Extension of the Bloch-Nordsieck model

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An analysis of infrared radiation in the potential scattering of a Dirac electron was presented by Bloch and Nordsieck [Phys. Rev. 52, 54 (1937)], in their now classic work, to illustrate how divergence difficulties are removed by nonperturbative treatment of the radiation of soft photons. It is shown here how an improved low-frequency approximation can be obtained which provides a small correction to the Bloch-Nordsieck sum rule for the scattering cross section, while still requiring as input only physical (on-shell) values of the amplitude for scattering in the absence of the radiation field. The calculation is based on a variational determination of the radiative amplitude with trial functions chosen in accordance with the Bloch-Nordsieck approximation. An extension of the model scattering problem is introduced which allows for potentials with a Coulomb tail. The applicability of the approximation procedure to the analogous problem of relativistic scattering in a low-frequency *external* field is pointed out.

I. INTRODUCTION

The infrared divergence problem was resolved many years ago by Bloch and Nordsieck[1] (BN) in the context of a relatively simple model. They considered the scattering of a Dirac electron by a short-range potential and allowed the electron to interact only with low-frequency photons—those with frequencies $\omega < \omega_s$, with $\hbar\omega_s$ small compared to the electron rest energy. In order to simplify the problem while still retaining its essential features, BN introduced the following two low-frequency approximations.

(i) Interaction with the radiation field is accounted for only in asymptotic states, that is, before and after the collision, not during the collision.

(ii) The asymptotic states, modified by the radiative interaction, are determined nonperturbatively, but in the approximation in which the Dirac matrices $\boldsymbol{\alpha}$ and β are replaced by the constants $\mathbf{v}/c$ and $(1-v^2/c^2)^{1/2}$, respectively, where $\mathbf{v}$ is the appropriate (initial or final) electron velocity.

The cross section, summed over all final photon states, which was calculated by BN on the basis of these approximations, was shown to be identical to the cross section for scattering with no interaction with the radiation field.

The removal of divergences was, of course, the main objective of the BN calculation. In view of the fundamental importance of this problem, however, it seems worthwhile to pursue the calculation one step further. We shall do so here, examining the leading terms which provide small corrections to the BN approximation. That progress along these lines should be possible, in the general context of relativistic collision physics, was suggested by Brown and Goble[2] in the course of their study of the relation between soft-photon approximations and the classical limit. While the present analysis is confined to the BN model, techniques and insights developed here might well prove useful in more realistic scattering problems. In this connection we note that infrared radiation in the presence of long-range Coulomb forces is a subject that has received proper mathematical treatment only relatively recently,[3] and few applications of the theory have appeared. Explicit treatment of this problem in a simple model may be of some heuristic value and for this reason we have generalized the model to allow for scattering potentials with a Coulomb tail, i.e., $V(r)\sim g/r$ as $r\rightarrow\infty$. It should also be mentioned that problems involving scattering in an intense, low-frequency external field (a laser field, for example) can be treated as a special case of the formalism and this provides additional motivation for attempting to improve the accuracy of the BN approximation. The external-field problem is discussed briefly later on, in Sec. IV.

We have previously studied a nonrelativistic version of the BN model and have derived corrections to the analog of the BN approximation.[4] There are two main sources of complication in a relativistic extension. Firstly, with v/c no longer treated as a small parameter, electron recoil effects, arising from the transfer of momentum to the field, must be dealt with more carefully. This can be done, as will be shown, although one is left with expressions for the correction terms which are rather cumbersome and difficult to evaluate explicitly compared to the nonrelativistic version. A second complication arises from the fact that the BN asymptotic states are only approximate solutions of the Dirac equation, the Dirac matrices having been replaced by constants [approximation (ii) mentioned above]. An iterative procedure for improving on this approximation was suggested by BN, but even if that were implemented there would still remain approximation (i) which introduces errors of comparable magnitude. We have found it more convenient to deal with both sources of error at the same time by employing a variational procedure. The BN approximation is now used as the basis for the choice of trial functions; long-range Coulomb effects can be included at this stage by imposition of the appropriate asymptotic boundary conditions. The variational principle provides an estimate of the transition amplitude which corrects for first-order errors in the trial function, leaving errors of second order. The transition amplitude obtained in this manner [see Eq. (3.19)] is of a simple product form. One factor contains effects of the radiation field. The other factor is similar in form to the matrix element for *single*-photon bremsstrahlung. A low-frequency approximation for this latter matrix element, sufficiently accurate so as to maintain the level of accuracy of the original variational approximation, can be obtained by an asymptotic evaluation in configuration space, as described in detail previously.[5] The variational construction of the transition amplitude is presented in Sec. III; this is then used to obtain a low-frequency approximation for the cross section summed over final photon states. Our results are discussed and summarized in Sec. IV. We begin, in Sec. II, with a brief review of the BN method in order to introduce notation and prepare for later developments.

II. BLOCH-NORDSIECK APPROXIMATION

The time-independent Dirac equation, in the absence of the scattering potential but including the radiative interaction, is written, in units such that $\hbar=c=1$, as

$$[\boldsymbol{\alpha}\cdot(\mathbf{p}_e-e\,\mathbf{A})+\beta m+H_F]\,|\,\psi\rangle=E\,|\,\psi\rangle\ . \qquad (2.1)$$

