Finite-nuclear-size corrections to the Uehling potential

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A study is made of finite-nuclear-size corrections to the Uehling potential for the ground state of hydrogenlike atoms ($Z=1$ and 80). The exact relativistic correction is compared to the results of perturbation theory and to nonrelativistic results based upon the approximations of Zemach. Perturbation theory and the nonrelativistic approximation are consistent with the exact calculation at $Z=1$, but not at $Z=80$. In the nonrelativistic approximation, this size correction is $\frac{1}{2}\alpha(Z\alpha)^5(R/\lambda\!\!\!^{-}_e)^2mc^2$, where R is the rms radius of the nucleus.

I. INTRODUCTION

There have been calculations of finite-nuclear-size corrections to the self-energy and vacuum polarization of various atoms. These size corrections are more important at high Z and must be included to avoid singularities at $Z\alpha\sim 1$.[1,2] In the case of high-Z muonic atoms, there have been calculations of self-energy[3] and vacuum polarization that include nuclear size, some of which include higher orders in the external potential.[4] Nuclear size has also been included in the calculation of self-energy[5] and vacuum polarization[6] for high-Z electronic atoms. Others have calculated these size corrections for low-Z muonic atoms.[7]

In contrast to the above-mentioned cases, size corrections to the self-energy and vacuum polarization for low-Z electronic atoms are much smaller, and therefore require a more precise analysis. Borie,[8] using a nonrelativistic approximation, has recently considered the size corrections to the lowest order self-energy and vacuum polarization in hydrogen. Lepage, Yennie, and Erickson[9] have argued that Borie's result is an overestimation as it is not valid to treat these size corrections nonrelativistically.

The present paper describes the calculation of the nuclear-size correction to the Uehling potential correction for the ground state of an electronic hydrogenlike atom with $Z=1$ or 80. The main focus is not so much on the results—for they are not within the limits of experimental accuracy[10]—as it is on the comparison of the exact results with the results of two approximate methods. The approximate methods are perturbation theory up to and including second order and a nonrelativistic treatment as described in Sec. III B. In this manner, the accuracy of perturbation theory and the nonrelativistic approximation is determined. A method suitable for calculating size corrections to the Uehling potential could serve as a guide to the calculation of size corrections to the self-energy for low-Z electronic atoms.

In Sec. II, the exact calculation, determined by numerically solving the Dirac equation, is described. Approximate methods are described in Sec. III. In Sec. III A, the numerical calculation in perturbation theory is discussed. The unperturbed eigenfunctions and energies are those determined from the Dirac equation for the Coulomb potential. In Sec. III B, the results of the nonrelativistic approximation are determined analytically with the aid of the approximations of Zemach.[11] All present calculations are based on a model in which the nuclear charge distribution is spherical and uniform.

II. EXACT CALCULATION

The radial Dirac equation can be written in the form[12]

$$\begin{bmatrix} 1+V(r)-E & -\frac{1}{r}\frac{d}{dr}r+\frac{\kappa}{r} \\ \frac{1}{r}\frac{d}{dr}r+\frac{\kappa}{r} & -1+V(r)-E \end{bmatrix}\begin{bmatrix} f_1(r) \\ f_2(r) \end{bmatrix}=0\ , \tag{2.1}$$

where $f_1(r)$ and $f_2(r)$ are the upper and lower components of the radial eigenfunction, respectively; E is the energy eigenvalue and $-\kappa$ is the eigenvalue of the operator $K=\beta(\sigma\cdot\mathbf{L}+1)$.[12] Since only radial potentials are considered here, the angular part of the Dirac equation can be factored as already has been done to obtain (2.1). The units chosen are those for which $\hbar=m_e=c=1$. The values $\alpha^{-1}=137.035\,963$ and $\lambda\!\!\!^{-}_e=0.386\,159\,05\times10^{-10}$ cm are used.

