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# Evidence from alkali-metal-atom transition probabilities for a phenomenological atomic supersymmetry

V. Alan Kostelecký* and Michael Martin Nieto†
*Theoretical Division, Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545*

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We review the proposal that relationships between physical spectra of certain atoms can be considered as evidence for a phenomenological supersymmetry. Next, a comparison is made between the supersymmetric and the hydrogenic approximations. We then present the calculation of low-$Z$ alkali-metal-atom transition probabilities between low-$n$ states, using supersymmetric wave functions. These probabilities agree more closely with accepted values than do those obtained with use of the hydrogenic approximation. This shows that, in simple radial Schrödinger theory, supersymmetry is a concept providing insight into the true, fermionic, many-body physics of these atoms.

## I. INTRODUCTION

Supersymmetry is a theoretical concept which has been applied to many fields of physics.[1] The application that received the first emphasis was in elementary particle physics. There, the implication is that there exist boson-fermion partner elementary particles;[2] for instance, there will be a bosonic "squark" partner to the fermionic "quark." Unfortunately, to date no supersymmetric partners of the known particles have been found. One place where it has been *proposed* that evidence suggests the presence of supersymmetry is in nuclear physics. There, supersymmetry would relate the properties of certain different nuclei to each other.[3]

Recently, we suggested[4] that distinct spectra of certain atoms and ions can be viewed as manifesting a *phenomenological* quantum-mechanical supersymmetry. That paper demonstrated that the phenomenological supersymmetric potential has the effect of modeling a closed inner orbital, yielding the same spectrum as the "hydrogenic approximation" but with the ground state removed. Thus, for an atom with a closed inner shell, supersymmetry partially models the effects of the inner electrons and the associated Pauli principle in a simple Schrödinger picture.

However, as a matter of principle, the supersymmetry does more than this. The supersymmetric wave functions are *not* the same as the hydrogenic ones. The supersymmetric wave functions are a *complete* set of states for the supersymmetric potential, whereas the hydrogenic wave functions are an *incomplete* set of states for the hydrogenic potential. The states excluded by the Pauli principle are needed for a complete set.

Thus, it is a pertinent question to ask which wave functions (supersymmetric or hydrogenic) yield better transition probabilities. The answer, which we present in this paper, is that for low-$Z$ alkali-metal atoms the supersymmetric wave functions do.

In Sec. II we review supersymmetry in quantum mechanics, specifically emphasizing its application to this problem. We proceed in Sec. III to discuss the properties of the supersymmetric model in atomic physics. The differences between the supersymmetric and the hydrogenic pictures are stressed.

In Sec. IV we present the results of transition probability calculations, demonstrating the interesting differences between the supersymmetric approximation for the low-$Z$ alkali-metal atoms and the hydrogenic approximation. The supersymmetric case is found to be superior. We comment on the physical implications of the supersymmetric picture as a useful model for viewing atoms, and we stress the importance of electron-electron interactions. A phenomenological model that seeks a calculationally useful approximation to the effects of these interactions will be the subject of a separate paper.[5]

We emphasize that our aim in this paper is to investigate the conceptual validity of the elementary physics associated with the supersymmetric approximation. We do this by comparing our results and those of another simple approach, the hydrogenic model, with the accepted values. It should be clear that we are not trying to get precise predictions with this approach. By incompletely incorporating the effects of the electron-electron interactions, we forfeit any possibility of obtaining *accurate* transition probabilities. Nonetheless, it is interesting to determine whether this amusing alternative provides a better approximation to the many-body physics of the alkali-metal atoms than does the simple hydrogenic model.

