11)$$

For the case of constant detuning, B is simply given by $(\omega_0-\omega)t$.

It can be verified that the following transformation:[15]

$$u=e^{-iB}a_1^*a_2+e^{iB}a_2^*a_1, \quad (12a)$$

$$v=i(e^{-iB}a_1^*a_2-e^{iB}a_2^*a_1), \quad (12b)$$

$$w=a_2^*a_2-a_1^*a_1=|a_2|^2-|a_1|^2, \quad (12c)$$

can be used to transform Eq. (10) into the familiar Bloch equations[4,16]

$$\begin{bmatrix}\dot u\\ \dot v\\ \dot w\end{bmatrix}=\begin{bmatrix}0 & -\dot B & 0\\ \dot B & 0 & \dot A\\ 0 & -\dot A & 0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}, \quad (13)$$

where u and v represent the components of the atomic dipole moment in phase and in quadrature with the incident laser field and w represents the population inversion for the atom, with the subscripts 1 and 2 referring to the lower and upper levels, respectively. Thus the atomic populations in the lower and upper levels at time t are given, respectively, by $|a_1(t)|^2$ and $|a_2(t)|^2$, and the initial condition of the atoms at $t=-\infty$ is usually assumed to be

$$|a_1(-\infty)|=1,\quad a_2(-\infty)=0. \quad (14)$$

In terms of the density-matrix formulation, the quantities u,v,w in Eq. (13) are related to the density-matrix elements ρ_{ij} of the atomic system by

$$u=\rho_{12}+\rho_{21}, \quad (15a)$$

$$v=-i(\rho_{12}-\rho_{21}), \quad (15b)$$

$$w=\rho_{22}-\rho_{11}. \quad (15c)$$

When the solution of $|a_1|$ and $|a_2|$ is determined from Eq. (10) [or equivalently when the solution of $|b_1|$ and $|b_2|$ or $|c_1|$ and $|c_2|$ is determined from Eqs. (6) or (1) since b_j and c_j differ from a_j only by a phase factor], the population inversion w can be determined from Eq. (12c) and the coherences v and u can be determined from Eq. (13) as

$$v=-\frac{\dot w}{\dot A},$$

and

$$u=\frac{\dot v-\dot A w}{\dot B}.$$

The above steps make the determination of the phases of a_1 and a_2 unnecessary for the physically measurable quantities given by Eqs. (15).

III. THE $(\alpha\beta\gamma)$ MODEL

The steps that led us to our analytic solution[13] of Eq. (10) for a general class of amplitude- and frequency-modulated pulses, which we shall call the $(\alpha\beta\gamma)$ model, followed closely that introduced by Rosen and Zener.[5] First, elimination of a_2 from Eq. (10) leads to the second-order differential equation

$$\ddot{a}_1+\left[i\dot{B}-\frac{\ddot{A}}{\dot{A}}\right]\dot{a}_1+\left[\frac{\dot{A}}{2}\right]^2 a_1=0 . \quad (16)$$

Then we introduce the change of variable from t, the time, to

$$z=z(t)\geq 0 \quad (17)$$

subject to the restriction that $\dot{z}\geq 0$ and

$$z(-\infty)=0,\ \ z(+\infty)=1 . \quad (18)$$

The transformation from t to z changes the range of the independent variable from $(-\infty,+\infty)$ to [0,1]. In terms of z, Eq. (16) can be written in the form

$$a_1''+\frac{1}{\dot{z}}\left[\frac{d}{dt}\ln\left[\frac{\dot{z}}{\dot{A}}\right]+i\dot{B}\right]a_1'+\left[\frac{\dot{A}}{2\dot{z}}\right]^2 a_1=0 , \quad (19)$$

where the prime denotes the derivative with respect to z, and it can be compared with the hypergeometric equation

$$a_1''+\frac{c-(a+b+1)z}{z(1-z)}a_1'-\frac{ab}{z(1-z)}a_1=0 . \quad (20)$$

A crucial feature of our solution is the introduction of an arbitrary function $h(z)$ in the relation between z and t, and in the pulse functions $\dot{A}$ and $\dot{B}$

$$\dot{z}=\frac{z(1-z)}{h(z)} , \quad (21a)$$

$$\dot{A}=\frac{\alpha}{\pi}\frac{[z(1-z)]^{1/2}}{h(z)} , \quad (21b)$$

$$\dot{B}=\frac{1}{\pi}\frac{\beta z+\gamma}{h(z)} ; \quad (21c)$$

and the subsequent proof that for a given initial condition, the level populations at time t of atoms subject to pulses of the type given by Eqs. (21) are independent of $h(z)$.[13]