With the aid of (2.1), a differential equation is obtained for the ratio of the upper and lower components of the radial eigenfunctions. By defining

$$G(r)=r^{-2\kappa}\frac{f_2(r)}{f_1(r)}\ , \tag{2.2}$$

one has

$$\frac{dG(r)}{dr}=\left[\frac{-2\kappa}{r}+\frac{1}{f_2(r)}\frac{df_2(r)}{dr}-\frac{1}{f_1(r)}\frac{df_1(r)}{dr}\right]G(r)\ . \tag{2.3}$$

Eliminating the derivatives of $f_1(r)$ and $f_2(r)$ which occur in (2.3) with the aid of (2.1), yields

$$\frac{dG(r)}{dr}=[V(r)+1-E]r^{-2\kappa}-[-V(r)+1+E][r^{\kappa}G(r)]^2\ . \tag{2.4}$$

Equation (2.4) can be solved numerically for the energy eigenvalue E for a given potential $V(r)$ and appropriate boundary conditions.

For the potentials of interest, the fourth-order Runge-Kutta method[13] is used to numerically integrate (2.4). As r approaches the origin, the potential for an extended nucleus approaches a constant. The free-particle solutions are also solutions for a constant potential. Therefore, the free-particle solutions substituted in (2.2) is employed to yield the initial condition $G(0)=0$ for $\kappa<0$. Starting at the origin, (2.4) is integrated outward. Equation (2.4) is integrated inwards starting at $r\geq 12$, where the Uehling potential is 12 orders of magnitude smaller than the Coulomb potential, so that $G(r)$ can be initialized using the Coulomb solutions. Both integrations are carried out to a convenient meeting point to 10-figure accuracy. The 10-figure accuracy is chosen so that E is sensitive to the nuclear-size effect. At the meeting point, the two integrated values for G are equal only at the correct energy eigenvalue. A Newton search is employed to find the zero of the difference of the two integrated values for G. The level of accuracy is determined from the convergence of the numerical results as the mesh size is made smaller. The accuracy of the eigenvalue search is determined from the difference of the two integrated values of G. The sensitivity of the integrated value of G to the chosen eigenvalue is also a factor in determining the reliability of the search.

Only the $1S_{1/2}$ state ($\kappa=-1$) is considered here. The shift in the energy eigenvalue (from the Coulomb energy) due to the extended nucleus in the absence of the Uehling potential is calculated first. The potential for the extended nucleus and energy shift are labeled $V_s(r)$ and δE_s. The Uehling potential for the extended nucleus $V_u(r)$, is then added to $V_s(r)$ and the corresponding energy shift δE_{su} is obtained. The difference, $\delta E_u=\delta E_{su}-\delta E_s$, is the energy shift due to the Uehling potential of the extended nucleus (size-Uehling potential). The size corrections due to the Uehling potential is determined by subtracting from δE_u the energy shift due to the Uehling potential for a point nucleus (point-Uehling potential).

The charge distribution normalized to unity

$$\rho(r)=\frac{3}{4\pi r_0^3}\Theta(r_0-r)\ ,$$
$$r_0=\sqrt{5/3}\langle r^2\rangle^{1/2}\ , \tag{2.5}$$

where $\langle r^2\rangle^{1/2}=0.86$ fm for $Z=1$ and $\langle r^2\rangle^{1/2}=5.2$ fm for $Z=80$, yields a potential

$$V_s(r)=\frac{Z\alpha}{2r_0^3}(r^2-3r_0^2)\Theta(r_0-r)-\frac{Z\alpha}{r}\Theta(r-r_0)\ . \tag{2.6}$$

The outward integration is carried to $r=r_0$ for $V(r)=V_s(r)$ in (2.4). For $r\geq r_0$, $G(r)$ is determined from the known Coulomb solutions regular as $r\to\infty$ given by[14]

$$f_1(r)=\frac{1+E}{r^{3/2}}[(\kappa+\gamma/c)W_{\nu-1/2,\lambda}(2cr)+W_{\nu+1/2,\lambda}(2cr)]\ , \tag{2.7}$$

$$f_2(r)=\frac{c}{r^{3/2}}[(\kappa+\gamma/c)W_{\nu-1/2,\lambda}(2cr)-W_{\nu+1/2,\lambda}(2cr)]\ ,$$

where

$$\gamma=Z\alpha,\quad \lambda=(\kappa^2-\gamma^2)^{1/2},\quad c=(1-E^2)^{1/2},\quad \nu=\frac{\gamma E}{c}\ , \tag{2.8}$$

and $W_{\alpha,\beta}(z)$ is the Whittaker function.[13] An integral representation[13]