## II. SUPERSYMMETRY IN ATOMIC QUANTUM MECHANICS

The Hamiltonian for supersymmetric quantum mechanics[6–9] may be written as $H_{\rm ss}=H_+\oplus H_-$, where $H_+$ and $H_-$ are given by

$$H_\pm\Psi_{\pm n}=\left[-\frac{d^2}{dx^2}+V_\pm(x)\right]\Psi_{\pm n}=\epsilon_n\Psi_{\pm n}\ . \tag{2.1}$$

In terms of a function $U(x)$, whose significance will be demonstrated shortly, the supersymmetric potential partners $V_+$ and $V_-$ are related by

$$V_+(x)=(\tfrac{1}{2}U')^2-\tfrac{1}{2}U''\ , \tag{2.2}$$

$$V_-(x)=(\tfrac{1}{2}U')^2+\tfrac{1}{2}U''\ , \tag{2.3}$$

where the prime means $d/dx$. Equation (2.2) automatically guarantees that the ground state of $H_+$ is normalized to zero energy, since a solution to Eq. (2.1) using $H_+$ is

$$\Psi_{+0}=N_0\exp(-\tfrac{1}{2}U),\quad \epsilon_0=0\ . \tag{2.4}$$

Thus, we see that $U\sim\ln\Psi_{+0}$, where $\Psi_{+0}$ is the ground state.

The Hamiltonians $H_+$ and $H_-$ can be written as

$$\begin{aligned} &H_+=A^\dagger A,\quad H_-=AA^\dagger\ ,\\ &A=-i(\partial_x+\tfrac{1}{2}U'),\quad A^\dagger=i(-\partial_x+\tfrac{1}{2}U')\ ,\\ &[A,A^\dagger]=U''\ . \end{aligned} \tag{2.5}$$

Note that if $V_+(x)$ is the harmonic-oscillator potential $x^2-1$ then in this convention $A=-i\sqrt{2}a$ and $A^\dagger=i\sqrt{2}a^\dagger$, where $a$ and $a^\dagger$ are the standard harmonic-oscillator lowering and raising operators.

The Hamiltonians $H_+$ and $H_-$ in (2.5) are the "bosonic" and "fermionic" supersymmetric partners. This can be seen by defining a two-component wave function and writing

$$Q=\begin{bmatrix}0&0\\A&0\end{bmatrix},\quad Q^\dagger=\begin{bmatrix}0&A^\dagger\\0&0\end{bmatrix}\ . \tag{2.6}$$

The supersymmetric Hamiltonian is then

$$H_{\rm ss}=Q^\dagger Q+QQ^\dagger=\begin{bmatrix}H_+&0\\0&H_-\end{bmatrix}\ . \tag{2.7}$$

The charges $Q$ and $Q^\dagger$ have the supersymmetric properties $Q^2=(Q^\dagger)^2=0$ and satisfy $[Q,H_{\rm ss}]=[Q^\dagger,H_{\rm ss}]=0$. The two Hamiltonians have the same spectra, except for the ground-state eigenvalue. Only the bosonic Hamiltonian $H_+$ has a normalizable ground state with eigenvalue $\epsilon_0=0$.

This procedure can be applied to any separable Schrödinger-type equation. Consider a three-dimensional Schrödinger equation with a central potential. Removing the angular variables yields a term $\sim[l(l+1)/r^2]$. This leaves an equation in one variable, $r$, upon which the supersymmetric procedure can be used.

In particular, consider the radial equation of the hydrogen atom. It can be written as

$$\left[-\frac{d^2}{dy^2}-E_n-\frac{1}{y}+\frac{l(l+1)}{y^2}\right]\chi_{nl}(y)=0\ , \tag{2.8}$$

where $E_n=-1/4n^2$, $y=(2\mu e^2/\hbar^2)r\equiv\beta r$, $n\geqq l+1$, and $\chi_{nl}(y)=rR_{nl}(y/n)$. If $\rho=y/n$, then

$$R_{nl}(\rho)=\frac{1}{n^2}\left[\frac{\beta^3\Gamma(n-l)}{2\Gamma(n+l+1)}\right]^{1/2}\rho^l\exp(-\tfrac{1}{2}\rho)L_{n-l-1}^{(2l+1)}(\rho) \tag{2.9}$$

is the solution to the usual radial equation.[10] Here, the associated Laguerre polynomials $L_j^{(\alpha)}(z)$ are defined by

$$L_j^{(\alpha)}(z)=\sum_{q=0}^{j}\frac{(-x)^q}{q!}\frac{\Gamma(j+\alpha+1)}{\Gamma(q+\alpha+1)\Gamma(j-q+1)}\ . \tag{2.10}$$