Here $\mathbf{p}_e$ is the electron momentum operator,

$$\mathbf{A}(\mathbf{r})=\sum_{i=0}^{s}\left(\frac{2\pi}{\omega_i\Omega}\right)^{1/2}\boldsymbol{\varepsilon}_i(a_i e^{i\mathbf{k}_i\cdot\mathbf{r}}+a_i^{\dagger}e^{-i\mathbf{k}_i\cdot\mathbf{r}})\ , \qquad (2.2)$$

represents the soft-photon contribution to the vector potential, and

$$H_F=\sum_{i=0}^{s}\omega_i a_i^{\dagger}a_i \qquad (2.3)$$

is the soft-photon energy operator. (The quantization volume Ω will eventually be allowed to become infinite, in the usual way.) In the absence of the electron-field interaction the solutions are of the form $|\,n\rangle\,|\,\mathbf{p}\rangle$, where the photon states, labeled by a collection of occupation numbers $n=\{n_0,n_1,\ldots,n_s\}$, satisfy $H_F\,|\,n\rangle=E_n\,|\,n\rangle$, with $E_n=\sum_i\omega_i n_i$, and where $|\,\mathbf{p}\rangle$ (we suppress spin indices) satisfies

$$(\boldsymbol{\alpha}\cdot\mathbf{p}_e+\beta m)\,|\,\mathbf{p}\rangle=E_{\mathbf{p}}\,|\,\mathbf{p}\rangle\ . \qquad (2.4)$$

We have $E_{\mathbf{p}}=(p^2+m^2)^{1/2}$ and

$$\langle\,\mathbf{r}\,|\,\mathbf{p}\rangle=(2\pi)^{-3/2}e^{i\mathbf{p}\cdot\mathbf{r}}\gamma(\mathbf{p})\ , \qquad (2.5)$$

with γ representing a four-component spinor. (Actually, such plane-wave states do not provide an adequate basis in the presence of Coulomb fields. The appropriate modification will be introduced in Sec. III.)

With the radiative interaction included we look for a solution of the form

$$|\,\Phi_{n\mathbf{p}}\rangle=W\,|\,n\rangle\,|\,\mathbf{p}\rangle\ , \qquad (2.6)$$

where W is a unitary wave operator. As shown by BN, an approximate solution, valid for $\omega_s\ll m$, can be obtained by replacing Eq. (2.1) with

$$[\mathbf{v}\cdot(\mathbf{p}_e-e\,\mathbf{A})+(1-v^2)^{1/2}m+H_F]W\,|\,n\rangle\,|\,\mathbf{p}\rangle=E_{n\mathbf{p}}W\,|\,n\rangle\,|\,\mathbf{p}\rangle\ . \qquad (2.7)$$

The total energy is represented as

$$E_{n\mathbf{p}}=E_n+E_{\mathbf{p}}+\Delta_{\mathbf{p}}\ . \qquad (2.8)$$

Conservation of total momentum implies, quite generally, that

$$[\mathbf{p}_e,W]=-[\mathbf{p}_F,W]\ , \qquad (2.9)$$

with $\mathbf{p}_F=\sum_i\mathbf{k}_i a_i^{\dagger}a_i$ representing the field momentum. Using this commutation relation, the eigenvalue equations satisfied by $|\,n\rangle$ and $|\,\mathbf{p}\rangle$, and the relation $E_{\mathbf{p}}=\mathbf{v}\cdot\mathbf{p}+(1-v^2)^{1/2}m$, we see that Eq. (2.7) is equivalent to

$$([H_F,W]-\mathbf{v}\cdot[\mathbf{p}_F,W])\,|\,n\rangle=(e\mathbf{v}\cdot\mathbf{A}+\Delta_{\mathbf{p}})W\,|\,n\rangle\ . \qquad (2.10)$$

Let us write

$$-e\mathbf{v}\cdot\mathbf{A}=\sum_i\eta_i\rho_i(a_i e^{i\mathbf{k}_i\cdot\mathbf{r}}+a_i^{\dagger}e^{-i\mathbf{k}_i\cdot\mathbf{r}})\ , \qquad (2.11)$$

with $\eta_i=\omega_i-\mathbf{v}\cdot\mathbf{k}_i$ and

$$\eta_i\rho_i=-e\left(\frac{2\pi}{\omega_i\Omega}\right)^{1/2}\boldsymbol{\varepsilon}_i\cdot\mathbf{v}\ . \qquad (2.12)$$

It is then easily seen that Eq. (2.10) will be satisfied if the commutation relations

$$[a_i,W]=-\rho_i e^{-i\mathbf{k}_i\cdot\mathbf{r}}W\ , \qquad (2.13a)$$

$$[a_i^{\dagger},W]=-\rho_i e^{i\mathbf{k}_i\cdot\mathbf{r}}W \qquad (2.13b)$$

hold and the level shift is chosen as

$$\Delta_{\mathbf{p}}=-\sum_i\eta_i\rho_i^2\ . \qquad (2.14)$$

Writing $W=W(\{\rho\})$, where $\{\rho\}$ represents the set $\{\rho_0,\rho_1,\cdots\rho_s\}$, one readily verifies that

$$W(\{\rho\})=\exp\left[\sum_i\rho_i(e^{i\mathbf{k}_i\cdot\mathbf{r}}a_i-e^{-i\mathbf{k}_i\cdot\mathbf{r}}a_i^{\dagger})\right] \qquad (2.15)$$

provides a solution of Eqs. (2.13) and hence of Eq. (2.7).