$$W_{\nu\pm1/2,\lambda}(z)=\frac{e^{-z/2}z^{\lambda-1/2}}{\Gamma(\lambda-\nu+\frac{1}{2}\mp\frac{1}{2})}\times\int_0^\infty e^{-x}(x/z)^{\lambda-\nu-1/2\mp1/2}\times(1+x/z)^{\lambda+\nu-1/2\pm1/2}dx\ ,$$
$$\mathrm{Re}(\lambda-\nu+\tfrac{1}{2}\mp\tfrac{1}{2})>0 \tag{2.9}$$

is employed to numerically calculate $G(r)$ for $r\geq r_0$. However, for $\kappa=-1$, the integral representation for $W_{\nu+1/2,\lambda}$ must first be analytically continued to $\mathrm{Re}(\lambda-\nu)>-1$ by adding and subtracting $e^{-x}(x/z)^{\lambda-\nu-1}$ in the integrand to obtain

$$W_{\nu+1/2,\lambda}(z)=e^{-z/2}\left[z^{\nu+1/2}+\frac{z^{\lambda-1/2}}{\Gamma(\lambda-\nu)}\int_0^\infty e^{-x}(x/z)^{\lambda-\nu-1}[(1+x/z)^{\lambda+\nu}-1]dx\right],\quad \mathrm{Re}(\lambda-\nu)>-1\ . \tag{2.10}$$

Equations (2.9) and (2.10) for $W_{\nu-1/2,\lambda}$ are numerically calculated to 12-figure accuracy by Gaussian quadrature. The value of $G(r_0)$ from the outward integration is matched to the value of $G(r_0)$ from the known Coulomb solutions to obtain δE_s, which is tabulated in Table I.

The Uehling potential for an arbitrary nuclear charge distribution $\rho(\mathbf{r})$ is given by

$$V_u(r)=-\frac{Z\alpha^2}{3\pi}\int_1^\infty dt(t^2-1)^{1/2}\left[\frac{2}{t^2}+\frac{1}{t^4}\right]\int d^3r'\,e^{-2|\mathbf{r}-\mathbf{r}'|t}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ . \tag{2.11}$$

For the charge distribution (2.5), the r' integral is carried out to give

$$V_u(r)=\frac{-Z\alpha^2}{16\pi r r_0^3}\int_1^\infty dt\frac{(t^2-1)^{1/2}}{t^3}\left[\frac{2}{t^2}+\frac{1}{t^4}\right]\{[4rt-(2r_0t+1)(e^{-2(r_0-r)t}-e^{-2(r_0+r)t})]\Theta(r_0-r)$$
$$+[(2r_0t-1)e^{2r_0t}+(2r_0t+1)e^{-2r_0t}]e^{-2rt}\Theta(r-r_0)\}\ . \tag{2.12}$$

For $r\geq r_0$, (2.12) is numerically calculated by Gaussian quadrature; however, for small r_0t the expression in the first set of square brackets is expanded in a power series in r_0t to avoid cancellations in the evaluation of the integrand. The cancellations are more severe for $r\leq r_0$, so the integrand in (2.12) for $r\leq r_0$ is rewritten to obtain

$$V_u(r)=\frac{-Z\alpha^2}{16\pi r r_0^3}\int_1^\infty dt\left[\frac{1-\frac{1}{2t^2}-\frac{1}{8t^4}}{t^2}+\frac{(1-1/t^2)^{1/2}-1+\frac{1}{2t^2}+\frac{1}{8t^4}}{t^2}\right]\left[\frac{2}{t^2}+\frac{1}{t^4}\right]$$
$$\times[4rt-(2r_0t+1)(e^{-2(r_0-r)t}-e^{-2(r+r_0)t})],\quad r\leq r_0\ . \tag{2.13}$$