Equations (21a) and (21b) show that we can make $\dot{A}$, the pulse amplitude into an arbitrary function of time; its area is $\alpha=\int_{-\infty}^{\infty}\dot{A}(t)dt$. Then $\dot{B}$ is a function that contains two adjustable parameters. In particular, for the initial condition that at $t=-\infty$, the atoms are in level 1 or the ground state, i.e.,

$$|a_1(-\infty)|=1,\ \ a_2(-\infty)=0 , \quad (22)$$

the solutions of a_1 and a_2 of Eq. (10) are given up to a phase factor by

$$a_1=F(a^*,b^*,c^*,z) , \quad (23a)$$

$$a_2=\frac{(-ab)^{1/2}}{|1-c|}z^{1-c}F(a-c+1,b-c+1,2-c,z) , \quad (23b)$$

where $F(a,b,c,z)$ denotes the hypergeometric function and where a,b,c are given, in terms of the parameters α,β,γ in the pulse functions (21) by

$$a=\frac{1}{2\pi}[(\alpha^2-\beta^2)^{1/2}-i\beta] , \quad (24a)$$

$$b=\frac{1}{2\pi}[-(\alpha^2-\beta^2)^{1/2}-i\beta] , \quad (24b)$$

$$c=\tfrac{1}{2}+i\frac{\gamma}{\pi} . \quad (24c)$$

The final level populations at $t=+\infty$ are given by

$$|a_1(+\infty)|^2=\operatorname{sech}\gamma\,\operatorname{sech}(\beta+\gamma)\{\sinh^2(\tfrac{1}{2}\beta+\gamma)+\cos^2[\tfrac{1}{2}(\alpha^2-\beta^2)^{1/2}]\} , \quad (25a)$$

$$|a_2(+\infty)|^2=\operatorname{sech}\gamma\,\operatorname{sech}(\beta+\gamma)\{\sinh^2(\tfrac{1}{2}\beta)+\sin^2[\tfrac{1}{2}(\alpha^2-\beta^2)^{1/2}]\} . \quad (25b)$$

The solution for the atomic excitation given by Eqs. (23)–(25) is the solution for our $(\alpha\beta\gamma)$ model in which the incident pulse functions given by Eqs. (21) can assume an infinite variety of amplitude and frequency modulations allowed by the arbitrary function $h(z)$ on whose presence the final outcome does not depend. The importance of the parameter $\Phi\equiv(\alpha^2-\beta^2)^{1/2}$ can be seen from observing the following circumstances:

(1) If $\beta=0$, then the population completely returns to the ground level at $t=+\infty$ when

$$\Phi=2n\pi,\ \ n=1,2,3,\ldots . \quad (26)$$

(2) If $\gamma=-\frac{1}{2}\beta$, then the population is completely excited to the upper level at $t=+\infty$ when

$$\Phi=(2n-1)\pi,\ \ n=1,2,3,\ldots . \quad (27)$$

Thus in the presence of frequency modulation, the parameter Φ, rather than the total area of pulse amplitude

$$A=\int_{-\infty}^{\infty}\dot{A}(t)dt=\alpha \quad (28)$$

should be identified with the dipole turning angle under the conditions specified above. We should mention that our solutions (23)–(25) are valid even if $\alpha<\beta$. By choosing $h(z)=\text{const}$ or $h(z)=\text{const}(\beta z+\gamma)$, we have the special cases considered by Rosen and Zener, Hioe, and Bambini and Berman. In the following section, we shall show that the square, Lorentzian, and Gaussian pulses, among others, are also cases of our $(\alpha\beta\gamma)$ model corresponding to particular choices of $h(z)$ in Eqs. (21).