If (following BN) we treat the scattering in Born approximation the transition amplitude to a particular final state $|\,n'\rangle\,|\,\mathbf{p}'\rangle$ can be written as

$$M^{\rm Born}_{n'\mathbf{p}';n\mathbf{p}}=\langle\Phi_{n'\mathbf{p}'}\,|\,V\,|\,\Phi_{n\mathbf{p}}\rangle=\langle n'\,|\,\langle\mathbf{p}'\,|\,W^{\dagger}(\{\rho'\})VW(\{\rho\})\,|\,n\rangle\,|\,\mathbf{p}\rangle\ , \qquad (2.16)$$

where the prime on the ρ_i parameters indicates the replacement of $\mathbf{v}$ with $\mathbf{v}'$. The expression (2.16) can be simplified by noting that V and W commute, and by making use of the identity[6]

$$W^{\dagger}(\{\rho'\})W(\{\rho\})=W(\{\rho-\rho'\})\ . \qquad (2.17)$$

This leads us to consider the matrix element $\langle n'\,|\,W(\{\rho-\rho'\})\,|\,n\rangle$. To proceed we observe that the identity

$$a_i e^{i\mathbf{k}_i\cdot\mathbf{r}}=e^{-i\mathbf{p}_F\cdot\mathbf{r}}a_i e^{i\mathbf{p}_F\cdot\mathbf{r}}\ , \qquad (2.18)$$

which follows directly from the basic commutation relations, can be used, along with its adjoint, to derive the useful representation

$$W(\{\rho-\rho'\})=e^{-i\mathbf{p}_F\cdot\mathbf{r}}W_0(\{\rho-\rho'\})e^{i\mathbf{p}_F\cdot\mathbf{r}}\ , \tag{2.19}$$

with

$$W_0(\{\rho-\rho'\})=\exp\left[\sum_i(\rho_i-\rho_i')(a_i-a_i^\dagger)\right]. \tag{2.20}$$

The matrix element of interest may now be expressed as

$$\langle n'\,|\,W(\{\rho-\rho'\})\,|\,n\,\rangle=e^{i(\mathbf{p}_n-\mathbf{p}_{n'})\cdot\mathbf{r}}\langle n'\,|\,W_0(\{\rho-\rho'\})\,|\,n\,\rangle\ , \tag{2.21}$$

where $\mathbf{p}_n=\sum_i\mathbf{k}_i n_i$ is the eigenvalue of $\mathbf{p}_F$ in the state $|\,n\,\rangle$. Assuming for simplicity that the potential is local, central, and spin independent, and defining

$$\widetilde{V}(\mathbf{q})=(2\pi)^{-3}\int d^3r\,e^{-i\mathbf{q}\cdot\mathbf{r}}V(r)\ , \tag{2.22}$$

we obtain

$$M^{\rm Born}_{n'\mathbf{p}';n\,\mathbf{p}}=\widetilde{V}(\mathbf{p}'+\mathbf{p}_{n'}-\mathbf{p}-\mathbf{p}_n)\langle n'\,|\,W_0(\{\rho-\rho'\})\,|\,n\,\rangle\ . \tag{2.23}$$

This amplitude vanishes in the limit of infinite quantization volume, corresponding to vanishing probability for emitting only a finite number of soft photons. To obtain a physically meaningful and useful result one may compute the cross section, summed over final states of the field; this is of the form

$$d\sigma=\frac{(2\pi)^4}{v}\sum_{n'}\int d^3p'\,\delta(E_{n'\mathbf{p}'}-E_{n\,\mathbf{p}})\,|\,M_{n'\mathbf{p}';n\,\mathbf{p}}\,|^2\ , \tag{2.24}$$

with M, in the present approximation, replaced by $M^{\rm Born}$. If we ignore the small shifts $E_{n'}-E_n$ and $\mathbf{p}_{n'}-\mathbf{p}_n$ corresponding to energy and momentum transferred to the field during the collision we can easily perform the sum over n' using the unitary property

$$\sum_{n'}|\,\langle n'\,|\,W_0(\{\rho-\rho'\})\,|\,n\,\rangle\,|^2=1\ . \tag{2.25}$$

This leads to the BN sum rule

$$\frac{d\sigma}{d\Omega}\cong\frac{d\sigma^{(0)}}{d\Omega}\ , \tag{2.26}$$

where $d\sigma^{(0)}/d\Omega$ is the field-free differential cross section. The restriction to the Born approximation can be removed, as first shown by Nordsieck[7] in the context of a nonrelativistic formulation. In the following a generalization of the sum rule is obtained which allows for potentials which are Coulombic at great distances and which includes corrections of first order in the cutoff frequency ω_s.

III. VARIATIONAL APPROXIMATION

The variational method provides a straightforward and convenient procedure for improving on the calculation outlined in Sec. II. If the scattering potential is of short range the BN wave functions may be adopted as the asymptotic states which form the basis for the construction of trial functions. To allow for potentials $V(r)$ which have the Coulombic form g/r for large r the plane waves must be modified; we define states

$$|\,\Phi^{(\pm)}_{n\,\mathbf{p}}\rangle=W\,|\,n\,\rangle\,|\,\mathbf{p}^{(\pm)}\rangle\ , \tag{3.1}$$

where, for $r\to\infty$,

$$\langle\,\mathbf{r}\,|\,\mathbf{p}^{(\pm)}\rangle\sim(2\pi)^{-3/2}\exp[i\mathbf{p}\cdot\mathbf{r}\pm i(g/v)\ln(pr\mp\mathbf{p}\cdot\mathbf{r})]\gamma(\mathbf{p})\ . \tag{3.2}$$

Consider now the Dirac equation

$$H\Psi^{(\pm)}_{n\,\mathbf{p}}=E_{n\,\mathbf{p}}\Psi^{(\pm)}_{n\,\mathbf{p}}\ , \tag{3.3}$$

with Hamiltonian

$$H=\boldsymbol{\alpha}\cdot(\mathbf{p}_e-e\,\mathbf{A})+\beta m+H_F+V\ , \tag{3.4}$$

and with solutions satisfying outgoing-wave $(+)$ or incoming-wave $(-)$ boundary conditions at infinity. Such solutions may be represented formally as $\Psi^{(\pm)}_{n\,\mathbf{p}}=\Phi^{(\pm)}_{n\,\mathbf{p}}+\widetilde{\Psi}^{(\pm)}_{n\,\mathbf{p}}$ with

$$\widetilde{\Psi}^{(\pm)}_{n\mathbf{p}}=\lim_{\epsilon\to0+}\widetilde{\Psi}_{n\,\mathbf{p}}(E_{n\,\mathbf{p}}\pm i\epsilon) \tag{3.5}$$

and

$$\widetilde{\Psi}_{n\,\mathbf{p}}(E)=(E-H)^{-1}(H-E_{n\,\mathbf{p}})\Phi^{(\pm)}_{n\,\mathbf{p}}\ . \tag{3.6}$$