The first term within the first set of parentheses in (2.13) is reduced to an integral of the type e^{-at}/t by successive integration by parts, and the second term has enough inverse powers of t, so that the expression within the last set of square brackets can be expanded for small r_0t and small rt to $O(t^7)$. The result for $r\leq r_0$ is

$$\begin{aligned}V_u(r)=\frac{-Z\alpha^2}{16\pi r r_0^3}\Bigg\{&\left[-\frac{16r_0^3-24rr_0^2+8r^3}{3}+\frac{12r_0^7-70rr_0^6+168r^2r_0^5-210r^3r_0^4+140r^4r_0^3-42r^5r_0^2+2r^7}{105}\right]\\&\times\int_1^\infty\frac{e^{-2(r_0-r)t}}{t}dt\\&+\left[\frac{16r_0^3+24rr_0^2-8r^3}{3}-\frac{12r_0^7+70rr_0^6+168r^2r_0^5+210r^3r_0^4+140r^4r_0^3+42r^5r_0^2-2r^7}{105}\right]\\&\times\int_1^\infty\frac{e^{-2(r_0+r)t}}{t}dt+\frac{153rr_0^2-83r^3}{9}-\frac{436rr_0^3-160r^3r_0}{15}+\frac{313rr_0^4+38r^3r_0^2-55r^5}{12}\\&-\frac{136rr_0^3+120r^3r_0^3-64r^5r_0}{9}+\frac{61180rr_0^6+105980r^3r_0^4-56364r^5r_0^2-8396r^7}{11025}\\&+\left[\left[8rr_0^2-\frac{8r^3}{3}\right]\left[2\ln2-\frac{35}{24}\right]-\frac{32}{3}rr_0^3\left[\frac{91}{40}-\frac{3\pi}{4}\right]-\left[\frac{3rr_0^4}{4}+\frac{r^3r_0^2}{2}-\frac{r^5}{20}\right]\right.\\&\left.-\left[\frac{64rr_0^5}{15}+\frac{64r^3r_0^3}{9}\right]\left[\frac{35}{24}-\frac{\pi}{2}\right]+\left[\frac{2rr_0^6}{3}+2r^3r_0^4+\frac{2r^5r_0^2}{5}-\frac{2r^7}{105}\right](1-2\ln2)+\cdots\right]\Bigg\},\quad r\leq r_0\end{aligned} \tag{2.14}$$

TABLE I. Energy shift for the $1S_{1/2}$ state calculated exactly and in perturbation theory. The entries are explained in the text and are in units of m_ec^2.

Energy shift	$Z=1$	$Z=80$
	Exact	
δE_s	$0.000\,938\,188\,3\times10^{-11}$	$0.988\,994\,132\,1\times10^{-4}$
δE_{su}	$-0.173\,206\,430\,5\times10^{-11}$	$0.111\,153\,576\,1\times10^{-4}$
$\delta E_u=\delta E_{su}-\delta E_s$	$-0.174\,144\,618\,8\times10^{-11}$	$-0.877\,840\,556\,0\times10^{-4}$
	Perturbation theory	
$\Delta E_u^{(1)}$	$-0.174\,143\,891\,1\times10^{-11}$	$-0.887\,479\,99\times10^{-4}$
$\Delta E_u^{(2)}$	$-0.000\,000\,749\,4\times10^{-11}$	$-0.001\,570\,54\times10^{-4}$
$\Delta E_{pu}^{(1)}$	$-0.174\,143\,911\,3\times10^{-11}$	$-0.893\,096\,075\,4\times10^{-4}$
$\Delta E_{pu}^{(2)}$	$-0.000\,000\,749\,4\times10^{-11}$	$-0.001\,678\,966\,4\times10^{-4}$
$\Delta E_{su}^{(2)}$	2.188×10^{-19}	$0.022\,06\times10^{-4}$

where the set of large square brackets is the expansion of the second term. The remaining integral in (2.14) is evaluated with the aid of the expansion[13]

$$\int_1^\infty dt\frac{e^{-\epsilon t}}{t}=-\gamma-\ln\epsilon-\sum_{n=1}^{\infty}\frac{(-\epsilon)^n}{n(n!)},\quad |\arg\epsilon|<\pi\ , \tag{2.15}$$

where $\gamma=0.577\ldots$ is Euler's constant. Using (2.12)—(2.15), $V_u(r)$ is numerically calculated to 12-figure accuracy.

