Multiplicative stochastic processes in nonlinear systems. II. Canonical and noncanonical effects

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We study a physical system consisting of a low-frequency nonlinear oscillator interacting both with a thermal bath at the temperature T_1 and a high-frequency linear oscillator which, in turn, interacts with a thermal bath at the temperature T_2 ($T_2 \geq T_1$). The interaction between slow and fast oscillator is nonlinear, thereby influencing the motion of the slow oscillator via fluctuations of a multiplicative nature. By means of a suitable procedure of elimination of the fast variables, a contracted description is obtained, which, at $T_1 = T_2$, exhibits precisely the same structure as that recently derived by Lindenberg and Seshadri [Physica (Utrecht) **109A**, 483 (1981)] from the Zwanzig Hamiltonian. Instead of the transition from the overdamped to the inertial case revealed in our earlier paper [S. Faetti *et al.*, Phys. Rev. A **30**, 3252 (1984)] in this series, it is shown that precisely the reverse effect takes place. This is confirmed by computer calculations and the reliability of the computer calculation, in turn, is confirmed in the inertial regime via analog simulation. The theory enabling us to explore the noise-induced transition to the overdamped regime is based on an improvement of the techniques of elimination of fast variables, which produces automatic resummation over infinite perturbation terms. At $T_2 > T_1$, a space-dependent diffusion term with the same structure as that involved with the multiplicative fluctuation of the "external" kind is proven to be added to the canonical multiplicative diffusion term exhibited by the case $T_1 = T_2$. Canonical and noncanonical effects, which have so far been the subject of separate investigations, may thus be described via one single picture. It is also shown that, in the purely canonical case, the ubiquitous character of the noncanonical diffusional form, ranging from Itô-like to Stratonovich-like structure, is lost and a unique form of diffusional equation occurs.

I. INTRODUCTION

This paper has to be regarded as the natural continuation of that of Ref. 1 with the same research strategy based on the joint use of theory, computer calculation, and analog simulation. In Ref. 1, henceforth referred to as I, we mainly addressed the following problems.

(i) First of all, we argued that it is dangerous to study a one-dimensional multiplicative differential stochastic equation without supplementing this investigation with additional information coming from a microscopic physical system behind the coarse-grained description provided by such an equation. This is indeed the general program of the reduced model theory (RMT),[2] which is currently finding a fertile field of application in the molecular dynamics of the liquid state.[3,4] The use of this general strategy enabled us to prove that the actual values of the physical observables range in a continuous way from the predictions of the Itô diffusion equation to those of the Stratonovich one. Similar conclusions have been reached in an independent way from Gardiner.[5] Recall that the microscopic model of I was given the form

$$\begin{aligned} \dot{x}(t) &= v(t) , \\ \dot{v}(t) &= -\gamma v(t) - x(t) y(t) - \frac{d\Phi(x)}{dx} + f(t) , \\ \dot{y}(t) &= -\Gamma y(t) + F(t) . \end{aligned} \tag{1.1}$$

The variable $\dot{y}$ is driven by its own Langevin equation and does not undergo any influence from the one-dimensional motion of the Brownian particle under study, with coordinate x and velocity v.

(ii) With increasing $\langle y^2 \rangle_{eq}$ and, therefore, the strength of the coupling between the Brownian particle and the variable y, at fairly large values of $\langle y^2 \rangle_{eq}$ the relaxation time of x was proven to decrease as a linear function of $1/\langle y^2 \rangle_{eq}$. It was shown that this ultimately provokes a transition from the overdamped to the inertial regime.

The major aim of the present paper is to address the same above-mentioned basic questions in the case when the stochastic variable y is also subjected to a back-reaction term which ensures detailed balance and, in suitable physical conditions, the attainment of a canonical equilibrium distribution.

The outline of this paper is as follows. In Sec. II we shall show how to derive the stochastic system here under study from a rigorous microscopic Hamiltonian description. In Sec. III we shall consider the case where some inertia is present. It will be shown that the constraint of detailed balance provokes an effect opposite to that of (ii): Upon increasing the coupling between the Brownian particle and y, a transition from the inertial to the overdamped regime is provoked. The theory which predicts this interesting effect will be checked via a joint use of computer calculation and analog experiment. Section IV will be devoted to answering the problem (i). We shall show that the constraint of canonical equilibrium distribution results in a unique kind of diffusion equation for the variable x.

Different forms of diffusion equation appear again when new physical conditions occur, which are incompatible with the attainment of such a canonical equilibrium distribution (Sec. V). Concluding remarks are found in Sec. VI.

II. ON THE DERIVATION OF THE LANGEVIN EQUATION UNDER STUDY FROM A HAMILTONIAN DESCRIPTION

As already done by some authors,[6-8] we shall derive the Langevin equation under study in this paper from a Hamiltonian description.

