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Misconception which led to the "material frame-indifference" controversy

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The confusion over the nature of the useful restriction on the allowable forms of constitutive relations ("material frame-indifference," or "objectivity") is due to the vague language of its formulation. In the final analysis, the confusion arises because the concept of general covariance of physical laws is applied in the inappropriate setting of the three-dimensional space instead of the four-dimensional space-time.

The debate over whether the "material frame-indifference" (MFI, also called "objectivity" or "rotational invariance") is a fundamental principle of classical physics, or a useful approximation that is very accurate under most circumstances, shows no signs of subsiding.[1] Constitutive equations for heat conduction, diffusion, and stress are usually required to satisfy the MFI, and so the question is quite important. We will see in a moment that the vague language of definitions is at the root of the controversy.

The mathematical formulation of the MFI is usually introduced as an axiom, whose validity is supported by a rather vague statement of the type

> (I) Any physical phenomenon (including material behavior) is independent of the motion of an observer (a frame of reference).

Now, this statement is undoubtedly correct (in classical physics) *if* it is understood to mean

> (II) Any physical phenomenon *happens* independently of the motion of an observer.

This is *not* the way, however, this statement is understood when it is used to justify the mathematical formulation of the MFI. In such cases, it is actually taken to mean

> (III) Any physical phenomenon is described by a mathematical relation, which, when written in terms of the quantities measured by each observer according to the same rules (i.e., *frame-indifferently defined* quantities), has the same *functional form* for all observers.

For example, Truesdell[2] writes (see Sec. IV.2)

> "We regard material properties as being . . . indifferent to the choice of frame. Since constitutive equations are designed to express idealized material properties, we require they shall be frame-indifferent. That is, if the constitutive equation
>
> $$\overleftrightarrow{T}(\chi(X,t),t)=\mathscr{F}(\chi^t;X,t) \qquad (1)$$
>
> is satisfied by the dynamic process $\{\chi,\overleftrightarrow{T}\}$, it is satisfied by every equivalent process $\{\chi^*,\overleftrightarrow{T}^*\}$. Formally, the consitutive mapping $\mathscr{F}$ in (1) must satisfy the identity
>
> $$\overleftrightarrow{T}^*(\chi^*(X,t^*),t^*)=\mathscr{F}(\chi^{*t^*};X,t^*)$$
>
> for all $\overleftrightarrow{T}^*$, χ^*, and t^* that may be obtained from $\overleftrightarrow{T}$, χ, and t by transformations . . ." from one frame (unstarred) to another (starred) of the motion χ, time t, and stress tensor $\overleftrightarrow{T}$, measured in each frame (by each observer) according to the same rules (i.e., *defined* frame-indifferently).

The assertion that the functional $\mathscr{F}$ is the same in both frames constitutes the mathematical content of the MFI and is used to restrict constitutive relations.

Now, this assertion would, of course, follow from (III). However, (III) is not equivalent to (II) in any sense. There is simply no reason for (II) and (III) to be equivalent. While (II) is the satement of the fact that (in classical physics) natural *phenomena* themselves are not influenced by how they are observed (measured, etc.), (III) is a statement about the mathematical form of the relationship between the measurements performed by different observers.

While (II) is universally accepted as an expression of the objective nature of physical reality, (III) is a nontrivial formal proposition, whose validity is not at all obvious, but must be checked through comparison with experimental data.

If such comparison *in all the cases studied* had confirmed proposition (III), one might have begun to think about tentatively elevating (III) to the status of the principle of nature. In reality, however, (III) is violated by one of the most basic laws of physics—Newton's Second Law, or momentum principle: force = (mass) × (acceleration).

There is no chance, therefore, for (III) to be universally valid. Those who would like to regard frame-indifference as a basic principle of nature are, naturally, rather unhappy about this. So they make an attempt to "render" the momentum principle frame-indifferent by defining the acceleration as that frame-indifferent vector field $\mathbf{a}$ which in the *inertial frame* reduces to the second time derivative of the position vector (see Ref. 2, p. 59). But then $\mathbf{a}$ is *not defined frame-indifferently* (because it cannot be measured by an observer in his own frame, without referring to another, inertial, frame), and thus force = (mass) × $\mathbf{a}$ will not satisfy (III) anyway. One can see how much confusion may arise from using vague statements like (I) instead of explicit (III).

To summarize, the MFI has nothing to do with objectivity of material behavior (despite "material objectivity" being its alternative name). The MFI is the proposition (III) applied to a constitutive relation, or its equivalent form[3] stating the invariance of the constitutive behavior under rigid-body motion of the *material.* There can be no other ways to check its validity than by experiment or derivation of a constitutive relation from the microscopic physics. The latter approach shows[1] that the MFI cannot be exactly true [because the microscopic physics obeys Newton's laws which do not satisfy (III)], but is a very good approximation for ordinary materials and circumstances (because the absolute accelerations due to the rigid-body motion are usually much smaller than the accelerations at the molecular scale). For example, the relative error of approximation for the stress in a Newtonian fluid is measured by $\Omega d^2/\nu$, where Ω is the angular velocity of (absolute) rotation, ν the kinematic viscosity, and d the characteristic length of fluid microstructure. Since d is very small for ordinary liquids, $O(10^{-8}$ cm), one needs $\Omega = O(10^{14}\ \text{sec}^{-1})$ in order to have an appreciable effect showing violation of the MFI. No wonder that experiments with ordinary fluids do not show such effects. Of course, if the characteristic length of the fluid microstructure is not so small (e.g., suspensions, polymer solutions), one must be more cautious; however, even in such cases the MFI is likely to be a good approximation especially since the rheology of such fluids involves more serious uncertainties.

