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Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics

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Recently a nonlinear Schrödinger equation (NLSE) with an inhomogeneous term proportional to $b\ln(|\psi|^2|a^3)\psi$ has been put forward. It has been proposed to apply it to atomic physics. Subsequent neutron interferometer experiments designed to test the physical reality of such a nonlinearity were not conclusive, thus rejecting it as unphysical. In the present paper it is pointed out that the different length scales a associated with atomic and nuclear physics, for example, lead to different typical energies b for these systems. Guided by the experience with phenomenological NLSE's, the constant b is for the following applications to nuclear physics identified with the compressibility of finite nuclear matter, $C=K/9$, i.e., $b\equiv C$. Thus we obtain consistent qualitative and quantitative answers related to the concepts of microworlds and mesoworlds as well as, e.g., the prediction $130\leq K\leq 250$ MeV. However, this necessitates the interpretation of the respective NLSE as an equation for extended objects.

I. INTRODUCTION

In spite of the fact that the linear Schrödinger equation (LSE) works extremely well for a wide range of nonrelativisitic quantum-mechanical problems, there are apparently some arguments in favor of nonlinear Schrödinger equations (NOSE, or NLSE; see Refs. 1–7, and references therein). A particularly interesting example is given by the NLSE which Białynicki-Birula and Mycielski (BBM)[6] recently proposed: namely,

$$\left[-\frac{\hbar^2}{2m}\nabla^2+U(\mathbf{x},t)-b\ln(|\psi|^2a^n)\right]\psi=-i\hbar\partial_t\psi(\mathbf{x},t)\quad,\qquad(1)$$

with the associated density $\rho(\mathbf{x},t)=|\psi(\mathbf{x},t)|^2$. This work of BBM was motivated by the desire to deal in a consistent way with quantum phenomena and in particular with the transition from microworlds to macroworlds (see further below). The notation is the conventional one with n standing for the dimensions of the problem. BBM suggest the length a to be the same universal constant for all systems and require the energy constant b to be very small, i.e., $2.5\times10^{-12}<b<4\times10^{-10}$ eV. The main reason for this requirement is that the deviations of (1) from the LSE are believed to be rather small (and that for larger b quantum-mechanical objects would behave like classical particles). BBM made it clear that they have no intention to abandon the conventional interpretation of quantum mechanics as a formalism for point particles. The response which (1) received and results of further formal and numerical discussions of its properties may be taken from Refs. 7–9.

BBM suggest that the NLSE (1) should be applied to atomic physics. Shimony proposed[10] a neutron interferometer experiment to test the physical reality of such a nonlinearity. Subsequently independent (and different) experiments carried out by Shull, Atwood, Arthur, and Horne[11] and by Gähler, Klein, and Zeilinger[12] resulted only in upper limits for b, i.e., $b<3.4\times10^{-13}$ and $b<3.3\times10^{-15}$ eV, respectively. The implications of these results are that there is no real basis for such a nonlinearity as far as atomic physics is concerned.[11–13]

Why should we reconsider this NLSE?

(i) Approximations to the linear Schrödinger equation (like Fermi theory, Hartree-Fock, the Thomas Fermi approach, nuclear fluid dynamics, analogies to classical fluid dynamics, etc.) are well known to lead to *non*linear Schrödinger equations that yield a lot of useful results.[14–17] But there are also approaches attempting to deal in a nonperturbative way with the *spatial extensions* or *self-interactions* of quantum-mechanical objects[2–4] as well as attempts to formulate the respective thermodynamical descriptions.[5] And all of them give rise to nonlinear Schrödinger-type equations. There is no arguing about the fact that the LSE works nicely in atomic physics with its tremendous interparticle separations (compared with the extensions of the particles involved, i.e., electrons and nuclei; in contrast with reality, within atomic physics the latter are usually treated as single entities). *If* nonlinearities are really important then they should manifest themselves most drastically in systems like atomic nuclei with their closely packed nucleons. [The possible objection that nucleons are not elementary particles, but made up of pointlike objects, does not invalidate these arguments; it simply insinuates that we follow the same level of sophistication as in atomic physics; i.e., now we ignore the substructures of the nucleons (and not of the nuclei, as in atomic physics). The substructures (e.g., nucleons in nuclei and quarks in nucleons) come only into play if we insist on a fully microscopic description at all levels. A qualified discussion of the associated problems is beyond the scope of this paper, so that references[1-5,18] should be consulted for further details.]

