New functional relation for vertex models

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A new functional relation for factorizable vertex systems that do not satisfy the so-called symmetry relation is proposed. Together with the unitary condition this relation enables one to obtain the free energy of these systems in the thermodynamic limit. These relations are then used to solve a particular case of the free-fermion eight-vertex model.

I. INTRODUCTION

In recent years there has been considerable growing interest on an exhaustive analysis of two-dimensional models that can be solved exactly. The efforts in this direction involve both spin and vertex systems.[1-3] The existence of nontrivial solutions of the factorization equations plays an essential role: When such solutions do exist, then two transfer matrices with different Boltzmann's weights commute.[4,5] From this commutation one can extract two functional relations for the partition function: the unitarity relation and the symmetry relation.[5-7] These two relations together with some analytical considerations are sufficient to determine the free energy per vertex (or per spin) in the thermodynamic limit.[8] Although this technique seems to be poorer than the inverse scattering method[9-11] (which allows in principle the determination of the whole eigenvalue spectrum of the transfer matrix, not only the largest one as in the functional technique) it has the advantage that the search for the pseudovacuum is not needed. Meanwhile, there are some models where the functional may fail. This usually happens when the model does not exhibit a 90° rotation symmetry (at the S-matrix level, this corresponds to models without crossing symmetry[5]). It is necessary then to find a substitute functional relation to supply the absence of the symmetry relation. This has already been done on the context of some $Z(N)\times Z(N)$ models,[12] but unfortunately (as we shall see in this paper) it does not work in some other cases. For this reason I propose in Sec. II a new functional relation which is applied, in Sec. III, to solve a special case of the free-fermion eight-vertex model.

II. THE FUNCTIONAL RELATION

Let $S_{i_1i_2}^{j_1j_2}$ be the two-particle scattering amplitude as shown in Fig. 1. The indices run from 0 to $q-1$ and correspond to the q kinds of particles. The parameter θ is the rapidity or spectral parameter.

The factorization equations or Yang-Baxter equations can be written as follows

$$S_{i_2i_2}^{k_1k_2}(\theta)S_{k_1i_3}^{j_1k_3}(\theta+\theta')S_{k_2k_3}^{j_2j_3}(\theta') = S_{i_2i_3}^{k_2k_3}(\theta')S_{i_1k_3}^{k_1j_3}(\theta+\theta')S_{k_1k_2}^{j_1j_2}(\theta)\ , \qquad (1)$$

where summation over repeated indices is understood. In Fig. 2 we give a graphical representation of these equations.

From Eq. (1) it is clear that if $\theta=\theta'=0$ then the so-called initial condition

$$S_{i_1i_2}^{j_1j_2}(\theta=0)=\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} \qquad (2)$$

is a trivial solution. If we put in the expression (1) $\theta+\theta'=0$ and use the initial condition we get

$$S_{i_1i_2}^{k_1k_2}(\theta)S_{k_2k_1}^{j_1j_2}(-\theta)=\rho(\theta)\delta_{i_1}^{j_2}\delta_{i_2}^{j_1}\ , \qquad (3)$$

where $\rho(\theta)$ is some even function of θ. The unitarity condition (3) can be extended to the partition function per vertex, $k(\theta)$, of the associated vertex model[5,6,13]

$$k(\theta)k(-\theta)=\rho(\theta)\ . \qquad (4)$$

The expression (4) is the unitarity (or inversion) relation. In order to solve the vertex it is necessary to aggregate another functional equation to the unitarity relation. This is usually done through the symmetry relation[6] which comes from a 90° rotation plus parity symmetries of the vertex model.[5] Naturally, there are models where such symmetries are absent, and a substitutional functional equation must be found, an example of which are

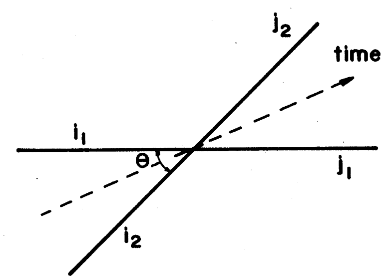

FIG. 1. S matrix in (1 + 1) dimensions.