The reader may ask at this point: "Are not you suggesting that vorticity may enter a constitutive relation?" The answer is no, vorticity in its usual meaning (i.e., defined as curl of the velocity with respect to the frame of an observer, which is not necessarily inertial) may not; however, the *absolute vorticity,* i.e., vorticity with respect to an inertial frame, may and sometimes will (as, e.g., in the Burnett order results for heat flux and stress tensor;[1] of course, for ordinary liquids such effects can be neglected). The famous "bathtub vortex" experiment (as shown, e.g., in the film *Vorticity* by A. H. Shapiro) is a good reminder of the very real (objective) nature of absolute vorticity.

Note also that the condition for the MFI to be a good approximation, e.g., $\Omega d^2/\nu \ll 1$, is independent of the "continuum assumption" $d/L \ll 1$ (where L is the macroscopic length scale). Hence it is not correct to regard the MFI as being automatically valid for continuum media.

It is worth emphasizing that the status of the MFI as an approximation does not diminish its usefulness and importance in any way. Indeed, the whole question would, perhaps, have been put to rest long ago if this simple fact had been recognized.

Let us now take a second look at the confusion between (II) and (III), criticized above. Actually, the intuitive feeling behind this confusion, i.e., the feeling that (III) should somehow follow from (II), is not without merit. But before further discussion, let us discard the artificial distinction between frames of reference and coordinate systems. The distinction was introduced to separate the purely spatial coordinate transformations (handled as a change of a coordinate system) from the coordinate transformations dependent on time (handled as a two-step procedure: first, a change of reference frame, which is effectively a time-dependent coordinate transformation restricted to rigid motions of the basis vectors; second, a spatial coordinate transformation). The reasons for introducing such a distinction will become apparent shortly; this distinction is an artifact of a particular formalism and is sometimes quite inconvenient, e.g., if a Lagrangian (material) coordinate system is used in a moving medium.

Then (III) can be reformulated as

(IV) Any physical law must be expressible in a form independent of a coordinate system.

Now, for purely spatial coordinate transformations, (IV) is never doubted and is easily achieved by writing all the mathematical relations in tensorial form. If the coordinate transformations are allowed to be time dependent, including changes of reference frames as a particular case, the validity of (IV) might seem less obvious for a moment. However, in this general case one recognizes (IV) as Einstein's principle of *general covariance,*[4] again easily achieved by writing all the mathematical relations in tensorial form, but now one must use *4-tensors in four-dimensional space-time.*

For example, 4-velocity is a 4-vector in space-time and is completely independent (i.e., has the same direction and magnitude) of a coordinate system in which it is measured (observed), as every *vector* should. It has, however, a time-like component, while what is normally called "velocity" is the projection of 4-velocity on the three-dimensional space-like hypersurface ("simultaneity"), which corresponds to our three-dimensional space. Clearly, this projection, i.e., 3-velocity, will behave (transform) as a 3-vector only if the coordinate transformations are confined to this hypersurface and do not involve the time coordinate, which is the case of purely spatial, time-independent coordinate transformations. In other words, 3-velocity is "frame-dependent." (For an extensive discussion of general covariance and four-dimensional formulation of Newtonian mechanics the reader is referred to Ref. 5; also very useful is Ref. 4, especially Chaps. 1, 6, and 12.)

The very existence of frame-dependent quantities is now seen as arising from an attempt to apply the basically correct idea of the invariance of physical laws in the framework (the three-dimensional space) that is not adequate to this purpose.

In four-dimensional formalism, every physical law obeys (IV), i.e., is frame-indifferent. This, however, does not provide such a convenient means of restricting constitutive relations as does the three-dimensional MFI discussed before. The three-dimensional MFI is very useful in describing the behavior of materials; it does not, however, follow directly from any principle, but is an approximation whose validity is intimately connected with the structure of matter and the character of its motion.

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[1]Some recent references are D. Jou and J. M. Rubi, J. Non-Equilib. Thermodynam. **5**, 125 (1980); W. G. Hoover, B. Moran, R. M. More, and A. J. C. Ladd, Phys. Rev. A **24**, 2109 (1981); L. C. Woods, J. Fluid Mech. **136**, 423 (1983); P. G. de Gennes *et al.*, Physica A **118**, 43 (1983); W. Band, Phys. Rev. A **29**, 2139 (1984); J. W. Dufty, *ibid.* **30**, 622 (1984).

[2]C. Truesdell, *A First Course in Rational Continuum Mechanics* (Academic, New York, 1977).

[3]C. Truesdell and W. Noll, *Handbuch der Physik* (Springer-Verlag, Berlin, 1965), Vol. 3/3, pp. 47 and 66.

[4]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 302.

[5]P. Havas, Rev. Mod. Phys. **36**, 938 (1964).