(ii) BBM suggested the quantity a in (1) to be a universal constant that may readily be scaled away. This is certainly true if a really *is* a constant as they assume. However, if it depends on the respective system or is even a function of $\mathbf{x}$ (as hinted by the discussion of similar hydrodynamical NLSE's[16,17]) then this is no longer possible.

Though (i) and (ii) do imply that it is sensible to reconsider the NLSE (1), they both violate the intentions of BBM[6,13] who did not want to leave the framework of the traditional quantum mechanics for point particles (certainly a convenient idealization).

To find out whether the concepts put forward by BBM are compatible with quantum-mechanical systems other than atomic physics, let us apply some of them to nuclear physics. Positive findings would not prove that (1) has to be reinterpreted as a NLSE for extended objects, but they would certainly give some impetus to further research trying to understand to what extent the results are accidental and to what extent they bespeak of some fundamental connections.

II. RECOLLECTIONS

For more details on the material gathered in this section the original paper of BBM[6] should be consulted. The prominent features of the NLSE (1) are contained in its soliton or "gausson" solutions,

$$\psi(\mathbf{x},t) = (l\pi^{1/2})^{-n/2}\exp\left[-\frac{i}{\hbar}\left(\frac{p^2 t}{2m}-\mathbf{p}\cdot\mathbf{x}\right)\right] \times \exp\left[-\left(\frac{\mathbf{x}-\mathbf{x}_0-\mathbf{v}t}{2l}\right)^2\frac{1}{2l^2}\right] , \quad (2)$$

where the relation $\mathbf{p}=m\mathbf{v}$ has been assumed to hold. The quantity

$$l \equiv \sqrt{\hbar^2/2mb} \quad (3)$$

that occurs in (2), characterizes the "gausson" solutions and represents a typical length of the system described by the NLSE (1). The center of mass "gausson" solution of a spherical object with radius R and mass density d is represented by

$$l=(3\hbar^2/8\pi bR^3 d)^{1/2} . \quad (4)$$

Defining now the quantity

$$R_0=(3\hbar^2/8\pi bd)^{1/5}=\left(\frac{\hbar^2}{2m}\frac{R^3}{b}\right)^{1/5} , \quad (5)$$

with $d=m/V=3m/4\pi R^3$, it is stated by BBM that $R \ll R_0$, $R \simeq R_0$, and $R \gg R_0$ classify the respective particles as micro-, meso-, and macro-objects, respectively. The properties of meso-objects are believed to be rather different from the ones of micro- and macro-objects. This lead BBM to specific estimates for the universal constant b and the typical spatial extensions of the system to which (1) might be applicable, namely,

$$2.5\times10^{-12}\ \mathrm{eV} < b < 4\times10^{-10}\ eV , \quad R_0 < 50\ \mathrm{\AA} = 50\times10^{-9}\ \mathrm{m} . \quad (6)$$

Another point of interest to the present discussion concerns the splitting or binding energy B released in the splitting of the wave function of an isolated system into nonoverlapping parts having essentially the same form as the initial wave function, say,

$$\psi(\mathbf{x}) \to \sum_{i=1}^{k}(p_i)^{1/2}\psi(\mathbf{x}-\mathbf{x}_i) ,$$

with $\sum_{i=1}^{k} p_i = 1$. The expression BBM provided for B is

$$B=-b\sum_{i=1}^{k} p_i \ln p_i . \quad (7)$$

III. APPLICATIONS TO NUCLEAR PHYSICS

Bearing in mind points (i) and (ii) we discuss now (a) whether the constant b really has to be very small, (b) whether (3)—alone and/or together with (5)—gives reasonable answers when applied to nuclear physics, and (c) whether the same holds for (7).