IV. SPECIFIC REALIZATIONS OF THE $(\alpha\beta\gamma)$ MODEL

In Eqs. (21) if we let

$$z=\tfrac{1}{2}[1+f(g(t))]\ , \tag{29}$$

where f is a function to be determined, then

$$\dot{z}\equiv\frac{dz}{dt}=\tfrac{1}{2}f'(g(t))\dot{g}(t)\ , \tag{30}$$

where the prime denotes the derivative with respect to $g(t)$. We now consider the set of pulse functions in which $h(z)$ assumes the following simple form:

$$h(z)=[\dot{g}(t)]^{-1}[z(1-z)]^{\delta}\ , \tag{31}$$

where δ may be positive or negative. Then from Eq. (29), $h(z)$ can be written as

$$h(z)=[\dot{g}(t)]^{-1}[\tfrac{1}{4}(1-f^2)]^{\delta}\ . \tag{32}$$

Substitutions of Eqs. (32) and (30) into Eq. (21a) gives us the following equation which the function f must satisfy:

$$\frac{1}{2}\frac{df}{dg}=[\tfrac{1}{4}(1-f^2)]^{1-\delta} \tag{33}$$

or

$$2^{1-2\delta}\int\frac{df}{(1-f^2)^{1-\delta}}=g+\text{const}\ . \tag{34}$$

Substitutions of Eqs. (32) and (33) into Eq. (21a) also gives the relation between z and f

$$\int\frac{dz}{z(1-z)}=2\int\frac{df}{1-f^2} \tag{35}$$

from which we get

$$\ln\left[\frac{z}{1-z}\right]=2\tanh^{-1}f+\text{const} \tag{36}$$

independent of δ.

There is an infinite number of choices for δ that would yield analytic expressions for the integral on the left-hand side of Eq. (34). There are, however, three particularly simple cases corresponding to the choices of $\delta=0,-\frac{1}{2},\frac{1}{2}$ that deserve special attention because for these cases, f can be expressed as simple functions of $g(t)$, which is still arbitrary but whose behavior at $t=-\infty$ and $+\infty$, or at $z=0$ and 1, must be required to be consistent with Eq. (36). The corresponding pulse functions are then given, from Eqs. (21b) and (21c), by

$$\dot{A}=\frac{\alpha}{\pi}\dot{g}(t)\{\tfrac{1}{4}[1-f^2(g(t))]\}^{1/2-\delta}\ , \tag{37a}$$

$$\dot{B}=\frac{1}{2\pi}\dot{g}(t)\frac{(\beta+2\gamma)+\beta f(g(t))}{\{\tfrac{1}{4}[1-f^2(g(t))]\}^{\delta}}\ . \tag{37b}$$

For all the different sets of pulse functions that we shall obtain in the following, which correspond to different values of δ and functions f and g, the degree of atomic excitation, given by our formulas (23)–(25), depends only on the three parameters α, β, and γ.

Case I: $\delta=0$.

Since

$$\int\frac{dx}{a^2-x^2}=\frac{1}{a}\tanh^{-1}\left[\frac{x}{a}\right]\ , \tag{38}$$

we get, from Eq. (34),

$$2\tanh^{-1}f=g(t)+\text{const}\ . \tag{39}$$

From Eq. (36), remembering that $z=0,1$ correspond to $t=-\infty,+\infty$, respectively, we are required to have $g(\pm\infty)=\pm\infty$. If we choose $g(0)=0$ for convenience, then we have

$$f=\tanh[\tfrac{1}{2}g(t)]\ . \tag{40}$$

The corresponding pulse functions, from Eqs. (37) are given by

$$\dot{A}=\frac{\alpha}{2\pi}\dot{g}(t)\text{sech}[\tfrac{1}{2}g(t)]\ , \tag{41a}$$

$$\dot{B}=\frac{1}{2\pi}\dot{g}(t)\{\beta+2\gamma+\beta\tanh[\tfrac{1}{2}g(t)]\}\ . \tag{41b}$$

These pulse functions have been given in our earlier paper,[13] and have been shown to give some earlier results for special choices of $g(t)$, $g(t)=t/\tau$ being one of the simplest choices.

Case II: $\delta=-\frac{1}{2}$.