Let us consider the Hamiltonian

$$\mathscr{H}=\tfrac{1}{2}M\dot{x}^2+U(x,y)+\tfrac{1}{2}k_x(x-\xi_1)^2+V(\xi_1,\xi_2,\dots)+\sum_{i=1}^{N_\xi}\mu_i\frac{\dot{\xi}_i^2}{2}+\tfrac{1}{2}m\dot{y}^2+\tfrac{1}{2}k_y(y-\eta_1)^2+\sum_{j=1}^{N_\eta}\nu_j\frac{\dot{\eta}_j^2}{2}+W(\eta_1,\eta_2,\dots)\,. \tag{2.1}$$

This means that the particle with mass M and coordinate x interacts with a particle of mass μ_1 and coordinate ξ_1, which in turn interacts with a large number $N_\xi-1$ of other particles with masses μ_i and coordinates ξ_i. The particle M via the potential $U(x,y)$ interacts also with a particle of mass m and coordinate y which is similarly coupled via a particle of mass ν_1 and coordinate η_1 with a thermal bath of particles with masses ν_j and coordinates η_j $(j>1)$.

We shall focus our attention on the case where the freedom degrees $\xi_i,\dot{\xi}_i$, $i=1,2,\dots$ and $\eta_j,\dot{\eta}_j$, $j=1,2,\dots$ are so fast as to make it legitimate to write an equation of motion concerning only x, y, $\dot{x}$, and $\dot{y}$ while simulating the influence of the fast variables via a suitable dissipation-fluctuation process. In the case $U(x,y)=0$ the two particles of interest do not interact with each other, thereby reducing the problem to one widely studied in the past, i.e., that concerning how to derive a standard Fokker-Planck equation for a single particle of interest immersed in one thermal bath. As well-known[9-11] explicit analytical results, including also non-Markovian effects[10,11] can be arrived at when assuming the number of particles of the thermal bath to be infinite, the thermal bath to be harmonic, and the coupling between particles of interest and thermal bath to be weak. A further significant assumption is that the whole system attains a canonical equilibrium distribution at the temperature T.[9-11] As a consequence of these assumptions, it can be shown[9-11] that also the Brownian particle must attain a canonical equilibrium state at the same temperature as its thermal bath. Since the freedom degrees $\xi_i,\dot{\xi}_i$, $i=1,2,\dots$ and $\eta_j,\dot{\eta}_j$, $j=1,2,\dots$ express two well-distinct thermal baths, for generality we shall assume the first to be characterized by the temperature T_1 and the second by the temperature T_2. The adiabatic elimination procedure (AEP) of Refs. 12–14, when applied to these two systems, regarded as being separated from each other, provides results in agreement with those of the former investigations.[14] If the two systems are brought into contact via the potential term $U(x,y)$, by application of the AEP of Refs. 12–14, we obtain for the probability distribution $\rho(x,v,y,w;t)$ (where $v\equiv\dot{x}$ and $w\equiv\dot{y}$) the following equation of motion:[15]

$$\frac{\partial}{\partial t}\rho(x,v,y,w;t)=\left[-v\frac{\partial}{\partial x}+\left[\frac{1}{M}\frac{\partial}{\partial x}U(x,y)\right]\frac{\partial}{\partial v}+\gamma\left[\frac{\partial}{\partial v}v+\frac{k_BT_1}{M}\frac{\partial^2}{\partial v^2}\right]-w\frac{\partial}{\partial y}+\left[\frac{1}{m}\frac{\partial}{\partial y}U(x,y)\right]\frac{\partial}{\partial w}+\Lambda\left[\frac{\partial}{\partial w}w+\frac{k_BT_2}{m}\frac{\partial^2}{\partial w^2}\right]\right]\rho(x,v,y,w;t)\,. \tag{2.2}$$

In the case when $T_2>T_1$ energy will flow from the second to the former thermal bath via the coupling $U(x,y)$ without affecting these thermal sources. This is a natural consequence of the assumptions $N_\xi=\infty$ and $N_\eta=\infty$, under which at $U(x,y)=0$ the standard Brownian motion is recovered.[14,4] In other words the sets of variables $\{\xi_i,\dot{\xi}_i\}$ and $\{\eta_j,\dot{\eta}_j\}$ can be thought of as being ideal heat baths.

The Langevin equation associated with Eq. (2.2) reads

$$\begin{aligned}\dot{x}&=v\,,\\ \dot{v}&=-\frac{1}{M}\frac{\partial}{\partial x}U(x,y)-\gamma v+f(t)\,,\\ \dot{y}&=w\,,\\ \dot{w}&=-\frac{1}{m}\frac{\partial}{\partial y}U(x,y)-\Lambda w+f'(t)\,,\end{aligned} \tag{2.3}$$

where $f(t)$ and $f'(t)$ are Gaussian white noises defined by

$$\langle f(0)f(t)\rangle_{\rm eq}=2\gamma\frac{k_BT_1}{M}\delta(t)\,, \tag{2.4}$$

$$\langle f'(0)f'(t)\rangle_{\rm eq}=2\Lambda\frac{k_BT_2}{m}\delta(t)\,. \tag{2.5}$$