(a) BBM did not *prove* a to be a constant; they, rather, inferred from their physical interpretation of (1) that this should be the case. However, if we interpret the $\rho=|\psi|^2$ due to (1) as representing the physical, experimentally measurable charge or mass density, then it might be sensible to consider a to be a function of $\mathbf{x}$.[13,16,17] Let us now follow the philosophy of hydrodynamical NOSE's and identify a^{-n} with the asymptotic undisturbed densities of two heavy ions (that are at $t=\pm\infty$ well separated so that they do not feel the presence of each other); i.e., $a^{-n} = \rho_0(\mathbf{x}) = \rho_0(\mathbf{x}, \pm\infty) = \rho_{01}(\mathbf{x}) + \rho_{02}(\mathbf{x})$, where the indices 1 and 2 refer to the two nuclei. The nonlinearity of (1) acquires then the form

$$b\ln(|\psi|^2 a^3) = b\ln[\rho(\mathbf{x},t)a^3] = b\ln[\rho(\mathbf{x},t)/\rho_0(\mathbf{x})] = b[\ln\rho(\mathbf{x},t)-\ln\rho_0(\mathbf{x})] . \quad (8)$$

For $\psi(\mathbf{x},t)\to 0$ Eq. (1) yields an infinitely strong binding, an unpleasant feature which does not necessarily pertain for the nonlinearity (8) where it may be counteracted by $\rho_0(\mathbf{x})$.

In the case of scattering events—as, e.g, discussed in Refs. 10–12—$\rho(\mathbf{x},t)\simeq\rho_0(\mathbf{x})$ holds for all times but the ones close to the climax of the scattering event, say, at $t\cong 0$. That is, for tightly bound pointlike particles possessing no internal strucutres (e.g., electrons and the nucleus as treated in atomic physics) the deviations of $\rho(\mathbf{x},t)$ from $\rho_0(\mathbf{x})$ will always be negligible. Hence, the nonlinearity will not be activated or only activated to an extremely small extent. According to our understanding, that should hold for atomic physics with its huge interparticle separations and also for the experiments discussed in Refs. 10–12. If the nonlinearity is something real, then it should manifest itself in a more obvious way in the scattering of heavy ions on each other (with distances of smallest approach that are comparable with their spatial extensions[16,17]). To an even larger extent this argument is believed to hold for the relations between nucleons bound together within a nucleus. Traditional nuclear physics denies, however, such nonlinearities, in spite of the extreme success of nonlinear methods like Hartree-Fock, Thomas Fermi, and the like.[14] Yet, it has to be admitted that even after 50 years one does not have a unified theory of the atomic nucleus and most (or all?) of the significant unresolved problems are related to short interparticle separations. And for short separations the nonlinearity should be of note, *if* it is something fundamental.

From this qualitative discussion it is inferred that the nonlinearity (and the constant b) should be rather small for systems with large interparticle separations, while it might be significant for strongly overlapping systems. Equation (8) indicates also that [for $a(\mathbf{x})$] the strength of the nonlinearity depends not just on b but on the combination of a and b. This is in line with the well established wisdom that typical lengths and energies vary from system to system; for example, in atomic physics they are of the order of angstroms and eV while nuclear physics requires fermis and MeV as suitable units. Our arguments are certainly sensible, but they are not strong enough to enforce the interpretation given. However, the same applies to the suggestion of BBM that a should be the same universal constant for all systems.[6]

(b) To find out whether or not the concept of separating the world into micro-, meso-, and macro-objects[6] does make sense when applied to nuclear physics, let us recall that meso-objects should exhibit properties distinctly different from the ones of micro- and macro-objects. This requirement seems to indicate that nucleons and α particles are the only candidates for meso-objects traditional nuclear physics can offer. To test this notion, we insert the well-determined charge radius of the proton, $R_p = 0.865$ fm, into (3) to obtain $b = 28$ MeV.