With the substitution $V(r)=V_s(r)+V_u(r)$, (2.4) is numerically integrated to obtain the energy shift δE_{su}. The outward integration is stable; however, the inward integration is unstable and small errors in the initial condition grow rapidly. A meeting point of $r=0.1$ is chosen to minimize errors and to obtain 10-figure accuracy for δE_{su}. The results are tabulated in Table I and are accurate to the values given. The energy shift, $\delta E_u=\delta E_{su}-\delta E_s$, which is the energy shift due to the size-Uehling potential, is also tabulated in Table I. The energy shift due to the point-Uehling potential is then subtracted from δE_u to obtain the size correction part of the Uehling potential energy shift. Because of singularities at the origin, the energy shift due to the point-Uehling potential is calculated in perturbation theory as discussed in Sec. III.

III. APPROXIMATE METHODS

A. Perturbation theory

The perturbing potential $\delta V(r)=\delta V_s(r)+V_u(r)$ is treated in perturbation theory, where $\delta V_s(r)=V_s(r)-V_c(r)$; $V_c(r)$ is the Coulomb potential. First- and second-order energy shifts are calculated for $V_u(r)$, and the second-order energy shift is calculated for the cross term between $\delta V_s(r)$ and $V_u(r)$.

First- and second-order energy shifts $\Delta E^{(1)}$ and $\Delta E^{(2)}$ for the $1S_{1/2}$ state are given by the equations

$$\Delta E^{(1)}=\int_0^\infty dr\, r^2[f_1^2(r)+f_2^2(r)]\delta V(r)\ , \tag{3.1}$$

$$\Delta E^{(2)}=-\int_0^\infty dr_1\int_0^\infty dr_2\, r_1^2 r_2^2\delta V(r_1)\delta V(r_2)\times\sum_{i,j=1}^{2}f_i(r_2)G_{-1}^{ij^R}(r_2,r_1,E_0)f_j(r_1), \tag{3.2}$$

valid for spherically symmetric potentials. In (3.1) and (3.2), $f_i(r)$, $i=1,2$, are the radial eigenfunctions for the Coulomb potential[12] and $G_{-1}^{ij^R}(r_2,r_1,E_0)$ is the radial reduced Dirac Green function for the Coulomb potential for the $1S_{1/2}$ state, which can be written in the form[15]

$$G_{-1}^{ij^R}(x_2,x_1,E_0)=[f_j(x_1)F_i(x_2)+f_i(x_2)G_j(x_1)]\Theta(x_1-x_2)+[f_i(x_2)F_j(x_1)+f_j(x_1)G_i(x_2)]\Theta(x_2-x_1)\ , \tag{3.3}$$

where

$$F_i(x)=\frac{f_i(x)}{\gamma^2}\left[\lambda(\lambda+\tfrac{1}{2}-\gamma x)+\frac{s_i}{2}+\int_0^1 dt\left[t^{2\lambda}\frac{e^{2\gamma x(1-t)}-1}{1-t}-s_i\lambda t^{2\lambda-1}e^{2\gamma x(1-t)}\right]\right], \tag{3.4a}$$

$$G_i(x)=\frac{f_i(x)}{\gamma^2}\left[\lambda(\lambda+\tfrac{1}{2}-\gamma x)+\frac{s_i}{2}+\psi(1)-\ln(2\gamma x)+e^{2\gamma x}\int_1^\infty dt\, e^{-2\gamma x t}\left[s_i\lambda t^{2\lambda-1}+\frac{t^{2\lambda}-1}{t-1}\right]\right],$$

$$s_i=(-1)^i,\quad E_0=\lambda\ . \tag{3.4b}$$

For $\delta V(r)\to V_u(r)$ in (3.1) and (3.2), one obtains the first- and second-order energy shifts $\Delta E_u^{(1)}$ and $\Delta E_u^{(2)}$ for the size-Uehling potential. These are calculated numerically by Gaussian quadrature and the results are listed in Table I. The numerical evaluation of the radial reduced Dirac Green function is discussed in Ref. 15. The first- and second-order energy shifts for the point-Uehling potential $V_{pu}(r)$ are treated similarly. The point-Uehling potential, as obtained from (2.11) with $\rho(\mathbf{r})=\delta^3(\mathbf{r})$, is

$$V_{pu}(r)=-\frac{Z\alpha^2}{3\pi}\int_1^\infty dt(t^2-1)^{1/2}\left[\frac{2}{t^2}+\frac{1}{t^4}\right]\frac{e^{-2tr}}{r}\ . \tag{3.5}$$

The results labeled $\Delta E_{pu}^{(1)}$ and $\Delta E_{pu}^{(2)}$ are also listed in Table I. Although the point-Uehling potential is singular at the origin, the integration in perturbation theory can be carried out because there are enough powers of r in the integrand of (3.1) and (3.2).