Throughout this paper we shall focus our attention either on the pair x,v or the variable x alone. The condition $T_2>T_1$ will serve the main purpose of keeping this system in a steady-state condition far from canonical equilibrium, and close to the illustrative scheme of I (see Fig. 2 of I). Nevertheless, the theoretical treatment of the present paper must be general enough as to include the case

$$T_1=T_2=T\,. \tag{2.6}$$

The Fokker-Planck equation of Eq. (2.2), when Eq. (2.6) is satisfied, admits the canonical equilibrium distribution

$$\rho_{\rm eq}(x,v,y,w) \propto e^{-Mv^2/2k_BT} e^{-m\omega^2/2k_BT} e^{-U(x,y)/k_BT} . \tag{2.7}$$

It is therefore worth remarking that in consequence of this canonical constraint, the variable x must be given the equilibrium distribution

$$\sigma_{\rm eq}(x) \propto e^{-\Phi_{\rm eff}(x)/k_BT} , \tag{2.8}$$

where

$$\Phi_{\rm eff}(x) = -k_BT \ln \left[A \int dy \exp[-U(x,y)/k_BT] \right] , \tag{2.9}$$

and A is a normalization factor, the actual value of which does not influence the calculations as resulting in an additive contribution to $\Phi_{\rm eff}$ independent of x.

Throughout this paper we shall assume that

(i) $U(x,y)$ has the form

$$U(x,y) = \Phi(x) + \tfrac{1}{2}\omega_0^2 y^2 + V(x,y) , \tag{2.10}$$

(ii) the masses m and M are

$$m = M = 1 . \tag{2.11}$$

This allows us to replace Eq. (2.3) with

$$\begin{aligned} &\dot{x} = v , \\ &\dot{v} = -\frac{d}{dx}\Phi(x) - \frac{\partial}{\partial x} V(x,y) - \gamma v(t) + f(t) , \\ &\dot{y} = w , \\ &\dot{w} = -\frac{\partial}{\partial y} V(x,y) - \Lambda w - \omega_0^2 y + f'(t) . \end{aligned} \tag{2.12}$$

To render the theoretical investigation still simpler we shall assume w to be the fastest variable of the system. This allows us to obtain from Eq. (2.12) the following reduced set of equations:

$$\begin{aligned} &\dot{x} = v , \\ &\dot{v} = -\frac{d}{dx}\Phi(x) - \frac{\partial}{\partial x} V(x,y) - \gamma v(t) + f(t) , \\ &\dot{y} = -\Gamma y - \frac{1}{\Lambda}\frac{\partial}{\partial y} V(x,y) + \frac{f'(t)}{\Lambda} , \end{aligned} \tag{2.13}$$

where

$$\Gamma \equiv \omega_0^2/\Lambda . \tag{2.14}$$

Note that the contraction on the variable w does not affect the canonical constraint leading to the effective potential of Eq. (2.9). It is worth remarking furthermore that the analog experiment of the next section simulates the complete set of equations (2.12), thereby also providing a check on the assumption that Eq. (2.12) can be replaced by Eq. (2.13).

Let us assume $V(x,y)$ to read

$$V(x,y) = \psi(x) y . \tag{2.15}$$

Later on we shall consider the case

$$\psi(x) = \tfrac{1}{2}\alpha x^2 . \tag{2.16}$$

If Eq. (2.16) were valid and the back-reaction term of x on y were disregarded, Eq. (2.13) would turn out to be exactly equivalent to Eq. (1.1). A major aim of this paper is to assess whether or not physical conditions do exist allowing the system of Eq. (2.13) to mimic the properties of that of Eq. (1.1) without disregarding the influence of such a back-reaction term. We shall return to this issue in Sec. V, where it will be shown that at $T_2 > T_1$ a diffusion term appears, the form of which ranges from the Itô to the Stratonovich one.

Equation (2.16) allows us to write $\Phi_{\rm eff}(x)$ of Eq. (2.9) as follows:

$$\Phi_{\rm eff}(x) = \Phi(x) - \psi^2(x)/2\omega_0^2 . \tag{2.17}$$

A central result of this paper is that the AEP naturally leads to the normalized potential of Eq. (2.17). A special caution, however, will be required to ensure that when $T_1 = T_2 = T$ the equilibrium distribution of x be canonical with respect to precisely this effective potential. We are now in a position to clearly outline the next two sections. Let us assume for a while that

$$\Phi_{\rm eff}(x) = \tfrac{1}{2}(\omega_0')^2 x^2 . \tag{2.18}$$

In addition to the lifetime of the system of interest, $1/\gamma$, and the lifetime of the variable y, $1/\Gamma$, we can then define a further relevant time scale, i.e., the oscillation time $1/\omega_0'$.

In the following sections we shall explore the following cases:

(a) $1/\omega_0' \sim 1/\gamma \gg 1/\Gamma$ (Sec. III),

(b) $1/\omega_0' \gg 1/\gamma \gg 1/\Gamma$

(c) $1/\omega_0' \gg 1/\Gamma \gg 1/\gamma$ } (Sec. IV).