Now, what is the physical interpretation of b in nuclear physics? Applying hydrodynamic concepts to elastic collisions of heavy ions similar NLSE's, such as (1), were arrived at. (That is, depending on the details of the derivation, they have a cubic or logarithmic nonlinearity.[16]) The strengths of the respective nonlinearities are in these cases given by the nuclear compressibility modulus, $C = K/9$, thus indicating that the nonlinearity is proportional to the elastic energy stored in compressed nuclear matter.[16,17] Detailed considerations of common points and of differences between these NLSE's and (1) are beyond the scope of this paper. Hence, let us simply take over the notion that in applications to nuclear physics there are good reasons for identifying the constant b with C, i.e., for the substitution $b = C = K/9$.

The numerical value $b = 28$ MeV arrived at for the proton implies thus $K = 9C = 9b = 250$ MeV. It corroborates nicely with conventional results. Evaluating now by the aid of (5) the radius R_0 of the proton we arrive at $R_p = 0.865$ fm $\simeq R_0(\text{proton}) = 0.8654$ fm. According to BBM such a coincidence implies indeed that protons (and, hence, also neutrons) are to be classified as meso-objects which are to correspond to the "gausson" solutions of (1). That is, our assumption is compatible with the resulting numbers. Extending the comparison via the global relation $R(A) = r_0 A^{1/3}$ with $r_0 = 1.2$ fm towards larger nuclei, Fig. 1 is arrived at. For heavier nuclei $R(A)$ (full curve) is obviously significantly larger than R_0 [dotted curve, (5)] so that we have to classify them as macro-objects. In the case of light nuclei the classification is more problematic. However, with the additional requirement[6] that the properties of meso-objects be distinctly different from the ones of micro- and macro-objects, we are lead back to nucleons and possibly α particles as feasible candidates for meso-objects or "gaussons."

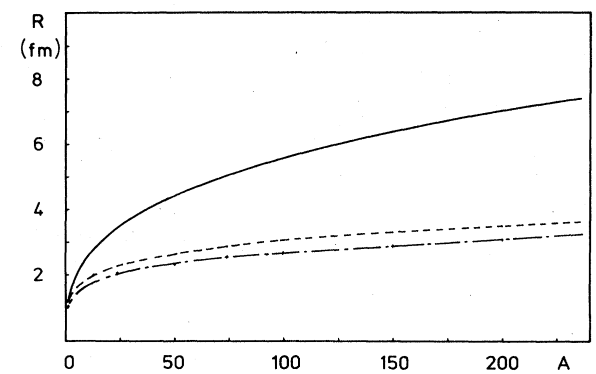

FIG. 1. Full curve represents the average nuclear radii as functions of the mass number, i.e., $R(A) = 1.2A^{1/3}$ fm. The R_0 of (5) with $K = 9C = 250$ MeV correspond to the dotted curve and the ones with $K = 9C = 130$ MeV to the broken curve.

At least for nucleons, this conclusion seems to be inescapable, so that they are to be identified with the "gausson" solutions of (1). Hence, we use now (3) with $b = C = K/9$ to evaluate the compressibility of the proton as a function of its radius (see Fig. 2). The absolute upper limit of K corresponding to the well-established charge radius of the proton is $K_{max} = 250$ MeV; its lower limit with $R = 1.2$ fm (as the largest possible radius for the nuclear matter distributions of nucleons, for which 1 fm should be more realistic) corresponds to $K_{min} = 130$ MeV. These two extreme limits are consistent with the range of values discussed in various approaches within nuclear physics.