The S matrix element is given by

$$S_{n'\mathbf{p}';n\,\mathbf{p}}=-(2\pi i)\delta(E_{n'\mathbf{p}'}-E_{n\,\mathbf{p}})M_{n'\mathbf{p}';n\,\mathbf{p}} \tag{3.7}$$

with

$$M_{n'\mathbf{p}';n\,\mathbf{p}}=\lim_{\varepsilon\to0+}i\varepsilon\langle\Phi^{(-)}_{n'\mathbf{p}'}\,|\,\widetilde{\Psi}_{n\,\mathbf{p}}(E_{n\,p}+i\varepsilon)\rangle\ . \tag{3.8}$$

(Here and in the following we make use of results—formulation of the scattering problem and derivation of a variational principle—previously obtained for scattering in the presence of long-range Coulomb interactions.[8] The formal generalization of those results to allow for the presence of a radiation field introduces no real complications and is incorporated in the present discussion without further comment.) The outgoing wave $\langle\,\mathbf{r}\,|\,\widetilde{\Psi}^{(+)}_{n\,\mathbf{p}}\rangle$ will, for $r\to\infty$, consist of a superposition of terms, each corresponding to a "channel" $\{n''\mathbf{p}''\}$ specified by a collection of photon occupation numbers $\{n''\}$ and by a momentum $\mathbf{p}''$, with $E_{n''\mathbf{p}''}=E_{n\,\mathbf{p}}$. If the right-hand side of Eq. (3.8) is to be nonvanishing the spatial integration must be divergent, and it is sufficient, in evaluating this singular integral, to consider only the contribution from the asymptotic domain. Closer analysis along these lines[8] reveals that $M_{n'\mathbf{p}';n\,\mathbf{p}}$ may be identified with the amplitude of the outgoing-wave component of $\Psi^{(\pm)}_{n\,\mathbf{p}}$ in channel $\{n'\mathbf{p}'\}$.

Let us now introduce trial functions $\Psi^{(\pm)}_{n\,\mathbf{p};t}=\Phi^{(\pm)}_{n\,\mathbf{p}}+\widetilde{\Psi}^{(\pm)}_{n\,\mathbf{p};t}$, with $\widetilde{\Psi}^{(\pm)}_{n\,\mathbf{p};t}$ defined by a limiting procedure analogous to that shown in Eq. (3.5), and a trial scattering amplitude

$$M^{t}_{n'\mathbf{p}';n\,\mathbf{p}}=\lim_{\varepsilon\to0+}i\varepsilon\langle\Phi^{(-)}_{n'\mathbf{p}'}\,|\,\widetilde{\Psi}_{n\,\mathbf{p};t}(E_{n\,\mathbf{p}}+i\varepsilon)\rangle\ . \tag{3.9}$$

The variational approximation[8] (a generalization of that first derived by Kohn[9]) then takes the form $M_{n'\mathbf{p}';n\,\mathbf{p}}\cong M^{v}_{n'\mathbf{p}';n\,\mathbf{p}}$ with

$$M^{v}_{n'\mathbf{p}';n\mathbf{p}}=M^{t}_{n'\mathbf{p}';n\mathbf{p}}+\langle\Psi^{(-)}_{n'\mathbf{p}';t}\,|\,H-E_{n\mathbf{p}}\,|\,\Psi^{(+)}_{n\mathbf{p};t}\rangle\ . \tag{3.10}$$

First-order errors in the trial functions will lead to errors in the estimate (3.10) which are only of second order.

Now let $u^{(\pm)}_{\mathbf{p}}(\mathbf{r})$ represent incoming- and outgoing-wave solutions of the Dirac equation

$$(\boldsymbol{\alpha}\cdot\mathbf{p}_e+\beta m+V)u^{(\pm)}_{\mathbf{p}}=E_{\mathbf{p}}u^{(\pm)}_{\mathbf{p}}(\mathbf{r})\ , \tag{3.11}$$

appropriate to scattering in the presence of the potential V but with no interaction with the radiation field. As trial functions we choose

$$\langle\mathbf{r}\,|\,\Psi^{(\pm)}_{n\mathbf{p};t}\rangle=W\,|\,n\,\rangle u^{(\pm)}_{\mathbf{p}}(\mathbf{r})\ , \tag{3.12}$$

which retains the factorized form of the BN wave functions but improves on it through inclusion of the effect of the potential. With this choice of trial function the trial scattering amplitude $M^{t}_{n'\mathbf{p}';n\mathbf{p}}$ vanishes for $n'\neq n$. This follows from the fact that p' and p are unequal for $n'\neq n$, a consequence of energy conservation, and in that case the spatial integration in Eq. (3.9) will be nonsingular.[10] The expression (3.9) then vanishes in the limit $\varepsilon\to 0$. A nonzero result, which is easily evaluated, is obtained for $n'=n$ but we shall eventually take the continuum limit (quantization volume $\Omega\to\infty$) and sum over final photon states. Then a single state $n'=n$ makes negligible contribution and we need not include it. (The situation is different in the case where the field is external and the photon states are discrete, as discussed in Sec. IV.) The variational approximation then becomes, for $n'\neq n$,

$$M^{v}_{n'\mathbf{p}';n\mathbf{p}}=\langle u^{(-)}_{\mathbf{p}'}\,|\,\langle n'\,|\,W^{\dagger}(\{\rho'\})(H-E_{n\mathbf{p}})W(\{\rho\})\,|\,n\rangle\,|\,u^{(+)}_{\mathbf{p}}\rangle\ . \tag{3.13}$$