The second-order cross term between $\delta V_s(r)$ and $V_{pu}(r)$ is calculated from (3.2). The two contributions to the cross term, $\delta V_s(r_1)V_{pu}(r_1)$ and $V_{pu}(r_1)\delta V_s(r_2)$, give equal results, because the rest of the integrand is symmetric under the interchange of the indices $1\leftrightarrow 2$. The integral is numerically evaluated by Gaussian quadrature and the result labeled $\Delta E_{su}^{(2)}$ is tabulated in Table I. All entries in the tables are numerically accurate to the values given.

B. Nonrelativistic approximation

The nonrelativistic results are obtained using the approximations of Zemach. All size corrections are treated in first-order perturbation theory, where the unperturbed Schrödinger equation is

$$\left[-\frac{\nabla^2}{2}+V_c(r)+V_{pu}(r)\right]\phi(\mathbf{r})=E\phi(\mathbf{r})\ . \tag{3.6}$$

The size correction from first-order perturbation theory is

$$\Delta E^{\rm nr}=\int d^3r\,\phi^*(\mathbf{r})\delta V_{fs}(r)\phi(\mathbf{r})\ , \tag{3.7}$$

where

$$\delta V_{fs}(r)=V_s(r)-V_c(r)+V_u(r)-V_{pu}(r)\ . \tag{3.8}$$

Following the procedure of Zemach,[11] $\phi(\mathbf{r})$ is approximated by

$$\begin{aligned}\phi(\mathbf{r})&\cong\phi_c(\mathbf{r})-\frac{1}{2\pi}\int d^3s\frac{1}{|\mathbf{r}-\mathbf{s}|}V_{pu}(\mathbf{s})\phi_c(\mathbf{s})\\&\equiv\phi_c(\mathbf{r})+\delta\phi(\mathbf{r})\ ,\end{aligned} \tag{3.9}$$

where $\phi_c(\mathbf{r})$ is the known Coulomb eigenfunction. Substituting (3.9) into (3.7) yields, to the leading order in α

$$\begin{aligned}\Delta E^{\rm nr}=&\int d^3r\,\phi_c^*(\mathbf{r})\delta V_{fs}(r)\phi_c(\mathbf{r})\\&+2\int d^3r\,\phi_c^*(\mathbf{r})\delta V_s(r)\delta\phi(\mathbf{r})\ ,\end{aligned} \tag{3.10}$$

where the fact that $V_u(r)-V_{pu}(r)$ is of order α smaller than $\delta V_s(r)$ is used to obtain (3.10). Since size corrections in the absence of the Uehling potential are not of interest here, the quantity $\Delta E_{su}^{\rm nr}$, the size correction to the Uehling-potential correction, is calculated from the definition

$$\begin{aligned}\Delta E_{su}^{\rm nr}\equiv&\Delta E^{\rm nr}-\int d^3r\,\phi_c^*(\mathbf{r})\delta V_s(r)\phi_c(\mathbf{r})\\=&\int d^3r\,\phi_c^*(\mathbf{r})[V_u(r)-V_{pu}(r)]\phi_c(\mathbf{r})\\&+2\int d^3r\,\phi_c^*(\mathbf{r})\delta V_s(r)\delta\phi(\mathbf{r})\ ,\end{aligned} \tag{3.11}$$

where the last step in (3.11) follows from (3.10).