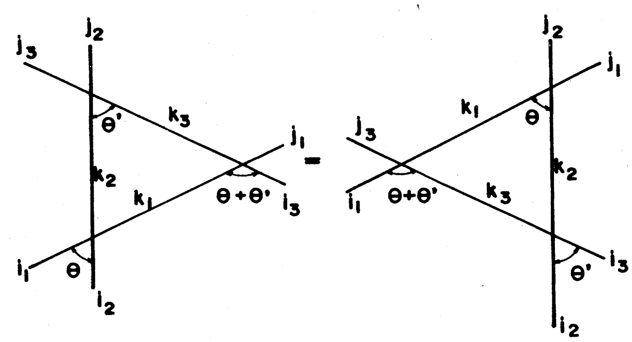

FIG. 2. Graphical representation of the factorization equations.

Belavin's solutions[14] of factorizable $Z(N)\times Z(N)$ models. Belavin and Zamolodchikov[12] found the following alternative form: if it is possible to find a number η such that

$$\sum_{i_1,j_1=0}^{N-1} S_{k_1i_1}^{j_1k_2}(2\eta)\psi_{j_1}^{i_1}=0,\quad k_1,k_2=0,1,\ldots,N-1$$

has as a *unique* solution $\psi_{j_1}^{i_1}\sim\delta_{j_1}^{i_1}$ then

$$\sum_{k_1,k_2=0}^{N-1} S_{i_1k_1}^{k_2j_1}(\eta-\theta)S_{i_2k_2}^{k_1j_2}(\eta+\theta)=\bar{\rho}(\theta)\delta_{i_2}^{j_1}\delta_{i_1}^{j_2}$$

satisfies the factorization equations and implies that the partition function per vertex obey the relation

$$k(\eta-\theta)k(\eta+\theta)=\bar{\rho}(\theta)\ .$$

However, Sogo *et al.*[2] have recently made an exhaustive analysis of all factorizable eight-vertex models and they found, in this category, a free-fermion eight-vertex model (they classify this model as type II, type I being the symmetric one solved by Baxter[15]). If we apply to this system the prescription developed by Belavin and Zamolodchikov we would have arrived at the conclusion (on account of the free-fermion condition) that the matrix ψ in diagonal form is *not* the unique solution and therefore their procedure cannot be used.

The factorization equations are invariant if we multiply the matrix $S(\theta)$ by a normalization function $\lambda(\theta)$, i.e., $S'(\theta)=\lambda(\theta)S(\theta)$ also satisfies the equations. Now one supposes that there exists a value $\theta=\bar{\theta}$ such that

$$S'^{\,j_1j_2}_{i_1i_2}(\bar{\theta})=\delta^{i_2}_{i_1}\delta^{j_2}_{j_1} \tag{5}$$

and

$$\sum_{k_1,k_2=0}^{q-1} S'^{\,i_2k_2}_{k_1i_1}(\theta+\bar{\theta})S'^{\,j_1j_2}_{k_1k_2}(\theta)=\rho'(\theta)\delta^{j_2}_{i_1}\delta^{j_1}_{i_2}\ , \tag{6}$$

then the factorization equations (1) are satisfied.

The introduction of the function $\lambda(\theta)$ comes only to take into account the fact that the value $\bar{\theta}$ may correspond to a pole of the original S matrix (as in the case that we shall study in the next section).

Returning to the original S matrix, we have

$$\sum_{k_1,k_2=0}^{q-1} S^{i_2k_2}_{k_1i_2}(\theta+\bar{\theta})S^{j_1j_2}_{k_1k_2}(\theta)=\bar{\rho}(\theta)\delta^{j_2}_{i_1}\delta^{j_1}_{i_2}\ , \tag{7}$$

where

$$\bar{\rho}(\theta)=\frac{\rho'(\theta)}{\lambda(\theta)\lambda(\theta+\bar{\theta})}\ .$$

Now what I would like to say is that the local condition (7) extends to the level of the partition function per vertex, i.e.,

$$k(\theta+\bar{\theta})k(\theta)=\bar{\rho}(\theta)\ . \tag{8}$$

This is a delicate point since if one tries to use the standard method to prove (8), the first step would be to show that the two transfer matrices $T(\theta)$ and $T(\theta+\bar{\theta})$ commute.[6] But the determinant of $S'(\theta)$ is zero, so the matrix is singular, and therefore the commutation cannot be proved from the factorization equations. However when Eq. (8) is applied to a particular case of the free-fermion eight-vertex model then at the level of the associated spin model (Ising model on a Union Jack lattice) it becomes exactly the symmetry relation. In any circumstance a more direct proof would be required.