A straightforward calculation, making use of Eqs. (2.8)–(2.10), along with the commutation relations (2.13) and the eigenvalue equations satisfied by $|\,n\,\rangle$ and $u^{(+)}_{\mathbf{p}}$, gives

$$(H-E_{n\mathbf{p}})W(\{\rho\})\,|\,n\rangle\,|\,u^{(+)}_{\mathbf{p}}\rangle =(\boldsymbol{\alpha}-\mathbf{v})\cdot\mathbf{P}W(\{\rho\})\,|\,n\rangle\,|\,u^{(+)}_{\mathbf{p}}\rangle\ , \tag{3.14}$$

with

$$\mathbf{P}\equiv-e\,\mathbf{A}-\mathbf{p}_F+W(\{\rho\})\mathbf{p}_F W^{\dagger}(\{\rho\})\ . \tag{3.15}$$

The commutation relations (2.13) may be used to rewrite this as

$$\mathbf{P}=-e\,\mathbf{A}+\sum_i \mathbf{k}_i\rho_i(a_ie^{i\mathbf{k}_i\cdot\mathbf{r}}+a_i^{\dagger}e^{-i\mathbf{k}_i\cdot\mathbf{r}}+\rho_i)\ . \tag{3.16}$$

Since $\mathbf{P}$ is a sum of terms, each proportional to either $a_ie^{i\mathbf{k}_i\cdot\mathbf{r}}$ or $a_i^{\dagger}e^{-i\mathbf{k}_i\cdot\mathbf{r}}$ it follows, as in the discussion leading to Eq. (2.19), that

$$W^{\dagger}(\{\rho'\})\mathbf{P}W(\{\rho\})=e^{-i\mathbf{p}_F\cdot\mathbf{r}}\mathbf{Q}e^{i\mathbf{p}_F\cdot\mathbf{r}}\ , \tag{3.17}$$

where

$$\mathbf{Q}=W^{\dagger}(\{\rho'\})\mathbf{P}W(\{\rho\})\,|_{\mathbf{r}=0}\ . \tag{3.18}$$

The variational expression then simplifies to

$$M^{v}_{n'\mathbf{p}';n\mathbf{p}}=\mathbf{m}_{\mathbf{p}'\mathbf{p}}(\mathbf{k}_{n'n})\cdot\langle n'\,|\,\mathbf{Q}\,|\,n\,\rangle\ , \tag{3.19}$$

with $\mathbf{k}_{n'n}=\mathbf{p}_{n'}-\mathbf{p}_n$ and

$$\mathbf{m}_{\mathbf{p}'\mathbf{p}}(\mathbf{k})=(2\pi)^{-3}\int d^3r\,[u^{(-)}_{\mathbf{p}'}(\mathbf{r})]^{\dagger}(\boldsymbol{\alpha}-\mathbf{v})u^{(+)}_{\mathbf{p}}(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\ . \tag{3.20}$$

An equivalent version of the variational expression may be obtained from Eq. (3.13) using an integration-by-parts procedure; it is of the same form as (3.19), but with $\mathbf{v}$ replaced by $\mathbf{v}'$ in Eq. (3.20) and ρ_i replaced by ρ_i' in Eq. (3.16).

As remarked earlier, useful results are obtained by summing the cross section over final states containing unobservable soft photons. Thus, generalizing the procedure which led to Eq. (2.26) we examine the expression (2.24) for the cross section with M replaced by the variational approximation (3.19), and with the initial state $|\,\hat{n}\,\rangle$ chosen for simplicity as the photon vacuum $|\,0\rangle$. Let us write

$$M^{v}_{n'\mathbf{p}';0\mathbf{p}}=\langle n'\,|\,U\,|\,0\rangle\ , \tag{3.21}$$

where, with $\mathbf{k}_{n'}$ replaced by the photon momentum operator in the argument of $\mathbf{m}_{\mathbf{p}'\mathbf{p}}$ we have

$$U=\mathbf{m}_{\mathbf{p}'\mathbf{p}}(\mathbf{p}_F)\cdot\mathbf{Q}\ . \tag{3.22}$$

The approximation procedure of Ref. 5 leads to a representation $\mathbf{m}_{\mathbf{p}'\mathbf{p}}=\mathbf{m}^{(0)}_{\mathbf{p}'\mathbf{p}}+\mathbf{m}^{(1)}_{\mathbf{p}'\mathbf{p}}$, where $\mathbf{m}^{(1)}_{\mathbf{p}'\mathbf{p}}$ provides a first-order correction to the leading term $\mathbf{m}^{(0)}_{\mathbf{p}'\mathbf{p}}$. Correspondingly, we have $U=U^{(0)}+U^{(1)}$. The leading term takes the particularly simple form

$$\langle n'\,|\,U^{(0)}\,|\,0\rangle=\langle n'\,|\,W_0(\{\rho-\rho'\})\,|\,0\rangle t(\hat{p}',\hat{p};p)\ , \tag{3.23}$$

where t is the (field-free) scattering amplitude associated with the solution $u^{(+)}_{\mathbf{p}}$ of Eq. (3.11). We put aside until later (Sec. IV) a discussion of the derivation of Eq. (3.23) and make use of this result here to simplify the expression for the total cross section. Thus, as the leading approximation to the cross section we have, with U replaced by $U^{(0)}$,

$$d\sigma^{(0)}=\frac{(2\pi)^4}{v}\sum_{n'}\int d^3p'\,|\,t(\hat{p}',\hat{p};p)\,|^2\langle 0\,|\,W_0^{\dagger}(\{\rho-\rho'\})\,|\,n'\rangle\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}+E_{n'})\langle n'\,|\,W_0(\{\rho-\rho'\})\,|\,0\rangle\ . \tag{3.24}$$