Since the point-Uehling potential is exponentially decreasing with a range of $\frac{1}{2}\lambda\!\!\!^{-}_e$, one approximates the Coulomb eigenfunction occurring in the integrand of (3.9) as its value at the origin to obtain the approximation

$$\begin{aligned}\delta\phi(\mathbf{r})\cong&\frac{Z\alpha^2}{6\pi^2}\phi_c(0)\int d^3s\frac{1}{|\mathbf{r}-\mathbf{s}|}\\&\times\int_1^\infty dt(t^2-1)^{1/2}\left[\frac{2}{t^2}+\frac{1}{t^4}\right]\frac{e^{-2st}}{s}\ .\end{aligned} \tag{3.12}$$

Since $\delta V_s(r)$ vanishes for $r>r_0$, $\delta\phi(\mathbf{r})$ is approximated by its value at the origin, which is evaluated as follows:

$$\begin{aligned}\delta\phi(0)\cong&\frac{Z\alpha^2}{6\pi^2}\phi_c(0)\int_1^\infty dt(t^2-1)^{1/2}\left[\frac{2}{t^2}+\frac{1}{t^1}\right]\int d^3s\frac{e^{-2st}}{s^2}\\=&\frac{Z\alpha^2}{3\pi}\phi_c(0)\int_1^\infty dt\frac{(t^2-1)^{1/2}}{t}\left[\frac{2}{t^2}+\frac{1}{t^4}\right]\\=&\frac{Z\alpha^2}{3\pi}\phi_c(0)\tfrac{1}{2}[2\beta(\tfrac{3}{2},\tfrac{1}{2})+\beta(\tfrac{3}{2},\tfrac{3}{2})]\\=&\tfrac{3}{16}Z\alpha^2\phi_c(0)\ .\end{aligned} \tag{3.13}$$

With these approximations, (3.11) becomes

$$\begin{aligned}\Delta E_{su}^{\rm nr}\cong&\int d^3r\,\phi_c^*(\mathbf{r})[V_u(r)-V_{pu}(r)]\phi_c(\mathbf{r})\\&+\tfrac{3}{8}Z\alpha^2|\phi_c(0)|^2\int d^3r\,\delta V_s(r)\ .\end{aligned} \tag{3.14}$$

The integral in the second term of (3.14) is evaluated for an arbitrary nuclear charge distribution $\rho(\mathbf{r})$ to give

$$\int d^3r\,\delta V_s(r)=\frac{2\pi Z\alpha}{3}\langle r^2\rangle\ , \tag{3.15}$$

where

$$\langle r^2\rangle=\int d^3r\,\rho(\mathbf{r})r^2,\quad\int d^3r\,\rho(\mathbf{r})=1\ . \tag{3.16}$$

The first term in (3.14) is evaluated with the approximation $\phi_c(\mathbf{r})\approx\phi_c(0)(1-Z\alpha r)$. Here it is necessary to retain the linear term in the expansion of $\phi_c(\mathbf{r})$ since from (2.11) and (3.5), it can be shown that

$$\int d^3r[V_u(r)-V_{pu}(r)]=0\ . \tag{3.17}$$

Keeping the term of lowest order in α, the first term in (3.14) is evaluated with the aid of

$$\begin{aligned}&\int d^3r\,\phi_c^*(\mathbf{r})[V_u(r)-V_{pu}(r)]\phi_c(\mathbf{r})\\&\approx-2|\phi_c(0)|^2Z\alpha\int d^3r\,r[V_u(r)-V_{pu}(r)]\\&=\frac{2\alpha(Z\alpha)^2|\phi_c(0)|^2}{3\pi}\int_1^\infty dt(t^2-1)^{1/2}\left[\frac{2}{t^2}+\frac{1}{t^4}\right]\int d^3r'\rho(\mathbf{r}')\int d^3r\,r\left[\frac{e^{-2|\mathbf{r}-\mathbf{r}'|t}}{|\mathbf{r}-\mathbf{r}'|}-\frac{e^{-2rt}}{r}\right]\\&=\frac{4\alpha(Z\alpha)^2|\phi_c(0)|^2}{3}\int_1^\infty dt(t^2-1)^{1/2}\left[\frac{2}{t^2}+\frac{1}{t^4}\right]\int d^3r'\rho(\mathbf{r}')\left[\frac{r'}{2t^2}+\frac{1-2r't-e^{-2r't}}{4r't^4}\right],\end{aligned} \tag{3.18}$$

where the integration over r is carried out to yield the last equation in (3.18). Since r' is restricted to lie within the nucleus, the expression in the last set of large parentheses in (3.18) is expanded in a power series in $r't$ keeping only the leading term (of order r'^2/t). This expansion, checked numerically, is found to give three-figure accuracy. With the aid of (3.13), (3.15), and (3.18), (3.14) becomes

$$\Delta E_{su}^{\rm nr}\cong\tfrac{1}{2}\pi\alpha(Z\alpha)^2|\phi_c(0)|^2\langle r^2\rangle\ . \tag{3.19}$$

The numerical value is listed in Table II for the $1S_{1/2}$ state.