Since

$$\int\frac{dx}{[(a^2-x^2)^3]^{1/2}}=\frac{x}{a^2[a^2-x^2]^{1/2}}\ , \tag{42}$$

we find, from Eq. (34),

$$\frac{4f}{[1-f^2]^{1/2}}=g(t)+\text{const}\ . \tag{43}$$

Choosing the constant to be zero, we find

$$f=\frac{g}{(16+g^2)^{1/2}}\ . \tag{44}$$

Consideration of Eq. (36) again requires $g(t)$ to have the property that $g(\pm\infty)=\pm\infty$. The pulse functions from Eqs. (37) are given by

$$\dot{A}=\frac{4\alpha}{\pi}\dot{g}(t)\frac{1}{16+g(t)^2}\ , \tag{45a}$$

$$\dot{B}=\frac{1}{\pi}\frac{\dot{g}(t)}{[16+g(t)^2]^{1/2}}\left[\beta+2\gamma+\frac{\beta g(t)}{[16+g(t)^2]^{1/2}}\right]\ . \tag{45b}$$

For the case $g(t)=\text{const}\,t$, $\dot{A}$ is the Lorentzian pulse, and $\dot{B}$ above gives the corresponding detuning function, and together they give an atomic excitation given by our formulas (23)–(25).

Case III: $\delta=\frac{1}{2}$.

Since

$$\int\frac{dx}{[a^2-x^2]^{1/2}}=\sin^{-1}\left[\frac{x}{a}\right]\ , \tag{46}$$

we find, from Eq. (34)

$$\sin^{-1}f=g(t)+\text{const}\ . \tag{47}$$

Choosing the constant to be zero, we find

$$f = \sin[g(t)] \ . \tag{48}$$

Consideration of Eq. (36) requires $g(t)$ to have the properties that

$$g(\pm\infty) = \pm\tfrac{1}{2}\pi \ . \tag{49}$$

The pulse functions from Eqs. (37) are given by

$$\dot{A} = \frac{\alpha}{\pi}\dot{g}(t) \ , \tag{50a}$$

$$\dot{B} = \frac{\dot{g}(t)}{\pi}\frac{\beta+2\gamma+\beta\sin[g(t)]}{\cos[g(t)]} \ . \tag{50b}$$

The especially simple form of $\dot{A}$ in terms of $\dot{g}$ in Eq. (50a) allows us to choose various forms of pulse functions of interest. In particular let us consider a square pulse, a Gaussian pulse, and an exponential pulse.

Let r be a positive parameter in all the following cases.

(i) A square pulse of area α.

We let

$$\dot{g}(t) = \begin{cases} r, & -\pi/2r \le t \le \pi/2r \\ 0, & \text{elsewhere} \ . \end{cases} \tag{51}$$

The pulse functions become

$$\dot{A} = \frac{\alpha r}{\pi} \ , \tag{52a}$$

$$\dot{B} = \frac{r}{\pi}\frac{\beta+2\gamma+\beta\sin(rt)}{\cos(rt)} \tag{52b}$$

in the time interval $-\pi/2r \le t \le \pi/2r$, and zero elsewhere.

(ii) A Gaussian pulse of area α.

We let

$$\dot{g}(t) = \sqrt{\pi}r\exp(-r^2t^2) \ , \tag{53}$$

and

$$g(t) = \sqrt{\pi}r\int_0^t \exp(-r^2t^2)dt = \tfrac{1}{2}\pi\,\mathrm{erf}(rt) \ . \tag{54}$$

The pulse functions are

$$\dot{A} = \frac{\alpha r}{\sqrt{\pi}}\exp(-r^2t^2) \ , \tag{55a}$$

$$\dot{B} = \frac{r\exp(-r^2t^2)}{\sqrt{\pi}}\frac{\beta+2\gamma+\beta\sin[\frac{1}{2}\pi\,\mathrm{erf}(rt)]}{\cos[\frac{1}{2}\pi\,\mathrm{erf}(rt)]} \ . \tag{55b}$$

Notice that the behavior of $\dot{B}$ as $t\to\pm\infty$ is

$$\lim_{t\to\pm\infty}\dot{B} = \frac{2r^2t}{\pi}(\beta+2\gamma\pm\beta) \ . \tag{56}$$

(iii) A symmetric exponential pulse of area α.