(The level-shift difference $\Delta_{p'}-\Delta_p$ has been ignored in the argument of the δ function since it is a quantity of higher order than terms ultimately retained.) The sum over states could be performed using closure were it not for the small photon energy $E_{n'}$ contributing to the energy conservation condition. It is easily verified, however,[4] that with terms of second order in the soft-photon energy ignored $E_{n'}$ may be replaced by the average value

$$R_{p'p}=\langle 0\,|\,W_0^\dagger(\{\rho-\rho'\})H_F W_0(\{\rho-\rho'\})\,|\,0\rangle = \sum_i \omega_i(\rho_i-\rho_i{}')^2 \,. \tag{3.25}$$

This expression, when evaluated in the continuum limit in which $\Omega^{-1}\sum_i$ is replaced by $(2\pi)^{-3}\int d^3k$, is just the energy which the electron would radiate classically in an instantaneous collision changing its momentum from $\mathbf{p}$ to $\mathbf{p}'$.[1] Equation (3.24) then leads to the sum rule

$$d\sigma^{(0)}=\frac{(2\pi)^4}{v}\int d^3p'\,\delta(E_{p'}-E_p+R_{p'p})\,|\,t(\hat{p}',\hat{p};p)\,|^2 \,. \tag{3.26}$$

This differs from the BN approximation (2.26) in the replacement of the Born amplitude by the exact t-matrix element and in the inclusion, to first order, of the energy loss due to radiation in the energy conservation condition.

An improved sum rule is obtained by including the correction term $\mathbf{m}^{(1)}_{p'p}$ in the bremsstrahlung matrix element $\mathbf{m}_{p'p}$ which appears in Eq. (3.22). One finds that, to first order,

$$d\sigma\cong\frac{(2\pi)^4}{v}\int d^3p'\,\delta(E_{p'}-E_p+R_{p'p})\,|\,T\,|^2 \,, \tag{3.27}$$

where

$$T=t(\hat{p}',\hat{p};p)+\langle 0\,|\,W_0^\dagger(\{\rho-\rho'\})\mathbf{m}^{(1)}_{p'p}(\mathbf{p}_F)\cdot\mathbf{Q}\,|\,0\rangle \,. \tag{3.28}$$

Since the photon momentum operator appears nonlinearly in the amplitude $\mathbf{m}^{(1)}_{p'p}(\mathbf{p}_F)$, evaluation of the vacuum expectation value in Eq. (3.28) is not trivial and additional approximations would be required to put the result in more explicit form. Rather than enter into such a calculation at this time let us examine the form of the result in the approximation in which the field momentum $\mathbf{p}_F$ is ignored altogether in the correction term; the electron is then permitted to exchange energy but not momentum with the field, as would be appropriate in the nonrelativistic limit. Then, with $\mathbf{m}^{(1)}_{p'p}(\mathbf{p}_F)\rightarrow\mathbf{m}^{(1)}_{p'p}(0)$ Eq. (3.28) becomes

$$T=t+\mathbf{m}^{(1)}_{p'p}(0)\cdot\langle 0\,|\,W_0^\dagger(\{\rho\})[-e\mathbf{A}(0)]W_0(\{\rho\})\,|\,0\rangle \,. \tag{3.29}$$

The vacuum expectation value is now readily evaluated with the aid of the commutation relations (2.13). With the sum over modes performed in the continuum limit Eq. (3.29) becomes

$$T=t-2e^2\left(\frac{2}{3\pi}\right)\omega_s\mathbf{m}^{(1)}_{p'p}(0)\cdot\mathbf{v} \,. \tag{3.30}$$

The correction term is seen to be proportional to $e^2\omega_s$, confirming the BN identification of the small parameter in the problem. Here we have obtained an expression for the coefficient function in an approximation which neglects the electron recoil momentum. An evaluation of $\mathbf{m}^{(1)}_{p'p}(0)$ in terms of the on-shell t-matrix element and its derivatives has been given previously.[5] In the following section we review the main features of that calculation and include some additional remarks bearing on the relationship between that study of single-photon bremsstrahlung and the infrared radiation problem of present concern.

IV. REMARKS ON THE LOW-FREQUENCY APPROXIMATION

The simplifying feature of the approximation (3.19) is the appearance of all multiphoton effects in a single factor $\langle n\,|\,\mathbf{Q}\,|\,n\rangle$ depending only on the radiation field. The remaining factor $\mathbf{m}_{p'p}(\mathbf{k})$ is similar in form to the matrix element for single-photon bremsstrahlung and a number of approximation techniques are available for the estimation of such matrix elements. In particular, a low-frequency approximation developed earlier[5] can be applied, with only minor changes, to the integral (3.20). We shall not reproduce the details of this calculation here but several general remarks are in order.

The calculational method developed in Ref. 5 for approximate evaluation of integrals of the form (3.20) is based on the fact that at low frequencies the dominant contribution to such an integral comes from the asymptotic domain. With the aid of asymptotic expansions of the wave functions $u_p^{(+)}$ and $u_{p'}^{(-)}$ this dominant contribution may be evaluated analytically. As opposed to other procedures (such as Low's[11]) this method is applicable even when the scattering potential has a Coulomb tail; one simply takes proper account of the logarithmic distortions of the asymptotic form of the wave function which arises as a result of the long-range Coulomb interaction.