We will mainly use $y$ instead of the standard variable $\rho=y/n$. Note that $y$ differs from atomic units by a factor of 2.

Equation (2.8) can be rewritten in the supersymmetric form (2.1) for $H_+$ by setting

$$V_{+l}=[2(l+1)]^{-2}-\frac{1}{y}+\frac{l(l+1)}{y^2} \tag{2.11}$$

$$\equiv(U')^2/4-U''/2\ , \tag{2.12}$$

$$E_n\to\epsilon_{nl}=\tfrac{1}{4}(l+1)^{-2}-\tfrac{1}{4}n^{-2}\ . \tag{2.13}$$

The potential $V_{+l}$ is defined so that $\epsilon_{l+1,l}$ is normalized to zero.

To obtain the supersymmetric partner of the hydrogen atom Hamiltonian for a fixed $l$ one must solve for $U$. This can be done either by solving (2.11) and (2.12), or more easily by writing $\chi_{l+1,l}$ in the form (2.4). The solution is

$$U(y)=y/(l+1)-2(l+1)\ln y\ . \tag{2.14}$$

Therefore, the supersymmetric partner $V_{-l}$ to the potential $V_{+l}$ is

$$V_{-l}=[2(l+1)]^{-2}-\frac{1}{y}+\frac{(l+1)(l+2)}{y^2}\ . \tag{2.15}$$

In Ref. 4 we considered possible applications of these potentials to atomic and ionic systems. We found many pairs of atoms and ions that have spectra satisfying supersymmetric orderings.

The first and simplest case occurs when $l=0$ ($s$ orbitals). The hydrogen $s$ levels are described by the potential

$$V_{+0}=\frac{1}{4}-\frac{1}{y}\ . \tag{2.16}$$

The partner potential $V_{-0}$ models the lithium atom, the spectrum of which starts at the $2s$ level. This potential is

$$V_{-0}=\frac{1}{4}-\frac{1}{y}+\frac{2}{y^2}\ . \tag{2.17}$$

Next, we can consider the case $l=1$ ($p$ orbitals). In Ref. 4 we pointed out that $V_{+1}$ could model the $p$ orbitals of the boron atom. However, it also can model the $p$ orbitals of the lithium atom, since we are considering a fixed $l$. The potential is

$$V_{+1}=\frac{1}{16}-\frac{1}{y}+\frac{2}{y^2} . \tag{2.18}$$

Then, the supersymmetric partner potential for the $p$ levels of aluminum or sodium atoms is

$$V_{-1}=\frac{1}{16}-\frac{1}{y}+\frac{6}{y^2} . \tag{2.19}$$

This allows only a $3p$ ground state.

Similarly, the first possible $d$-orbital supersymmetry among the alkali-metal atoms could appear between potassium and rhubidium, since the $3d$ orbitals of rhubidium are filled. This $d$-orbital supersymmetry will not play a role in the remainder of this paper, however, as we shall only be concerned with the alkali-metal atoms hydrogen, lithium, and sodium.