To get a feel for the impact that such a small value of K, such as 130 MeV, has on the discussion of Fig. 1, the calculations were repeated with $C = K/9 = 14.4$ MeV to obtain essentially the same result as before (see the broken curve in Fig. 1).

(c) Recalling that α particles fulfill, at least to a reasonable extent, the requirements for meso-objects, let us use

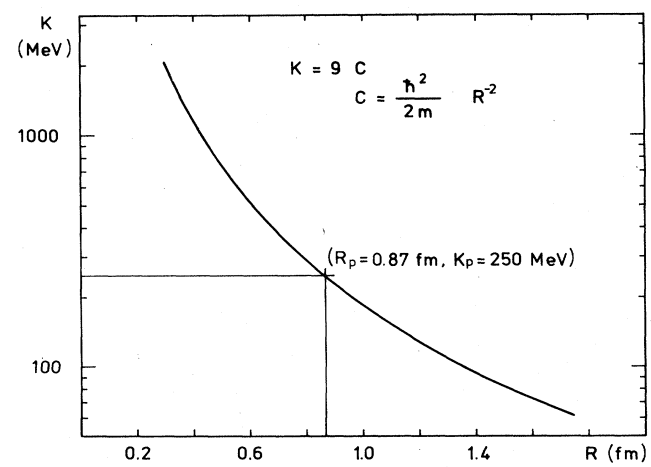

FIG. 2. Nuclear compressibility modulus $K = 9C = 9(\hbar^2/2m)/R^2$ is plotted as a function of the proton radius [see Eq. (3)].

(7) to evaluate the splitting or binding energy B for the breakup of an α "gausson" solution into four identical nucleon-"gausson" solutions. Since such a crude picture ignores spin and isospin, it is understood that one should at best expect order of magnitude estimates. Using the two extreme values for K as discussed above (i.e., $K=9C=9b=250$ and 130 MeV) and the more reasonable one for $R=1$ fm with 189 MeV, we obtain for the binding energy of the α particle the estimates $B(K=130)=-20$ MeV, $B(K=189)=-29$ MeV, and $B(K=250)=-39$ MeV. Bearing in mind that the experimentally established $B_\alpha=-28.3$ MeV yields $K=184\ \text{MeV}=9\times 20.4$ MeV, all these numbers are surprisingly close to the truth.

IV. SUMMARY

It is concluded that the concepts of BBM as developed by them in the context of their NLSE[6] (1) are compatible with nuclear physics. That holds, e.g., as well for the binding energy of "gausson" solutions as for the division of the world into micro-, meso-, and macro-objects. Taken seriously, the implications are, e.g., that within nuclear physics nucleons and α particles are meso-objects and that the compressibility of (finite) nuclear matter should be found in the interval $130 \leq K\,[\text{MeV}] \leq 250$.

However, in view of the negative experimental results of Refs. 11 and 12, and bearing in mind the discussion of Sec. III and of Refs. 2–4, 16, and 17, it has to be realized that the only consistent interpretation of the NLSE (1) would then be the one of an equation for extended objects (and not for point particles as originally suggested by BBM). This new interpretation emerges if the a of (1) is taken to be a different constant for different systems or even a function of the spatial variables. We would like to caution the reader against premature far-reaching conclusions. But we believe that the present discussion indicates that it is sensible to continue more detailed studies into (1) and similar NLSE's to find out to what extent our tentative conclusions will survive further tests and to see to what extent it is possible to marry the field theoretical NLSE (1) with hydrodynamical and thermodynamical concepts.

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[1]B. Mielnik, Commun. Math. Phys. **37**, 221 (1974); P. Pearle, Phys. Rev. D **13**, 857 (1975).

[2]E. W. Mielke, Fortschr. Phys. **29**, 551 (1981); H. Matsumoto, N. J. Papastamatiou, H. Umezawa, and M. Umezawa, Phys. Rev. D **23**, 1339 (1981); M. Umezawa, Prog. Theor. Phys. **72**, 606 (1984); A. F. Rañada and L. Vazquez, An. Fis. **76**, 139 (1980); L. Garcia and A. F. Rañada, J. Phys. A **13**, 141 (1980); T. A. Brody, Rev. Mex. Fis. **29**, 461 (1983), and references therein.