(d) $1/\omega_0' \gg 1/\Gamma \sim 1/\gamma$

III. INERTIAL CASE

A. Theory

In the inertial case (a) we must regard both x and v as being variables of interest. This means that we are allowed to eliminate from the set of Eq. (2.13) only the variable y provided that the parameter Γ is large enough.

We shall apply the adiabatic elimination procedure (AEP) of Refs. 12–14 [the same as that which must be applied to the original Hamiltonian for Eq. (2.2) to be obtained]. We shall review a basic aspect of this procedure. First of all, we must write the Fokker-Planck equation to be associated with the set of Langevin equations of interest, i.e., in the present case, Eq. (2.13) supplemented by Eq. (2.15). This reads

$$\frac{\partial}{\partial t}\rho(x,v,y;t)=\mathscr{L}\rho(x,v,y;t)\equiv\left[-v\frac{\partial}{\partial x}+[\Phi'(x)+\psi'(x)y]\frac{\partial}{\partial v}+\gamma\left[\frac{\partial}{\partial v}v+k_BT_1\frac{\partial^2}{\partial v^2}\right]+\Gamma\frac{\partial}{\partial y}y+\frac{1}{\Lambda}\psi(x)\frac{\partial}{\partial y}+\frac{k_BT_2}{\Lambda}\frac{\partial^2}{\partial y^2}\right]\rho(x,v,y;t)\ , \tag{3.1}$$

where

$$\Phi'(x)\equiv\frac{d}{dx}\Phi(x)$$

and

$$\psi'(x)\equiv\frac{d}{dx}\psi(x)\ .$$

The central step of the AEP (Refs. 12–14) is precisely to reach a decision on how to divide the operator $\mathscr{L}$ into a perturbation, $\mathscr{L}_1$, and an unperturbed part, $\mathscr{L}_0$. The final result significantly depends on this choice. In this paper we shall show that a proper choice (suggested by the physical nature of the system under study) may correspond to a resummation on the infinite terms generated by a seemingly more natural choice, determined by the mere remark that $\mathscr{L}_1$ must be defined as that part of the full operator which concerns both relevant and irrelevant variables (and only that part). This touches the current debates among several groups as to the most suitable procedure of elimination of fast variables.[16–20] It has been pointed out that to bypass resummation difficulties it is necessary to develop procedures alternative to the projection operator methods.[16–18] On the contrary, we believe that these difficulties can be overcome within the theoretical background of the AEP of Refs. 12–14, which indeed relies on the Zwanzig projection operator method,[21] provided that the preliminary problem of how to divide $\mathscr{L}$ into perturbed and unperturbed part is solved. We shall came back to this basic issue in Sec. IV.

As to the operator $\mathscr{L}$ defined by Eq. (3.1), the most natural choice would be

$$\mathscr{L}_0\equiv\Gamma\left[\frac{\partial}{\partial y}y+\frac{k_BT_2}{\Gamma\Lambda}\frac{\partial^2}{\partial y^2}\right]\ ,$$
$$\mathscr{L}_1\equiv\mathscr{L}-\mathscr{L}_0\ , \tag{3.2}$$

with the associated projection operator

$$P\rho(x,v,y;t)\equiv\rho_{\rm eq}(y)\sigma(x,v;t)\equiv\rho_{\rm eq}(y)\int dy\,\rho(x,v,y;t)\ , \tag{3.3}$$

where

$$\rho_{\rm eq}(y)\propto\exp(-\omega_0^2y^2/2k_BT_2)\ . \tag{3.3'}$$

Via the mere application of the standard rules of the AEP (Refs. 12–14), we are then led by this choice to

$$\frac{\partial}{\partial t}\sigma(x,v;t)=\left[-v\frac{\partial}{\partial x}+\gamma\left[\frac{\partial}{\partial v}v+k_BT_1\frac{\partial^2}{\partial v^2}\right]+\Phi'_{\rm eff}\frac{\partial}{\partial v}+\frac{k_BT_2}{\Gamma\omega_0^2}\frac{\partial^2}{\partial v^2}[\psi'(x)]^2\right]\sigma(x,v;t)\ , \tag{3.4}$$

where $\Phi'_{\rm eff}\equiv(d/dx)\Phi_{\rm eff}$ and $\Phi_{\rm eff}$ is precisely the effective potential defined by Eq. (2.19). This appealing feature of the AEP, which is proven to be independent of the preliminary division of $\mathscr{L}$ into $\mathscr{L}_0$ and $\mathscr{L}_1$, is unfortunately joined to the fact that the last term on the right-hand side (rhs) of Eq. (3.4) makes the steady state of the system deviate from the canonical distribution

$$\sigma_{\rm eq}(x)\propto\exp(-\Phi_{\rm eff}/k_BT_1)\ .$$

Note that in the case $T_1=T_2=T$ this must be actually attained by the system. Note that when $\psi(x)$ is given the form of Eq. (2.16) and Φ of Eq. (1.1) is identified with $\Phi_{\rm eff}$, Eq. (3.4) is no longer distinguishable from the Fokker-Planck equation associated with Eq. (1.1). This means that when $\Phi_{\rm eff}$ is assumed to be a harmonic potential, Eq. (3.4) should result in precisely those instability phenomena which have been widely studied recently by the Jolla group.[22] These instability processes do not have any plausibility in the canonical case $T_1=T_2=T$ (as it will be confirmed below by further theoretical remarks). This leads us to choose a more proper kind of division of $\mathscr{L}$ (and a corresponding kind of projection operator). We would like to note, however, that by using precisely this new kind of projection operator, in Sec. V we shall show that the condition $T_2>T_1$ makes a diffusion term appear which is typical of the pumping processes (see paper I), thereby providing a sort of justification for the fourth term on the rhs of Eq. (3.4).