In Ref. 4 we also considered extensions of the above, to supersymmetries of $k$ orders. Up to a zero-point energy, a fermionic partner potential was used as the bosonic potential for a new supersymmetry, and then the new fermionic partner potential was found. In equation form, the $k$th-order supersymmetry potential is

$$V^{(k)}_{+l}=[2(l+k+1)]^{-2}-1/y+(l+k)(l+k+1)/y^2 , \tag{2.20}$$

$$V^{(k)}_{-l}=V^{(k)}_{+l}+2(l+k+1)/y^2 . \tag{2.21}$$

Operationally,

$$V^{(k)}_{-l}\to V^{(k+1)}_{+l}-\epsilon^{(k)}_{l+k+2,l} , \tag{2.22}$$

where

$$\epsilon^{(k)}_{nl}=\tfrac{1}{4}[(l+k+1)^{-2}-n^{-2}] . \tag{2.23}$$

For instance, $k=1$ can correspond to the sodium $s$ levels being the supersymmetric partners of the lithium $s$ levels, which themselves were previously taken to be the $l=0$ supersymmetric partners of the hydrogen $s$ levels. We shall use this example below.

The effect of supersymmetry is always to increase the effective angular-momentum barrier of $V^{(k)}_{+l}$ by one unit in going to $V^{(k)}_{-1}$, i.e.,

$$(l+k)(l+k+1)/y^2\to(l+k+1)(l+k+2)/y^2 . \tag{2.24}$$

The extra $1/y^2$ dependence is a repulsive potential that prevents the valence electron of the fermionic partner atom from attaining the ground state present in the bosonic atom. The fermionic Hamiltonian therefore has a built-in approximation to the effects of the Pauli principle. We shall discuss this further in Sec. III.

## III. COMPARISON OF THE SUPERSYMMETRIC AND THE HYDROGENIC MODELS

Any precise calculation of atomic systems has to take into account the Pauli principle and the many-body forces that exist among the electrons. Nonetheless, one may ask whether, in a simple Schrödinger-picture model, *some* of the complex features of atomic systems can be seen as manifestations of a symmetry. This is the question we answer positively with supersymmetry.

In this section we focus on hydrogen and lithium. Owing to the relatively small number of electrons in these atoms, the physical evidence for supersymmetry would be best here.

The lithium ground-state configuration of $1s^2 2s^1$ consists of a filled inner $1s$ shell with the ground state of the valence electron being the $2s$ state. Physically, the effective principal quantum number (as used in quantum-defect theory) of the $2s$ electron is 1.6, not 2.0. This is because the inner three protons and two electrons, with a total hydrogenic charge of one, do not have the same net effect as a *central* charge of one. The electrons interact, and the orbitals are quantum-mechanical "clouds."

The Schrödinger hydrogenic model has to treat this by hand.[11] The hydrogenic model would still give a ground state $n_0=1$ if the Pauli principle were not put in by hand to give $n_0=2$. In contrast, the supersymmetric model gives $n_0=2$ directly. It automatically takes care of the Pauli principle via its effective potential. Granted, the effective potential overcompensates, since $n_0=2$ instead of 1.6, but it heads in the correct direction.

So, the first thing supersymmetry does for a simple Schrödinger theory is to incorporate the effects of the Pauli principle automatically,[12] instead of having to do it by hand. The reader might think that, although this is intriguing, it is not really significant since the spectra of the supersymmetric and hydrogenic models are the same.

However, there is more. The supersymmetric model uses a *complete* set of states, which as a matter of principle is different from the *incomplete* set of states of the hydrogenic model. The eigenfunctions of $V_{+l}$ are $R_{nl}$, $n\geqq l+1$, so those of the hydrogen $s$ levels are $R_{n0}$, $n\geqq 1$. However, for the hydrogenic model of lithium, the $s$-orbital eigenstates are $R_{n0}$, $n\geqq 2$. This is *not* a complete set of states, since the ground state $R_{10}$, excluded by the Pauli principle, is needed for a complete set of states. Contrariwise, the supersymmetric eigenfunctions of $V_{-l}$ are $R_{n,l+1}$, $n\geqq 2$, which *do* form a complete set of states, in particular for the lithium $s$ levels.

Note that the hydrogenic eigenfunctions *do* form a complete set whenever all the levels for a fixed value of $l$ are accessible to the valence electron. In such cases, the supersymmetric and hydrogenic models coincide.