We let

$$\dot{g}(t) = \frac{\pi r}{2}\exp(-r\,|\,t\,|) \tag{57}$$

and

$$g(t) = \begin{cases} -\frac{1}{2}\pi[1-\exp(-r\,|\,t\,|)], & t<0 \\ \frac{1}{2}\pi[1-\exp(-r\,|\,t\,|)], & t>0 \ . \end{cases} \tag{58}$$

The pulse functions are

$$\dot{A} = \frac{\alpha r}{2}\exp(-r\,|\,t\,|) \ , \tag{59a}$$

$$\dot{B} = \tfrac{1}{2}r\exp(-r\,|\,t\,|)\left[\frac{\beta+2\gamma\mp\beta\sin[\frac{1}{2}\pi(1-e^{-r|t|})]}{\cos[\frac{1}{2}\pi(1-e^{-r|t|})]}\right], \quad t \lessgtr 0 \ . \tag{59b}$$

The behavior of $\dot{B}$ as $t\to\mp\infty$ is

$$\lim_{t\to\mp\infty}\dot{B} = \frac{r}{\pi}(\beta+2\gamma\mp\beta) \ . \tag{60}$$

V. THE $(\alpha\beta)$ MODEL

For the case when the pulse functions $\dot{A}$ and $\dot{B}$ have the same time dependence except for possibly a multiplicative constant, the solution of the coupled equations (10) can be shown to be reducible to the solution first given by Rabi.[8] We shall refer to this case as the $(\alpha\beta)$ model for reasons that will become apparent.

Instead of considering Eq. (10), we may consider the original coupled equations (1) or (6), where the solutions c_1,c_2; b_1,b_2; and a_1,a_2 of the same subscripts differ from each other only by a phase factor. Thus up to a phase factor, Eqs. (1) and (6) can be written, in terms of $\dot{A}$ and $\dot{B}$, which have the usual meanings specified below Eq. (10), as

$$i\begin{bmatrix}\dot{a}_1\\ \dot{a}_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-\dot{B} & -\dot{A}\\ -\dot{A} & \dot{B}\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} \ . \tag{61}$$

If $\dot{A}$ and $\dot{B}$ have the same time dependence, we may write

$$\dot{A} = \alpha\dot{\phi}(t) \ , \tag{62a}$$

$$\dot{B} = \beta\dot{\phi}(t) \ , \tag{62b}$$

where α and β are positive constants. Let us also denote

$$\Gamma = (\alpha^2+\beta^2)^{1/2} \ , \tag{63}$$

and define

$$\tau = \phi(t) = \int_{-\infty}^{t}\dot{\phi}(t)dt \ . \tag{64}$$

In terms of this new time scale τ, Eq. (61) reduces to

$$i\frac{d}{d\tau}\begin{bmatrix}a_1\\ a_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-\beta & -\alpha\\ -\alpha & \beta\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} \tag{65}$$

in which the matrix elements are now constants. Since we shall be only interested in pulse functions for which the area A of the pulse is finite, we shall assume that $\dot{\phi}(t)$ and $\phi(t)$ have the properties that $\dot{\phi}(\pm\infty) = \phi(-\infty) = 0$, and thus from Eq. (64), $\tau=0$ will correspond to $t=-\infty$. The solution of Eq. (65) corresponding to the initial condition