Nevertheless, the energy pumping structure of Eq. (3.4) is certainly an artifact which must be avoided via a more proper division of the operator $\mathscr{L}$. Such a proper choice is precisely the same as that recently proposed by Haake,[19] which in the present case reads

$$\mathscr{L}_0\equiv\Gamma\left[\frac{\partial}{\partial y}y+\frac{k_BT_2}{\Lambda\Gamma}\frac{\partial^2}{\partial y^2}\right]+\frac{\psi(x)}{\Lambda}\frac{\partial}{\partial y}\ ,$$
$$\mathscr{L}_1\equiv\mathscr{L}-\mathscr{L}_0\ . \tag{3.5}$$

This leads us to the projection operator

$$P\rho(x,v,y;t)\equiv\rho_{\rm eq}(y\mid x)\sigma(x,v;t)\equiv\rho_{\rm eq}(y\mid x)\int dy\,\rho(x,v,y;t)\ , \tag{3.6}$$

where

$$\rho_{\rm eq}(y\mid x)\propto\exp\left[-\left[y+\frac{\psi(x)}{\omega_0^2}\right]^2\frac{\omega_0^2}{2k_BT}\right]\ . \tag{3.7}$$

The physical meaning of this choice is that, as a consequence of the back-reaction term, the equilibrium state attained by y depends also on the state of the variable of interest.

Note that this choice, rather than being a proposal of general validity, is mainly dictated by the purpose of fulfilling the canonical constraint of Eq. (2.8). As a matter of fact, in a case explicitly dealt with by Haake,[19] precisely the choice of Eq. (3.5) was shown to lead to an unwanted noncanonical equilibrium distribution.[23] Although Titulaer[20] argued that, despite this flaw, Haake's proposal in certain conditions leads to an improved description of the transient processes, we believe that preference should be given to the fulfillment of the canonical constraint. This means that in certain cases we do not share Haake's point of view.

The demonstration that the choice of Eqs. (3.5) and (3.6), in the case under study in this paper, allows the canonical constraint to be fulfilled, leads to some technical difficulties resulting precisely from the fact that $\rho_{\rm eq}(y\mid x)$ depends also on x. To bypass this difficulty, let us make the change of variables

$$\tilde{x}=x\ ,\qquad \tilde{y}=y+\psi(x)/\omega_0^2\ , \tag{3.8}$$

which means

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \tilde{x}}+\frac{\psi'(\tilde{x})}{\omega_0^2}\frac{\partial}{\partial \tilde{y}}\ ,\qquad \frac{\partial}{\partial y}=\frac{\partial}{\partial \tilde{y}}\ . \tag{3.9}$$

By replacing Eq. (3.9) for Eq. (3.5) we obtain

$$\mathscr{L}_1=[\Phi'_{\rm eff}+\psi'(x)\tilde{y}]\left[\frac{\partial}{\partial v}+\frac{v}{k_BT_1}\right]-v\left[\frac{\Phi'_{\rm eff}}{k_BT_1}+\frac{\partial}{\partial x}\right]$$
$$-v\frac{\psi'(x)}{\omega_0^2}\left[\frac{\omega_0^2\tilde{y}}{k_BT_2}+\frac{\partial}{\partial \tilde{y}}\right]+\gamma\left[\frac{\partial}{\partial v}v+k_BT_1\frac{\partial^2}{\partial v^2}\right]\ , \tag{3.10}$$
$$\mathscr{L}_0=\Gamma\left[\frac{\partial}{\partial \tilde{y}}\tilde{y}+\frac{k_BT_2}{\Lambda\Gamma}\frac{\partial^2}{\partial \tilde{y}^2}\right]\ .$$

Thus, the projection operator of Eq. (3.6) recovers a suitable form enabling us to apply the usual rules of Refs. 12–14. Then, by making a second-order perturbation calculation, we obtain

$$\frac{\partial}{\partial t}\sigma(x,v;t)=\mathscr{L}_{\rm eff}\sigma(x,v;t)$$
$$\equiv\left[-v\frac{\partial}{\partial x}+\gamma\left[\frac{\partial}{\partial v}v+k_BT_1\frac{\partial^2}{\partial v^2}\right]+\Phi'_{\rm eff}\frac{\partial}{\partial v}+\frac{\langle y^2\rangle_{\rm eq}}{\Gamma}[\psi'(x)]^2\frac{\partial}{\partial v}\left[\frac{\partial}{\partial v}+\frac{v}{k_BT_2}\right]\right]\sigma(x,v;t)\ , \tag{3.11}$$

where

$$\langle y^2\rangle_{\rm eq}=k_BT_2/\omega_0^2\ . \tag{3.12}$$

It can be easily checked that when $T_1=T_2=T$, Eq. (3.11) attains the equilibrium distribution

$$\sigma_{\rm eq}(x,v;\infty)\propto e^{-v^2/2k_BT}e^{-\Phi_{\rm eff}(x)/k_BT}\ , \tag{3.13}$$

which satisfies indeed the canonical constraint of Eq. (2.8).