III. SOLUTION OF A FREE-FERMION EIGHT-VERTEX MODEL

Sogo *et al.*[2] studied a factorizable free-fermion eight-vertex model where the allowed vertex configurations and the respective weights are shown in Fig. 3. It is a particular case of the one solved by Fan and Wu[16] using the method of dimers, and it corresponds to the Ising model defined on a Union Jack lattice.[17,18]

The free fermion condition reads

$$h_1h_2+h_t^2=1+h_a^2\ . \tag{9}$$

Through the factorization equations Sogo *et al.*[2] found the following parametrization (see Felderhof[19] who also solved the free-fermion model using elliptical function parametrizations)

$$h_1(\theta)=\frac{\mathrm{cn}(\theta,k)}{\mathrm{dn}(\theta,k)}+\frac{\gamma\epsilon\,\mathrm{sn}(\theta,k)}{(1-\gamma^2)^{1/2}},\quad \epsilon=\pm1\ , \tag{10a}$$

$$h_2(\theta)=h_1(-\theta)\ , \tag{10b}$$

$$h_t(\theta)=\frac{\epsilon\,\mathrm{sn}(\theta,k)}{(1-\gamma^2)^{1/2}}\ , \tag{10c}$$

$$h_a(\theta)=\frac{\delta k\,\mathrm{sn}(\theta,k)\mathrm{cn}(\theta,k)}{\mathrm{dn}(\theta,k)},\quad \delta=\pm1\ , \tag{10d}$$

$$h_r(\theta)=1\ , \tag{10e}$$

where θ is the rapidity, k is the modulus of the Jacobian elliptic functions,[20] and γ is an additional parameter.

First I define

$$(1-\gamma^2)^{1/2}=\mathrm{sn}(\xi,k) \tag{11}$$

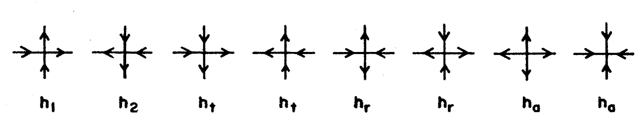

FIG. 3. Vertices of the free fermion model and their respective activities.

and for convenience, I will change all the elliptic functions to the modulus $\bar{k}$

$$\bar{k}=\frac{ik}{k'} \tag{12}$$

where k' is the complementary modulus.

In this way, Eqs. (10) can be rewritten (putting $\epsilon=\delta=1$)

$$h_1=\mathrm{cn}(u,\bar{k})+\frac{\mathrm{cn}(\eta,\bar{k})\,\mathrm{sn}(u,\bar{k})}{\mathrm{sn}(\eta,\bar{k})\mathrm{dn}(u,\bar{k})}\ , \tag{13a}$$

$$h_2=\mathrm{cn}(u,\bar{k})-\frac{\mathrm{cn}(\eta,\bar{k})\mathrm{sn}(u,\bar{k})}{\mathrm{sn}(\eta,\bar{k})\mathrm{dn}(u,\bar{k})}\ , \tag{13b}$$

$$h_t=\frac{\mathrm{dn}(\eta,\bar{k})\mathrm{sn}(u,\bar{k})}{\mathrm{sn}(\eta,\bar{k})\mathrm{dn}(u,\bar{k})}\ , \tag{13c}$$

$$h_a=\frac{-i\bar{k}\,\mathrm{sn}(u,\bar{k})\mathrm{cn}(u,\bar{k})}{\mathrm{dn}(u,\bar{k})}\ , \tag{13d}$$

$$h_r=1\ , \tag{13e}$$

where $u=k'\theta$ and $\eta=k'\xi$.