$$|\,a_1(\tau=0)\,| = 1, \quad a_2(\tau=0) = 0 \ , \tag{66}$$

can be easily found to be

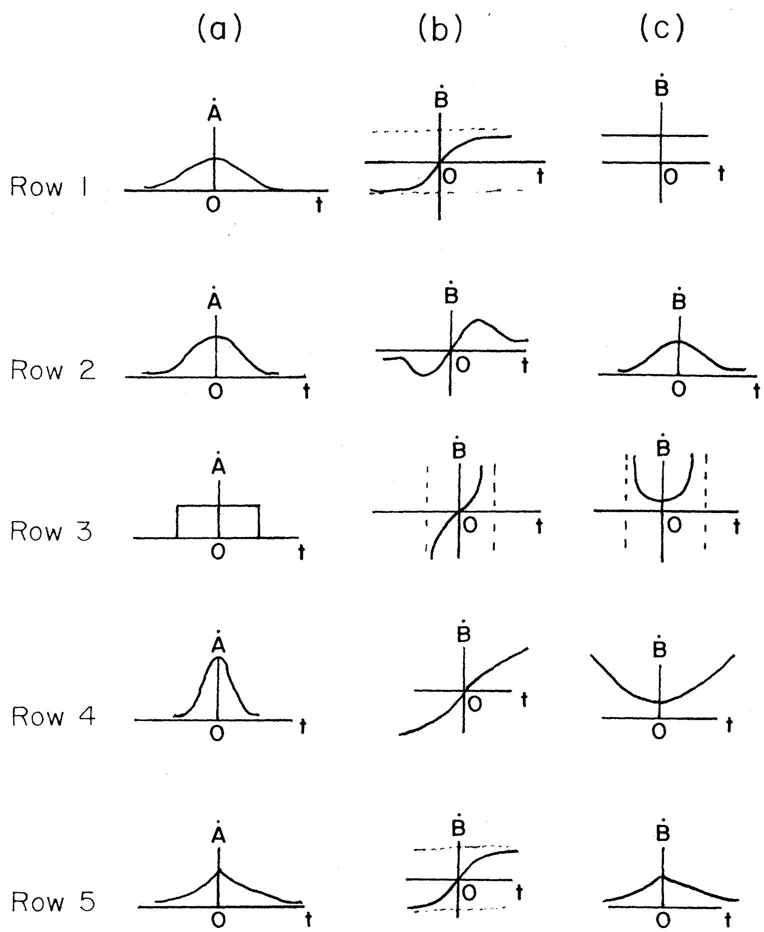

FIG. 1. We present in column (a) sketches of some special amplitude modulation functions; in column (b) sketches of their corresponding antisymmetric frequency-modulation functions obtained under the assumption that $\gamma=-\frac{1}{2}\beta$ in the $(\alpha\beta\gamma)$ model; and in column (c) sketches of their symmetric frequency-modulation functions obtained under the assumption that $\beta=0$ in the $(\alpha\beta\gamma)$ model.

	Column (a)	Column (b)	Column (c)
Row 1	$\dot{A}=\frac{\alpha}{\pi}\,\mathrm{sech}t$	$\dot{B}=\frac{\beta}{\pi}\tanh t$	$\dot{B}\frac{2\gamma}{\pi}$
Row 2	$\dot{A}=\frac{\alpha}{2\pi}\frac{1}{1+t^2}$	$\dot{B}=\frac{\beta}{2\pi}\frac{t}{1+t^2}$	$\dot{B}=\frac{\gamma}{\pi}\frac{1}{(1+t^2)^{1/2}}$
Row 3	$\dot{A}=\begin{cases}\frac{\alpha}{\pi}, -\frac{\pi}{2}\le t\le\frac{\pi}{2}\\ 0,\ \text{elsewhere}\end{cases}$	$\dot{B}=\begin{cases}\frac{\beta}{\pi}\tan t\\ 0\end{cases}$	$\dot{B}=\begin{cases}\frac{2\gamma}{\pi}\sec t\\ 0\end{cases}$
Row 4	$\dot{A}=\frac{\alpha}{\sqrt{\pi}}e^{-t^2}$	$\dot{B}=\frac{\beta}{\sqrt{\pi}}e^{-t^2}\tan[\frac{1}{2}\pi\,\mathrm{erf}(t)]$	$\dot{B}=\frac{2\gamma}{\sqrt{\pi}}e^{-t^2}\sec[\frac{1}{2}\pi\,\mathrm{erf}(t)]$
Row 5	$\dot{A}=\alpha e^{-\lvert t\rvert}$	$\dot{B}=\mathrm{sgn}(t)\frac{\beta}{4}e^{-r\lvert t\rvert}\tan[\frac{1}{2}\pi(1-e^{-r\lvert t\rvert})]$	$\dot{B}=\frac{\gamma}{2}e^{-r\lvert t\rvert}\sec[\frac{1}{2}\pi(1-e^{-r\lvert t\rvert})]$

$$a_1(\tau)=\cos(\tfrac{1}{2}\Gamma\tau)+i\frac{\beta}{\Gamma}\sin(\tfrac{1}{2}\Gamma\tau)\ , \tag{67a}$$

$$a_2(\tau)=i\frac{\alpha}{\Gamma}\sin(\tfrac{1}{2}\Gamma\tau)\ , \tag{67b}$$

where Γ is given by Eq. (63). In terms of the time-dependent pulse functions, we have

$$a_1(t)=\cos[\tfrac{1}{2}\Gamma\phi(t)]+i\frac{\beta}{\Gamma}\sin[\tfrac{1}{2}\Gamma\phi(t)]\ , \tag{68a}$$

TABLE I. Various realizations of the $(\alpha\beta\gamma)$ and $(\alpha\beta)$ models.