Let us assume $\Phi_{\rm eff}(x)$ to have the form

$$\Phi_{\rm eff}(x)=(-1)^s\tfrac{1}{2}(\omega_0')^2x^2+G(x)\quad(s=0,1)\ , \tag{3.14}$$

thus being characterized by a "harmonic" $(-1)^s\frac{1}{2}(\omega_0')^2x^2$, and anharmonic part, $G(x)$. When $\psi(x)$ is given by Eq. (2.16), the Langevin equation corresponding to Eq. (3.11) reads

$$\dot{x}=v\ ,$$
$$\dot{v}=-[(-1)^s(\omega_0')^2+\eta(t)]x-\gamma v-\lambda x^2v-G'(x)+f(t)\ , \tag{3.15}$$

where $\eta(t)$ and $f(t)$ are uncorrelated white noises defined by

$$\langle f(0)f(t)\rangle=2\gamma k_BT_1\delta(t)\ , \tag{3.16}$$

$$\langle\eta(0)\eta(t)\rangle=2\lambda k_BT_2\delta(t)\ , \tag{3.17}$$

and the parameter λ is defined by

$$\lambda\equiv\frac{\alpha^2}{\Gamma}\frac{1}{\omega_0^2}\ . \tag{3.18}$$

$f(t)$ coincides with the stochastic force appearing in the second equation of the set of Eq. (2.3), whereas $\eta(t)$ simulates the fluctuation of the potential driving the Brownian particle of interest which depends on the coupling with the fast harmonic oscillator.

Let us consider for a while the case $s=0$. At $T_1=T_2=T$, Eq. (3.15) is precisely of the same kind as that derived from the Zwanzig Hamiltonian by Lindenberg and Seshadri.[7] The only minor difference is that the stochastic forces $f(t)$ and $\eta(t)$ are uncorrelated with each other. This is so because these are to simulate two distinct thermal baths. Lindenberg and Seshadri,[7] on the contrary, derived also a xv term, which expresses the effect of the interference between $\eta(t)$ and $f(t)$, noises resulting indeed in Ref. 7 from one and the same thermal bath.

This appearing result means that the canonical constraint forces a unique kind of equation to appear, no matter which is the true "microscopic" Hamiltonian behind the "macroscopic phenomenon" we are interested in. The microscopic system described by Eq. (2.1) has, in fact, a structure completely different from that of the Zwanzig Hamiltonian.[6]

This is by itself a result of some interest since it is related in some way to the problem (i) posed in the Introduction. The wide (and continuous) set of results between the Itô and Stratonovich limit is allowed by a lack of infor-

mation (see paper I). In a sense, at $T_1=T_2=T$ the basic information provided by Eq. (2.8) removes completely any uncertainty: as a further significant effect of this, in the next sections we will show that the diffusion equation, leading the motion of x when this is the slowest variable of the system, is characterized by only one possible form.

As already done by Lindenberg and Seshadri,[7] it is tempting to replace the term $-\lambda x^2 v$ appearing in Eq. (3.15) with $-\lambda\langle x^2\rangle_{eq}v$. This leads immediately to stating that, as an effect of the coupling with the fast oscillator, the effective friction affecting the Browian particle of interest increases. In the case $G=0$ and $s=0$ Lindenberg and Seshardi[7] supported that statement via rigorous arguments based on the Stratonovich energy envelope method.[24] This means therefore that these are bounded by the physical condition required for significant inertial properties to be exhibited, i.e.,

$$\gamma+\lambda\frac{k_BT}{(\omega_0')^2}\ll\omega_0' . \tag{3.19}$$

Via a joint use of theory, computer calculation, and analog experiment in the remainder of this section, we shall address the question of whether or not this mean-field approximation can also apply when the relationship of Eq. (3.19) is not fulfilled. In the next section, furthermore, via purely theoretical arguments, we shall account for the deviations from the predictions of this mean-field approximation.

Upon increasing the coupling strength or the temperature T, the condition

$$\gamma+\lambda\frac{k_BT}{(\omega_0')^2}\sim 2\omega_0' \tag{3.20}$$

can be attained, at which a transition from the inertial to overdamped regime will be shown to take place. Note that the relaxation rate of x, which will be denoted by k, will be shown to be different from

$$k=(\omega_0')^2/[\gamma+\lambda k_BT/(\omega_0')^2] , \tag{3.21}$$

which would imply the mean-field approximation $-\lambda\langle x\rangle_{eq}v$ to be valid also in the overdamped regime. Nonetheless, such a transition from the inertial to overdamped regime does actually take place. This interesting effect is precisely the reverse of that in the problem (ii) addressed in Sec. I.