By retaining the first *two* terms in the asymptotic expansion, rather than just the leading term, one includes a correction of higher order in the frequency. This additional accuracy is in fact required for consistency since our chief motivation for adopting the variational formulation was precisely to retain such higher-order corrections. One might be concerned that by including an additional term (of order $1/r$ compared to the leading term) in the asymptotic form of the wave functions appearing in Eq. (3.20) one would be required to introduce a cutoff radius R in the radial integration to avoid a divergence at the origin. However, the contribution from the interior region $r<R$ may be suppressed, thereby allowing us to obtain results independent of R. Thus, we may write

$$\boldsymbol{\alpha}=i[H_0,\mathbf{r}] \tag{4.1}$$

where $H_0=\boldsymbol{\alpha}\cdot\mathbf{p}_e+\beta m+V$. Then, using the Dirac equation (3.11) and the relation $[H_0,e^{-i\mathbf{k}\cdot\mathbf{r}}]=\boldsymbol{\alpha}\cdot\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{r}}$ we observe that $\boldsymbol{\alpha}$ in Eq. (3.20) may be replaced by $i\mathbf{r}(E_{p'}-E_p+\boldsymbol{\alpha}\cdot\mathbf{k})$. The additional power of $\mathbf{r}$ introduced by this substitution allows us to obtain a higher-order correction using only asymptotic properties of the wave functions.[5] There is an additional integral in Eq. (3.20), not present in our previous calculation,[5] involving $\mathbf{v}$ rather than $\boldsymbol{\alpha}$. Analogous suppression of the interior contribution to this integral may be achieved by invoking the orthogonality of the wave functions to allow the replacement of $e^{-i\mathbf{k}\cdot\mathbf{r}}$ with $e^{-i\mathbf{k}\cdot\mathbf{r}}-1$. This in turn may be written as

$$e^{-i\mathbf{k}\cdot\mathbf{r}}-1=-i\mathbf{k}\cdot\mathbf{r}\int_0^1 e^{-is\mathbf{k}\cdot\mathbf{r}}ds \; . \tag{4.2}$$

One carries out the radial integration first, the extra factor of $\mathbf{r}$ having the desired effect, as described above. The subsequent integration over s causes no difficulty, the s dependence being of a very simple form.

Since only the on-shell amplitude for scattering in the absence of the radiation field appears in the asymptotic form of the wave function it is evident that not only the leading term but also the first-order correction term in the low-frequency approximation for the bremsstrahlung matrix element can be expressed in terms of the physical scattering amplitude. If, following Rose,[12] we denote this field-free amplitude as $\chi'^{\dagger}A(\hat{p}',\hat{p};p)\chi$, where χ and χ' represent initial and final two-component spin states, then the field-free differential cross section is given by $|\chi'^{\dagger}A(\hat{p}',\hat{p};p)\chi|^2$ and the amplitude of the "large"-component projection of the outgoing-wave part of $u_{\mathrm{p}}^{(+)}(\mathbf{r})$ is given by $A(\hat{r},\hat{p};p)\chi$. The low-frequency approximation derived in Ref. 5 was expressed in terms of $\chi'^{\dagger}A\chi$ and its angular derivatives. We note that the relation between this amplitude and the conventionally defined t-matrix element[13] is

$$-(2\pi)^{-2}\chi'^{\dagger}A(\hat{p}',\hat{p};p)\chi=E_{\mathrm{p}}t(\hat{p}',\hat{p};p) \; . \tag{4.3}$$

If we keep only the leading term in the asymptotic expansion of $u_{\mathrm{p}}^{(+)}$ and of $u_{\mathrm{p}'}^{(-)}$ we obtain a first approximation for the amplitude $\mathbf{m}_{\mathrm{p'p}}(\mathbf{k})$ and hence of $M_{n'\mathrm{p}';n\mathrm{p}}$ from Eq. (3.19). The calculation of this first approximation, which is straightforward and will not be reproduced here, makes use of the integration techniques described above. In addition Eq. (2.10), rewritten with the aid of Eq. (3.15) as

$$(\mathbf{v}\cdot\mathbf{P}+H_F)W|n\rangle=(E_n+\Delta_{\mathrm{p}})W|n\rangle \; , \tag{4.4}$$

is used along with the symmetry property mentioned in the sentence immediately following Eq. (3.20). We find

$$\begin{aligned}M_{n'\mathrm{p}';n\mathrm{p}}\cong{}&(p-p')^{-1}\langle n'|W_0^{\dagger}(\{\rho'\})(E_{n'}+\Delta_{\mathrm{p}'}-H_F)W_0(\{\rho\})|n\rangle(E_{\mathrm{p}}/p')t(p)B(p',p)\\&-(p-p')^{-1}\langle n'|W_0^{\dagger}(\{\rho'\})(E_n+\Delta_{\mathrm{p}}-H_F)W_0(\{\rho\})|n\rangle(E_{\mathrm{p}'}/p)t(p')B(p,p') \; .\end{aligned} \tag{4.5}$$

Here we have suppressed the angular dependence of the t matrix. The function $B(p',p)$, defined in Ref. 5, can be represented here to the required accuracy as

$$B(p',p)=1+iy\delta\ln(\tfrac{1}{2}|\delta|) \tag{4.6}$$

with $\delta=(p-p')/p$ and $y=-gE_{\mathrm{p}}/p$. It should be noted that we have *not* ignored the radiated momentum $\mathbf{k}_{n'n}$ in deriving Eq. (4.5). This momentum appears initially in nearly singular denominators of the form $(p'-p+\hat{p}\cdot\mathbf{k}_{n'n})^{-1}$ and $(p-p'-\hat{p}'\cdot\mathbf{k}_{n'n})^{-1}$, so that its neglect is not justified at the outset, but it drops out of the final result. Equation (4.5) may be simplified by introducing the approximation

$$t(p)\cong t(\bar{p})+\tfrac{1}{2}(p-p')\frac{\partial t}{\partial p} \; , \tag{4.7}$$

where $\bar{p}\equiv(p+p')/2$, with a similar expansion for $t(p')$. Then, if we ignore the logarithmic corrections in the functions $B(p',p)$ and $B(p,p')$, as well as terms of still higher order we find, using energy conservation along with the relation $(E_{\mathrm{p}}-E_{\mathrm{p}'})E_{\mathrm{p}}\cong(p-p')p$, the form

$$M_{n'\mathrm{p}';n\mathrm{p}}\cong\langle n'|W_0(\{\rho-\rho'\})|n\rangle t(\bar{p}) \; . \tag{4.8}$$