TABLE II. Finite-size corrections to the Uehling potential for the $1S_{1/2}$ state (units of m_ec^2).

Z	Exact	Perturbation theory	nr approx.
1	4.19×10^{-19}	4.21×10^{-19}	3.74×10^{-19}
80	1.69×10^{-6}	2.78×10^{-6}	4.49×10^{-8}

IV. CONCLUSIONS

The two corrections to the Coulomb potential included here are the Uehling potential and the change from the Coulomb potential due to an extended nuclear charge distribution (size potential). Furthermore, the Uehling potential depends on the nuclear charge distribution. Hence, one distinguishes between a point-Uehling potential and a size-Uehling potential in the obvious way. The purpose of the present calculation is to determine the nuclear-size correction to the Uehling potential correction, and also to determine the validity of approximate methods of calculation. The basic idea is to isolate the combined effects of the finite nuclear size and Uehling potential from the effect of each considered separately.

The results are summarized in Table II. The exact energy shift due to the Uehling potential for a finite nucleus is determined by subtracting the energy due to the size potential from the energy due to the size-Uehling plus size potentials. The difference is denoted δE_u in Table I. To isolate the effect of nuclear size on the Uehling potential, the energy shift due to the point-Uehling potential is subtracted from the previous difference and the result appears under the heading "exact" in Table II. It should be noted here that the exact point-Uehling energy shift is assumed to be the same as the point-Uehling energy shift calculated in perturbation theory ($\Delta E_{pu}^{(1)}$ and $\Delta E_{pu}^{(2)}$ of Table I). To the accuracy of the present calculation, the convergence of perturbation theory for the point-Uehling potential is rapid enough to validate this approximation.

Since the main objective of this exercise is to check the methods of approximation, no contribution beyond second-order perturbation theory is included here even though it may be important. Therefore, to isolate the effect of nuclear size on the Uehling potential, the energy shift due to the size-Uehling minus the point-Uehling is calculated up to and including second-order perturbation theory. To the desired accuracy, the only other second-order effect that must be added to this result is the cross term between the point-Uehling and size potentials. The total result appears under the heading "perturbation theory" in Table II. The nonrelativistic results are obtained from (3.19).

In comparing these results, it is found that at $Z=1$, perturbation theory is a valid method of calculation and the nonrelativistic approximation agrees with the exact results within 11%. At $Z=80$, neither approximation is valid. It is not surprising that perturbation theory is not valid at high Z, since at high Z, perturbation theory converges slowly for the size potential in the absence of the Uehling potential. If third-order perturbation theory were included in the present calculation, then the results of perturbation theory at $Z=80$ would better agree with the exact results. The agreement (within 11%) of the nonrelativistic results with the exact results is a check of Zemach's approximation for $Z=1$.

Equation (3.19) disagrees with the nonrelativistic calculation made by Borie. The present result is smaller by a factor on the order of the rms radius of the nucleus divided by $\lambda\!\!\!^{-}_e$. Borie's starting point is based on an approximation which is valid in evaluating the point-Uehling correction, but is not valid in calculating size corrections to the Uehling-potential correction as the present results indicate. One step in the nonrelativistic calculation is to approximate the wave functions by their values at the origin. For the present problem, this approximation is valid only if the change in the wave function due to the Uehling potential is first calculated. This corrected wave function at the origin is then used to calculate in perturbation theory the energy shift due to the nuclear size. Approaching the problem in the reverse order is found not to be valid, i.e., taking the corrected wave function due to the size correction evaluated at the origin to calculate the energy shift due to the Uehling potential. This is due to the different effective ranges of the two perturbations. The size potential has an effective range on the order of the charge radius of the nucleus, while the Uehling potential has an effective range on the order of the Compton wavelength of the electron. Both approaches, of course, are identical if the corrected wave functions are not approximated by their values at the origin.

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