Note that the computer calculation of subsection III B refers to the system of Eq. (3.11) (when the condition $T_1=T_2=T$ is attained). Therefore, the corresponding results cannot be thought of as being a check on the elimination procedure which led us from Eq. (2.13) to Eq. (3.11). They only serve the purpose of shedding light on the transition from the inertial to the overdamped regime.

On the contrary, the analog experiment of subsection III C simulates the complete set of equations (2.12) [supplemented by Eqs. (2.15) and (2.16)]. Therefore, the corresponding results can be though of as a reliable check on the whole approach leading from Eqs. (2.12) to Eq. (3.11). For technical reasons it was not possible to explore the overdamped regime in this manner.

We can therefore conclude this subsection by saying that the analog experiment of subsection III C mainly provides a check on Eq. (3.11), while the computer calculation of the next subsection, in a sense, is a check on the diffusion equation which will be derived in Sec. IV. Both subsections B and C will concern the case defined by $s=0$, $G(x)=0$, $T_1=T_2\equiv T$.

B. Computer calculation

In this subsection we want to test the reliability of the mean-field approximation which consists of replacing the term $-\lambda x^2 v$ in Eq. (3.15) with $-\lambda\langle x^2\rangle_{eq}v$. This means that for $G=0$ and $s=0$, the normalized equilibrium correlation function of x, $\Phi_x(t)$, is

$$\Phi_x(t)\equiv[\langle xx(t)\rangle_{eq}-\langle x\rangle_{eq}^2]/(\langle x^2\rangle_{eq}-\langle x\rangle_{eq}^2)$$
$$=A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t} , \tag{3.22a}$$

where

$$A_1\equiv 1-A_2=\lambda_2/(\lambda_1-\lambda_2) , \tag{3.22b}$$

$$\lambda_i=\{-\gamma_0+(-1)^i[\gamma_0^2-4(\omega_0')^2]^{1/2}\}/2 \quad (i=1,2) . \tag{3.22c}$$

$$\gamma_0=\gamma+\lambda k_BT/(\omega_0')^2 . \tag{3.22d}$$

In order to test Eqs. (3.22) we calculate the "exact" value of $\Phi_x(t)$ by using the continued fraction procedures (CFP).[25] The version used in the present paper consists of evaluating the Mori states over which the Fokker-Planck operator of Eq. (3.11) must be expanded and determining the expansion parameters of the continued fraction directly from these states. Notice that in this case Eq. (3.22a) to Eq. (3.22d) can be obtained by limiting the expansion of the continued fraction to the first two Mori states.

Figure 1 shows the correlation function $\Phi_x(t)$ at different values of the thermal energy. These results have been obtained by using 14 Mori states (i.e., 14 steps of the continued fraction). The estimated convergence of the continued fraction is better than 5%. The values of the parameters of Eq. (3.11) have been chosen in such a way that the transition from the oscillatory to the overdamped regime is expected to take place at $k_BT=3.9\times10^6$. This prediction is in a satisfactory agreement with the computational results of Fig. 1.

However, we note that a slight quantitative difference is found when one compares the exact time dependence of $\Phi_x(t)$ obtained with a large number of Mori states ($n>14$) with that predicted by the mean-field approximation ($n=2$). This difference increases as the temperature of the thermal bath is increased. Figure 2 shows an example of the time dependence of $\Phi_x(t)$ at different values of the number of Mori states. The temperature has been chosen slightly exceeding the value of the transition to the overdamped regime. Computer calculation at $n>2$ leaves virtually unchanged the analytical form of $\Phi_x(t)$ [Eqs. (3.22a) and (3.22b)] while slightly affecting the actual values of λ_1 and λ_2 which thus exhibit small deviations from the predictions of Eq. (3.22d). This is shown by Fig. 3, which illustrates the dependence on temperature of the frequency and damping of this effective damped oscilla-