[Of course $t(\bar{p})$ could be replaced by $t(p)$, as in Eq. (3.23), if one were not interested in keeping track of higher-order corrections.] It might be supposed that the leading correction to the lowest-order approximation (4.8) originates from the logarithmic term in Eq. (4.6). It turns out, however, that this correction cancels, to order δ, in the construction of the cross section. Thus, the effect of the logarithmic correction is to replace Eq. (4.8) by

$$M_{n'\mathrm{p}';n\mathrm{p}}\cong\langle n'|W_0^{\dagger}(\{\rho-\rho'\})|n\rangle t(p)+(iyE_{\mathrm{p}}/p^2)\ln(\tfrac{1}{2}|\delta|)I_{n'\mathrm{p}';n\mathrm{p}}t(p) \; , \tag{4.9}$$

where

$$I_{n'\mathrm{p}';n\mathrm{p}}\equiv\langle n'|W_0^{\dagger}(\{\rho'\})(\Delta_{\mathrm{p}}+\Delta_{\mathrm{p}'}+E_n+E_{n'}-2H_F)W_0(\{\rho\})|n\rangle \tag{4.10}$$

is a quantity of first order in the soft-photon frequency. Matrix elements of W_0 are real so that the cross term vanishes in taking the absolute square of the amplitude (4.9). This leaves a correction of order $(\delta\ln\delta)^2$. The net result of these considerations is that with corrections of order δ ignored the only effect of the Coulomb tail in the scattering potential is to modify the t-matrix element in the approximation (3.26) for the cross section.

The formalism described above may be applied to the problem of scattering in the presence of a low-frequency external field; one simply chooses photon occupation numbers defining the initial and final states to be sufficiently large. With spontaneous emission ignored and, for simplicity, with only a single mode assumed to be occupied one has

$$\exp[\rho_j(a_j-a_j^\dagger)]\,|\,n_j\rangle \cong \sum_{l=-\infty}^{\infty} J_l(2\rho_j\sqrt{n_j})\,|\,n_j-l\rangle \ . \tag{4.11}$$

This result is established by expanding the exponential, neglecting the commutator $[a_j,a_j^\dagger]$, and ignoring photon depletion effects, so that, e.g.,

$$a_j^\dagger\,|\,n_j-l\rangle \cong \sqrt{n_j}\,|\,n_j-l+1\rangle \ ;$$

one also makes use of the series expansion of the Bessel function J_l. In establishing the variational approximation (3.19) we neglected the trial amplitude $M^t_{n'\mathbf{p}';n\,\mathbf{p}}$ since it is nonvanishing only for $n'=n$ and omission of a single state in establishing the cross section sum rule gives negligible error in the continuum limit. In the external field problem, on the other hand, the amplitude for scattering to a particular photon state is nonvanishing and physically meaningful. We must therefore complete the variational approximation (3.19) by adding the trial amplitude. One finds, using the form (3.12) for the trial function along with the definition (3.9), the result

$$M^t_{n_j'\,\mathbf{p}';n_j\mathbf{p}} = J_0(2(\rho_j-\rho_j')\sqrt{n_j})\,t(\hat{p}',\hat{p};p)\,\delta_{n_j'\,n_j} \ , \tag{4.12}$$

where we have replaced $\langle n_j\,|\,W_0\,|\,n_j\rangle$ by the zeroth order Bessel function in line with the approximation (4.11). The analog of Eq. (4.8) in the external field problem is

$$M_{n_j'\,\mathbf{p}';n_j\mathbf{p}} \cong J_{n_j'-n_j}(2(\rho_j-\rho_j')\sqrt{n_j})\,t(\hat{p}',\hat{p};p) \ , \tag{4.13}$$

valid for all n_j'. For $n_j'=n_j$ it is only the trial amplitude (4.12) which contributes; the external-field analog of the variational correction term (3.19) vanishes for $n_j'=n_j$. To verify this let us write the field-dependent factor in the analog of Eq. (3.19) as

$$\langle n_j\,|\,\mathbf{Q}\,|\,n_j\rangle = -e\langle n_j\,|\,W_0^\dagger(\{\rho'\})\mathbf{A}(0)W_0(\{\rho\})\,|\,n_j\rangle \ . \tag{4.14}$$

Here we replaced $\mathbf{P}$ by $-e\mathbf{A}$ since, according to Eq. (3.15), the difference arises only from commutators of field operators and these are neglected in the present approximation. For the same reason the right-hand side of Eq. (4.14) may be written as

$$-e\langle n_j\,|\,\mathbf{A}(0)W_0(\{\rho-\rho'\})\,|\,n_j\rangle \ .$$

Then, inserting the expansion (2.2) for $\mathbf{A}(0)$ and using the approximation (4.11) this latter matrix element is seen to be proportional to the vanishing quantity $J_1(x)+J_{-1}(x)$, $x=2(\rho_j-\rho_j')\sqrt{n_j}$. The approximation (4.13) for the stimulated bremsstrahlung amplitude can be improved through a more accurate treatment, along the lines described above, of the *single*-photon bremsstrahlung matrix element $\mathbf{m}_{\mathbf{p}',\mathbf{p}}(\mathbf{k})$.[14]

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