Thus, the wave functions of the two models are different, even if the spectra are not. Although the eigenfunctions $\Psi_{nlm}(y,\theta,\phi)$ of the hydrogen atom remain

$$\Psi_{+nlm}(y,\theta,\phi)=Y_{lm}(\theta,\phi)R_{nl}(y/n) , \tag{3.1}$$

where $Y_{lm}(\theta,\phi)$ represents the spherical harmonics, the supersymmetric partner eigenfunctions are

$$\Psi_{-nlm}(y,\theta,\phi)=Y_{lm}(\theta,\phi)R_{n,l+1}(y/n) . \tag{3.2}$$

Similarly, the eigenfunctions of the $k$th supersymmetric pair are

$$\Psi^{(k)}_{+nlm}(y,\theta,\phi)=Y_{lm}(\theta,\phi)R_{n,l+k}(y/n) \tag{3.3}$$

and

$$\Psi^{(k)}_{-nlm}(y,\theta,\phi)=Y_{lm}(\theta,\phi)R_{n,l+k+1}(y/n) . \tag{3.4}$$

Observe that the node structure of the eigenfunctions (3.2)–(3.4) is unusual. Normally, the number of nodes in the radial eigenfunction of a state designated $(n,l)$ is[10] $n-l-1$. Thus, for fixed $l$, the ground-state eigenfunction $n=l+1$ has zero nodes. However, the ground state of the first member of a supersymmetric pair has $n-l-k-1$ nodes, while its supersymmetric partner has $n-l-k-2$ nodes. For instance, the $2s$ level in lithium has one node in the hydrogenic approximation but none in the supersymmetric picture.

Intuitively, one might expect such a radical change in the structure of the wave functions to be in conflict with experimental data, e.g., on transition probabilities. Indeed, the eigenfunctions (3.1)–(3.4) can be used to calculate transition probabilities for the valence electrons of the alkali-metal atoms. These predictions and those[10] of the simple hydrogenic approximation can then be compared to the accepted values.[13] We shall demonstrate in Sec. IV that the phenomenological supersymmetry provides greater overall accuracy than the hydrogenic approximation for low-$Z$ alkali-metal atoms. Note that it is for low-$Z$ atoms that one expects both the supersymmetric and the hydrogenic approximations to be most applicable. In such cases, there are fewer electrons to give large noncentral forces. Also, for low $Z$, relativistic (spin) effects are negligible.

A final observation is that the supersymmetric and hydrogenic pictures have different viewpoints. The hydrogenic picture is motivated by the physically reasonable assumption that in sufficiently excited states the valence electron of certain atoms feels forces approximating those in the hydrogen atom. Although, as we have shown, the supersymmetric picture incorporates this idea, it goes a step further in that the supersymmetric partners are treated as components of a *single* theory rather than as two separate systems.

Thus, for example, hydrogen and lithium are viewed in the supersymmetric scheme as complements, without which the scheme fails. The presence of supersymmetry admits the existence of lithium, given the existence of hydrogen. In contrast, the hydrogenic model treats the two systems independently, even though a relationship between them is physically expected in one-body Schrödinger theory.

## IV. TRANSITION PROBABILITIES

The transition probability $T_{fi}$ for the transition from a level $(n_i,l_i)$ to a level $(n_f,l_f)$ is[10]

$$T_{fi}=\frac{\pi}{6}\alpha^3\nu_R F_l\left[\frac{1}{n_f^2}-\frac{1}{n_i^2}\right]^3 |\langle R_{n_f l_f}|y|R_{n_i l_i}\rangle|^2 , \qquad (4.1)$$

where the fine-structure constant $\alpha$ and the Rydberg frequency $\nu_R$ are given by

$$\alpha=e^2/\hbar c,\quad \nu_R=\mu e^4/(4\pi\hbar^3) . \qquad (4.2)$$

The $l$-dependent quantity $F_l$, given by

$$F_l=\begin{cases} l_i/(2l_i+1), & l_f=l_i-1 \\ (l_i+1)/(2l_i+1), & l_f=l_i+1 \end{cases} \qquad (4.3)$$

arises from the contribution of the angular part of the eigenfunctions. Using Eq. (4.1), we have computed selected transition probabilities of lithium and sodium, with supersymmetric eigenfunctions and with hydrogenic eigenfunctions. We emphasize that the exact hydrogenic energy eigenvalues, not the experimental values for each atom, were used in all calculations.