Model	Specific description	Equations in text	$\dot{A}$ (Rabi frequency)	$\dot{B}$ (Atom-field detuning)	Solution
$(\alpha\beta\gamma)$	General $h(z)$	Eqs. (21)	$\frac{\alpha}{\pi}\frac{[z(1-z)]^{1/2}}{h(z)}$	$\frac{1}{\pi}\frac{\beta z+\gamma}{h(z)}$	
	$h(z)=[\dot{g}(t)]^{-1}[z(1-z)]^{\delta}$				
	$\delta=0,\ g(\pm\infty)=\pm\infty$	Eqs. (41)	$\frac{\alpha}{2\pi}\dot{g}(t)\,\mathrm{sech}[\frac{1}{2}g(t)]$	$\frac{1}{2\pi}\dot{g}(t)\{\beta+2\gamma+\beta\tanh[\frac{1}{2}g(t)]\}$	
	$\delta=-\frac{1}{2},\ g(\pm\infty)=\pm\infty$	Eqs. (45)	$\frac{4\alpha}{\pi}\dot{g}(t)\frac{1}{16+[g(t)]^2}$	$\frac{1}{\pi}\frac{\dot{g}(t)}{\{16+[g(t)]^2\}^{1/2}}\left\{\beta+2\gamma+\frac{\beta g(t)}{[16+g(t)^2]^{1/2}}\right\}$	
	$\delta=\frac{1}{2},\ g(\pm\infty)=\pm\frac{1}{2}\pi$	Eqs. (50)	$\frac{\alpha}{\pi}\dot{g}(t)$	$\frac{\dot{g}(t)}{\pi}\frac{\beta+2\gamma+\beta\sin[g(t)]}{\cos[g(t)]}$	
	Special cases of $\delta=0$:				Eqs. (23)–(25)
	Square pulse	Eqs. (52)	$\begin{cases}\frac{\alpha r}{\pi}, & -\frac{\pi}{2r}\le t\le\frac{\pi}{2r}\\ 0, & \text{elsewhere}\end{cases}$	$\begin{cases}\frac{r}{\pi}\frac{\beta+2\gamma+\beta\sin(rt)}{\cos(rt)}, & -\frac{\pi}{2r}\le t\le\frac{\pi}{2r}\\ 0, & \text{elsewhere}\end{cases}$	
	Gaussian pulse	Eqs. (55)	$\frac{\alpha r}{\sqrt{\pi}}\exp(-r^2t^2)$	$\frac{r\exp(-r^2t^2)}{\sqrt{\pi}}\frac{\beta+2\gamma+\beta\sin[\frac{1}{2}\pi\,\mathrm{erf}(rt)]}{\cos[\frac{1}{2}\pi\,\mathrm{erf}(rt)]}$	
	Exponential pulse	Eqs. (59)	$\frac{\alpha r}{2}\exp(-r\lvert t\rvert)$	$\frac{r}{2}\exp(-r\lvert t\rvert)\frac{\beta+2\gamma\mp\beta\sin[\frac{1}{2}\pi(1-e^{-r\lvert t\rvert})]}{\cos(\frac{1}{2}\pi)(1-e^{-r\lvert t\rvert})},\quad t\lessgtr 0\,.$	
$(\alpha\beta)$	Similarly modulated $\dot{A}$ and $\dot{B}$: $\dot{\phi}(\pm\infty)=0$	Eqs. (62)	$\alpha\dot{\phi}(t)$	$\beta\dot{\phi}(t)$	Eqs. (68) and (69)
	Special cases of similarly modulated $\dot{A}$ and $\dot{B}$	Eq. (71)	$\alpha\exp(-r^2t^2)$	$\beta\exp(-r^2t^2)$	Eqs. (68), (69), and (72)
		Eq. (73)	$\alpha\exp(-r\lvert t\rvert)$	$\beta\exp(-r\lvert t\rvert)$	Eqs. (68), (69), and (74)