The physical regime when all weights are positive in the ordered phase corresponds to $0<\bar{k}<1$, $u=iv$, $\eta=i\tau$, $0<v<\tau$, $0<\tau<\bar{K}'$, and $\bar{K}'-v<\tau$ ($\bar{K}'$ is the complete elliptic integral of the first kind of the complementary modulus $\bar{k}'$). It is easy to verify that condition (5) is satisfied for $\bar{u}=i\bar{K}'$ and $\lambda(u)=[\mathrm{cn}(u,\bar{k})]^{-1}$. Using Eqs. (3), (4), (7), and (8) one can derive the functional relations for the reduced free energy per vertex f ($f=-\beta F$)

$$f(u)+f(-u)=\ln\left[\frac{\mathrm{sn}^2\eta-\mathrm{sn}^2u}{\mathrm{sn}^2\eta\,\mathrm{dn}^2u}\right], \tag{14a}$$

$$f(u+i\bar{K}')=\ln\left[\frac{\mathrm{cn}(u+i\bar{K}')}{\mathrm{cn}u}\right]+f(-u)\ , \tag{14b}$$

where I have omitted the explicit dependence on $\bar{k}$ of the elliptic functions.

The logarithms have the following expansion for $\bar{k}<1$:

$$\ln\left[\frac{\mathrm{sn}^2\eta-\mathrm{sn}^2u}{\mathrm{sn}^2\eta\,\mathrm{dn}^2u}\right]=-\sum_{n=1}^{\infty}\frac{16q^n}{n(1-q^{2n})}\sin\left[\frac{\pi n}{2\bar{K}}(\eta+\bar{K}-i\bar{K}')\right]\times\sin\left[\frac{\pi n}{2\bar{K}}(\eta-\bar{K}-i\bar{K}')\right]\sin^2\left[\frac{\pi nu}{2\bar{K}}\right], \tag{15a}$$

$$\ln\left[\frac{\mathrm{cn}(u+i\bar{K}')}{\mathrm{cn}u}\right]=-2\sum_{n=1}^{\infty}\left[\frac{2q^n+(-1)^n(1+2q^n)}{n(1-q^{2n})}\right]\sin\left[\frac{\pi ni\bar{K}'}{2\bar{K}}\right]\sin\left[\frac{\pi n}{\bar{K}}\left[u+\frac{i\bar{K}'}{2}\right]\right], \tag{15b}$$

where

$$q=\exp\left[-\frac{\pi\bar{K}'}{\bar{K}}\right].$$

From these expressions and Eqs. (14) one can derive the reduced free energy per vertex in the ordered phase (for the disordered phase, i.e., $\bar{k}>1$, it is necessary to change $\bar{k}\to(\bar{k})^{-1}$ and derive the new expansion for the logarithms),

$$f(u)=-\sum_{n=1}^{\infty}\frac{1}{n(1-q^{2n})\cos\left[\dfrac{\pi ni\bar{K}'}{2\bar{K}}\right]}[g_n(u)+h_n(u)] \tag{16}$$

with

$$g_n(u)=[2q^n+(-1)^n(1+q^{2n})]\sin\left[\frac{\pi ni\bar{K}'}{2\bar{K}}\right]\sin\left[\frac{\pi nu}{\bar{K}}\right]$$

and

$$h_n(u)=8q^n\sin\left[\frac{\pi n}{2\bar{K}}(\eta+\bar{K}-i\bar{K}')\right]\sin\left[\frac{\pi n}{2\bar{K}}(\eta-\bar{K}-i\bar{K}')\right]\sin\left[\frac{\pi nu}{2\bar{K}}\right]\sin\left[\frac{\pi n}{2\bar{K}}(u-i\bar{K}')\right].$$

IV. CONCLUSION

For factorizable vertex models that do not satisfy the symmetry relation I have presented an alternative functional relation. This relation is verified to be true for a free-fermion eight-vertex model which corresponds to the Ising model defined on a Union Jack lattice; nevertheless a more direct demonstration is still lacking.

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