Table I contains the results for lithium. As noted above, the $s$-orbital spectrum of lithium may be interpreted as the supersymmetric partner of the $s$-orbital spectrum of hydrogen. However, the $p$ orbitals of lithium *cannot* be related to those of hydrogen by a supersymmetry, since in both atoms the $2p$ orbitals are accessible to the outermost electron. Thus, for this atom, the phenomenological supersymmetry occurs solely in the $s$ orbitals. We have therefore calculated only the $s\rightarrow p$ and $p\rightarrow s$ transition probabilities for lithium, using the supersymmetric radial eigenfunctions $R_{n1}$ for both the $s$ and the $p$ orbitals. For other transition probabilities, the hydrogenic and supersymmetric eigenfunctions are identical, as we discussed in Sec. III.

Table II contains the results for sodium. As we have already discussed [Eqs. (2.20)–(2.24)], the sodium $s$ orbitals are the supersymmetric partners of the lithium $s$ orbitals. Their radial eigenfunctions are therefore $R_{n2}$ in the supersymmetric picture. Also, from Eq. (2.19), the sodium $p$ orbitals are supersymmetric partners of the lithium $p$ orbitals, because the sodium ground-state configuration has the $2p$ orbitals filled. It follows that the appropriate radial eigenfunctions for the sodium $p$ orbitals are also $R_{n2}$. The $d$ orbitals of sodium, however, remain unaffected in the supersymmetric approximation, since the $3d$ orbitals are accessible to the valence electron. Thus, for sodium, the phenomenological supersymmetry is expected to appear in the $s$ and $p$ orbitals alone. We have therefore calculated $s\rightarrow p$, $p\rightarrow s$, $p\rightarrow d$, and $d\rightarrow p$ transition probabilities in this case, using $R_{n2}$ as the radial eigenfunctions for the $s$, $p$, and $d$ orbitals.

For purposes of comparison, we have included in Tables I and II the accepted values[13] for the transition probabilities and also the results obtained using standard hydrogenic radial eigenfunctions. Note that in Ref. 13 the majority of the quoted values are obtained calculationally rather than experimentally. Typically, these values are expected to be accurate to within about 10% in the lithium case and to within about 25% in the sodium case.

At this point, we turn to a discussion of the results. We begin by examining the data for lithium, given in Table I. For all levels the supersymmetric approximation yields results closer to the accepted values[13] than the hydrogenic approximation. However, both approximations are extremely poor for certain levels. This is only to be expected, since neither approximation correctly takes into account the effects of electron-electron interactions. For instance, both predict zero transition probabilities for $np\rightarrow ns$, although these are nonzero in the real lithium atom.

Turning to the results for sodium, we see several noteworthy features of Table II. First, the supersymmetric predictions for the $s\rightarrow p$ transitions are all better than the hydrogenic ones, typically by a factor of 2.