$$a_2(t)=i\frac{\alpha}{\Gamma}\sin[\tfrac{1}{2}\Gamma\phi(t)] . \quad (68b)$$

The atomic populations in levels 1 and 2 are thus given by

$$|a_1(t)|^2=\frac{\alpha^2+2\beta^2}{2\Gamma^2}+\frac{\alpha^2}{2\Gamma^2}\cos[\Gamma\phi(t)] , \quad (69a)$$

$$|a_2(t)|^2=\frac{\alpha^2}{2\Gamma^2}\{1-\cos[\Gamma\phi(t)]\} . \quad (69b)$$

Notice that unlike in the case of the $(\alpha\beta\gamma)$ model in which the amplitude area is normalized to be always equal to α, we have not normalized the amplitude area in Eq. (62a) to be equal to α; instead the amplitude area is seen to be equal to $\alpha\phi(+\infty)$.

If we denote

$$\Phi(t)=\Gamma\phi(t) , \quad (70)$$

it is seen that the population completely returns to the ground level when $\Phi(t)=2n\pi$, $n=1,2,\ldots$, and that the population is never completely excited to the upper level unless $\beta=0$, in which case a complete population inversion occurs when $\Phi(t)=(2n-1)\pi$.

As an example, if $\dot{A}$ and $\dot{B}$ are both Gaussian, then

$$\dot{\phi}(t)=\exp(-r^2t^2) , \quad (71)$$

and the use of Eq. (64) gives

$$\phi(t)=\int_{-\infty}^{t}\exp(-r^2t^2)dt=\frac{\sqrt{\pi}}{2r}[1+\mathrm{erf}(rt)] , \quad (72)$$

which we use in Eqs. (68) and (69) for our solution. Another example is if $\dot{A}$ and $\dot{B}$ are both exponential, then we have

$$\dot{\phi}(t)=\exp(-r|t|) \quad (73)$$

and

$$\phi(t)=\begin{cases}\frac{1}{r}\exp(-r|t|), & t<0\\ \frac{1}{r}[2-\exp(-r|t|)], & t>0\end{cases} \quad (74)$$

for our solution given by Eqs. (68).

The solutions with three input parameters were treated in the previous section. The input parameters are α, β, and γ; we do not count r, the time-scaling parameter. They involve hypergeometric functions that appear as solutions of differential equations with three regular singular points. One can simplify the calculations by demanding two regular singular points. This condition leads to the $\alpha\beta$ model, which has been treated in this section.

VI. SUMMARY

We have presented analytic solutions to the two-state problem, Eqs. (23)–(25) for the $(\alpha\beta\gamma)$ model and Eqs. (68) and (69) for the $(\alpha\beta)$ model in which the pulse functions given by Eqs. (21) and (62), respectively, allow an infinite variety of amplitude and frequency modulations. Our results are summarized in Table I for the general cases involving arbitrary functions of a general type, as well as some special realizations including the hyperbolic secant, Lorentzian, square, Gaussian, and exponential amplitude-modulation functions together with their corresponding frequency-modulation functions. In Fig. 1, we present in column (a) sketches of these special amplitude-modulation functions; in column (b) sketches of their corresponding antisymmetric frequency-modulation functions obtained under the assumption that $\gamma=-\frac{1}{2}\beta$ in the $(\alpha\beta\gamma)$ model; and in column (c) sketches of their symmetric frequency modulation functions obtained under the assumption that $\beta=0$ in the $(\alpha\beta\gamma)$ model. For given values of α, β, and γ, the final outcome of the degree of atomic excitation is independent of the type of the pulse functions used. For the $(\alpha\beta)$ model, the frequency-modulated function is assumed to have the same shape, except for a multiplicative constant, as the amplitude-modulated function and the final outcome again only depends on the values of α and β but not on the shape of the pulse functions.

We should also mention those solutions that can be derived from the confluent hypergeometric equation, the Landau-Zener solution being one of them. We have recently succeeded[17,18] in extending the Landau-Zener solution to the three-level problem by finding an appropriate generalization of the confluent hypergeometric function.

It should be pointed out that the importance of these analytic solutions goes beyond the two- and three-level problems because various many-level problems possessing certain types of dynamic symmetries can be reduced to the two- or three-level problem.[19]

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