[3]P. B. Burt, *Quantum Mechanics and Nonlinear Waves* (Harwood Academic, Chur-London-New York, 1981).

[4]E. F. Hefter, J. Phys. (Paris) Colloq. **45**, C6-67 (1984); Prog. Theor. Phys. **69**, 329 (1983); Phys. Lett. **141B**, 5 (1984); Acta Phys. Pol. **A65**, 377 (1984); E. Hefter and K. A. Gridnev, Prog. Theor. Phys. **72**, 549 (1984); E. F. Hefter, M. de Llano, and I. A. Mitropolsky, Phys. Rev. C **30**, 2042 (1984), and references cited in these papers.

[5]T. A. Minelli, A. Pascolini, and G. Pozzato, contributed to the Terzo Convegno sui Fenomeni Nonlineari, Amantea, Italy, 1984 (unpublished).

[6]I. Białynicki-Birula and I. Mycielsky, Ann. Phys. (N.Y.) **100**, 62 (1976).

[7]T. F. Morris, Can. J. Phys. **56**, 1405 (1978); I. Ventura and G. C. Marquez, J. Math. Phys. **19**, 838 (1978); T. W. B. Kibble, Commun. Math. Phys. **64**, 73 (1978); R. Haag and U. Bannier, *ibid.* **65**, 189 (1979).

[8]I. Białynicki-Birula and J. Mycielski, Phys. Scr. **20**, 539 (1979); I. Białynicki-Birula, in *Nonlinear Problems in Theoretical Physics,* edited by A. F. Rañada, Lecture Notes in Physics, Vol. 98 (Springer, Berlin-Heidelberg-New York, 1979), p. 15.

[9]T. A. Minelli and A. Pascolini, Lett. Nuovo Cim. **27**, 413 (1980).

[10]A. Shimony, Phys. Rev. A **20**, 394 (1979).

[11]C. G. Shull, D. K. Atwood, J. Arthur, and M. A. Horne, Phys. Rev. Lett. **44**, 765 (1980).

[12]R. Gähler, A. G. Klein, and A. Zeilinger, Phys. Rev. A **23**, 1611 (1981).

[13]I. Białynicki-Birula (private communication).

[14]P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, Berlin-Heidelberg-New York, 1980).

[15]E. Madelung, Z. Phys. **40**, 322 (1926); K.-K Kan and J. J. Griffin, Nucl. Phys. **A301**, 258 (1978), and references therein.

[16]D. S. Delion, K. A. Gridnev, E. F. Hefter, and V. M. Semjonov, J. Phys. G **4**, 125 (1978); K. A. Gridnev, K. Mikulaš, V. M. Semjonov, and E. F. Hefter, Izv. Akad. Nauk SSSR, Ser. Fiz. **45**, 134 (1981) [Bull. Acad. Sci. USSR, Phys. Ser. **45**, 117 (1981)]; K. A. Gridnev, E. F. Hefter, K. Mikulaš, V. M. Semjonov, and V. B. Subbotin, Aust. J. Phys. **36**, 155 (1983).

[17]E. F. Hefter, K. A. Gridnev, S. Saad, V. M. Semjonov, and V. B. Subbotin, J. Phys. (Paris) Colloq. **45**, C6-241 (1984); R. Stock, R. Bock, R. Brockmann, A. Dacal, J. W. Harris, M. Maier, M. E. Ortiz, H. G. Pugh, R. E. Renford, A. Sandoval, L. S. Schröder, H. Ströbele, and K. L. Wolf, Phys. Rev. Lett. **49**, 1236 (1982).

[18]F. Capura, Am. J. Phys. **47**, 11 (1979), and references therein.