TABLE I. A comparison of accepted values (Ref. 12) of lithium transition probabilities with those calculated using the hydrogenic approximation and using the supersymmetric model.

| Transition | Hydrogenic approximation ($10^8$ Hz) | Supersymmetric approximation ($10^8$ Hz) | Accepted value ($10^8$ Hz) |
|---|---|---|---|
| $3s \rightarrow 2p$ | 0.063 | 0.225 | 0.349 |
| $4s \rightarrow 2p$ | 0.026 | 0.097 | 0.101 |
| $4s \rightarrow 3p$ | 0.0184 | 0.0435 | 0.0746 |
| $5s \rightarrow 2p$ | 0.0129 | 0.0495 | 0.0460 |
| $5s \rightarrow 3p$ | 0.0091 | 0.0228 | 0.0276 |
| $5s \rightarrow 4p$ | 0.0065 | 0.0124 | 0.0225 |
| $6s \rightarrow 2p$ | 0.0074 | 0.0286 | 0.0250 |
| $6s \rightarrow 3p$ | 0.0051 | 0.0132 | 0.0144 |
| $6s \rightarrow 4p$ | 0.003 58 | 0.007 31 | 0.009 22 |
| $6s \rightarrow 5p$ | 0.002 68 | 0.004 54 | 0.008 48 |
| | | | |
| $3p \rightarrow 2s$ | 0.2246 | 0.0749 | 0.0117 |
| $4p \rightarrow 2s$ | 0.0967 | 0.0322 | 0.0142 |
| $4p \rightarrow 3s$ | 0.0307 | 0.0145 | $3.69\times10^{-5}$ |
| $5p \rightarrow 2s$ | 0.0495 | 0.0165 | 0.0107 |
| $5p \rightarrow 3s$ | 0.0164 | 0.0076 | |
| $5p \rightarrow 4s$ | 0.007 38 | 0.004 13 | $2.07\times10^{-5}$ |
| $6p \rightarrow 2s$ | 0.028 60 | 0.009 53 | 0.006 97 |
| $6p \rightarrow 3s$ | 0.009 56 | 0.004 40 | |
| $6p \rightarrow 4s$ | 0.004 46 | 0.002 44 | $8.38\times10^{-5}$ |
| $6p \rightarrow 5s$ | 0.002 43 | 0.001 51 | $1.72\times10^{-5}$ |

However, they are still in disagreement with accepted values by a factor of 3 or so. Second, for $p \rightarrow s$ transitions the supersymmetric predictions, although perhaps slightly superior, alternate with the hydrogenic predictions in accuracy. Third, the hydrogenic approximation agrees better with accepted values of low-lying $p \rightarrow d$ and $d \rightarrow p$ transitions that are available in the literature.

We have also computed transition probabilities for potassium. As the reader might anticipate, the situation for potassium is less clear. It appears impossible to draw any precise conclusions concerning the relative merit of the two approximations from the potassium data. The matter is further complicated by the accuracy of the accepted values, which is believed to be only within 50% for many

TABLE II. A comparison of accepted values (Ref. 12) of sodium transition probabilities with those calculated using the hydrogenic approximation and using the supersymmetric model.

| Transition | Hydrogenic approximation ($10^8$ Hz) | Supersymmetric approximation ($10^8$ Hz) | Accepted value ($10^8$ Hz) |
|---|---|---|---|
| $4s \rightarrow 3p$ | 0.018 | 0.035 | 0.251 |
| $5s \rightarrow 3p$ | 0.009 | 0.017 | 0.072 |
| $5s \rightarrow 4p$ | 0.0065 | 0.0116 | |
| $6s \rightarrow 3p$ | 0.0051 | 0.0094 | 0.0330 |
| $6s \rightarrow 4p$ | 0.0036 | 0.0066 | 0.0173 |
| $6s \rightarrow 5p$ | 0.0027 | 0.0044 | |
| | | | |
| $4p \rightarrow 3s$ | 0.0307 | 0.0117 | 0.0292 |
| $5p \rightarrow 3s$ | 0.0164 | 0.0057 | 0.0060 |
| $5p \rightarrow 4s$ | 0.0074 | 0.0039 | 0.0074 |
| $6p \rightarrow 3s$ | 0.009 56 | 0.003 13 | 0.002 26 |
| $6p \rightarrow 4s$ | 0.004 46 | 0.002 20 | 0.002 31 |
| $6p \rightarrow 5s$ | 0.002 43 | 0.001 47 | |
| | | | |
| $4p \rightarrow 3d$ | 0.003 48 | 0.023 47 | 0.001 57 |
| | | | |
| $4d \rightarrow 3p$ | 0.070 | 0.014 | 0.131 |
| $5d \rightarrow 3p$ | 0.034 | 0.007 | 0.050 |
| $5d \rightarrow 4p$ | 0.0149 | 0.0047 | 0.0260 |
| $6d \rightarrow 3p$ | 0.0188 | 0.0038 | 0.0257 |
| $6d \rightarrow 4p$ | 0.0086 | 0.0026 | 0.0130 |
| $6d \rightarrow 5p$ | 0.0045 | 0.0018 | |

levels.

Taken overall, the data in Tables I and II allow certain conclusions to be drawn. First, for low-$n$ transitions, the supersymmetric approximation is superior in the case of the lithium atom. We take this as evidence in favor of the phenomenological supersymmetry. Further, despite their unorthodox structure, the supersymmetric eigenfunctions yield transition probabilities for the low-$Z$ alkali-metal atoms that are better overall than those from the hydrogenic approximation. Therefore, intuitive arguments against the phenomenological supersymmetry based on the unusual effective potentials and wave functions are not compelling *a priori*.

We conclude that the simple supersymmetric description of the lighter alkali-metal atoms does indeed seem to be in closer agreement with nature than the simple hydrogenic picture. As expected, the more complex electron-electron interactions occurring in the heavier alkali-metal atoms cause the breakdown of both the supersymmetric and the hydrogenic approximations. It is clear that any application of supersymmetry that would be calculationally useful to the practicing atomic physicist would necessitate the incorporation of the effects of these interactions. Work on this is currently in progress.[5]

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[*] Address after September 1, 1985: Department of Physics, Indiana University, Bloomington, IN 47405. Address during calender year 1986: Theory Division, CERN, 1211 Genève 23, Switzerland.

[†] Address during academic year 1985—1986: Niels Bohr Institutet, Blegdamsvej 17, 2100 København ø, Denmark.

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[10] See, for example, H. A. Bethe and E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic, New York, 1957), or R. D. Cowan, *The Theory of Atomic Structure and Spectra* (University of California Press, Berkeley, 1981).

[11] The expression "by hand" means that the mathematically allowed ground state must be excluded *ad hoc.*

[12] From a more formal viewpoint, one can speculate as to the reasons for the mocking of the Pauli principle by supersymmetry. Recall that the spin-statistics theorem is a field-theoretical result, whereas in quantum mechanics it has to be inserted by hand in the form of the Pauli principle. Now, the supersymmetry generators in field theory are spinors, while those in quantum mechanics are scalars. Further, the quantum-mechanical supersymmetry algebra (2.7) is (Ref. 6) the limit of zero space dimensions of the field-theory supersymmetry algebra, osp(1/4). Note that field theory in zero space dimensions is equivalent to quantum mechanics in one space dimension. It is thus reasonable to conjecture that supersymmetry is useful in mocking the Pauli principle because the Pauli principle is the quantum-mechanical remnant of the Fermi-Dirac statistics of field theory.

[13] W. L. Wiese, M. W. Smith, and B. M. Glennon, *Atomic Transition Probabilities, Vol. I,* Natl. Bur. Stand. (U.S.) Natl. Stand. Ref. Data Ser. No. 4 (U.S. GPO, Washington, D.C., 1966); *Atomic Transition Probabilities, Vol. II,* Natl. Bur. Stand. (U.S) Natl. Stand. Ref. Data Ser. No. 22 (U.S. GPO, Washington, D.C., 1969). The reader may also consult the *CRC Handbook of Chemistry and Physics,* edited by R. C. Weast, M. J. Astle, and W. H. Beyer (CRC, Boca Raton, Florida, 1984), 65th ed.