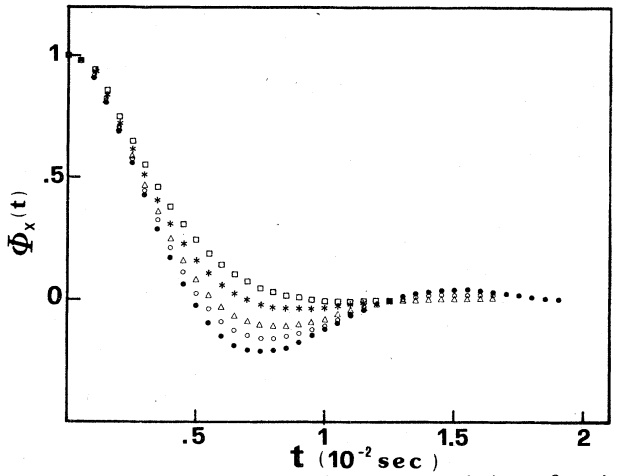

FIG. 1. Normalized stationary correlation function $\Phi_x(t)=[\langle xx(t)\rangle_{eq}-\langle x\rangle^2_{eq}]/(\langle x^2\rangle_{eq}-\langle x\rangle^2_{eq})$ at different values of the thermal energy ($T_1=T_2\equiv T$): $k_BT=10^5$ erg (●), $k_BT=5\times10^5$ erg (○), $K_BT=10^6$ erg (△), $k_BT=2\times10^6$ erg (*), and $k_BT=4\times10^6$ erg (□). $\Phi_x(t)$ is obtained by using 14 Mori states of the continued fraction of Ref. 25. The calculation relies on the Fokker-Planck operator of Eq. (3.11) with Φ_{eff} given by Eq. (3.14) with $G=0$ and $s=0$. The parameters of Eq. (3.11) are $\gamma=400$ sec^{-1}, $\omega_0'=460$ sec^{-1}, and $\lambda=29.4$ (sec$\times$cm^2)$^{-1}$ [see Eq. (3.18)]. A transition to the overdamped regime is expected at $k_BT=3.9\times10^6$ erg.

tor. Solid circles and triangles correspond to the exact behavior obtained with 15 Mori states, while the dashed and solid lines correspond to the predictions of the mean-field approximation. We point out that in the temperature range of Fig. 3 the exact and the approximate results are very close. Unfortunately, in this regime the CFP algorithm becomes less stable as the temperature increases. Therefore we cannot investigate the validity of Eq. (3.22a) for $k_BT>2\times10^6$.

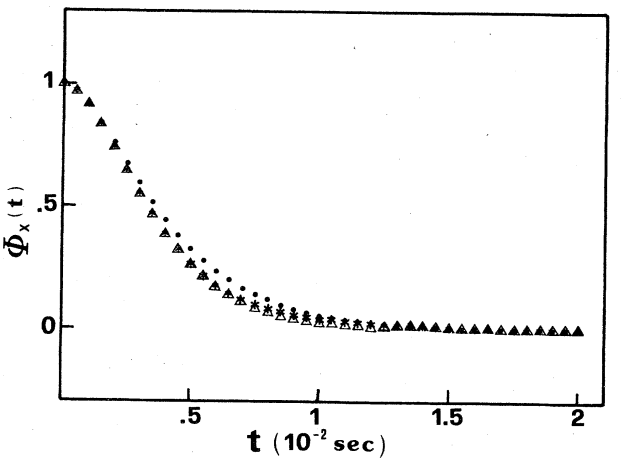

FIG. 2. Dependence of the calculated $\Phi_x(t)$ on the number of Mori states of the continued fraction: ● (2 states), * (8 states), and △ (12 states). The parameters of Eq. (3.11) are the same of Fig. 1, while the thermal energy $k_BT=4\times10^6$ erg ($T_1=T_2\equiv T$) has been chosen slightly exceeding the threshold of the transition to the overdamped regime ($k_BT=3.9\times10^6$ erg).

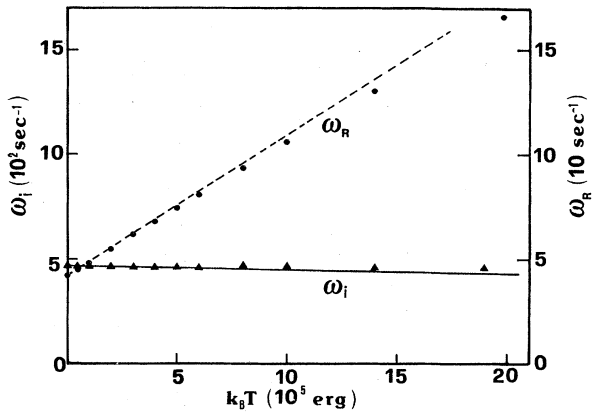

FIG. 3. Frequency and damping of the effective oscillator [Eqs. (3.22a) and (3.22b)] as a function of the thermal energy k_BT. ω_R is defined via $\omega_R\equiv-\mathrm{Re}\lambda_2$, ω_i is defined via $\omega_i\equiv\mathrm{Im}\lambda_2$. The parameters of Eq. (3.15) are $\gamma=83.3$ sec^{-1}, $\omega_0'=460$ sec^{-1}, and $\lambda=29.4$ (sec$\times$cm^2)$^{-1}$. At $T=0$ these values correspond to a fully inertial regime. The dashed and solid lines correspond to the mean-field approximation of Eq. (3.22a), while the solid circles and triangles correspond to the result of the CFP calculation with 15 Mori states. In this latter case we report only the decay rate of the eigenstate with largest weight.

Figure 4 shows the decay time $T_0\equiv\int_0^\infty\Phi_x(t)dt$ in the overdamped regime versus the thermal energy. In this case, also, the mean-field results (dashed line) approximate with satisfactory precision the exact ones (dots in Fig. 4) in the low-temperature range ($k_BT<5\times10^6$). However, a large discrepancy between these two results is obtained at higher temperatures ($k_BT>5\times10^6$). The solid line in Fig. 4 represents the theoretical prediction of a simple mean-field calculation which will be discussed in the fol-

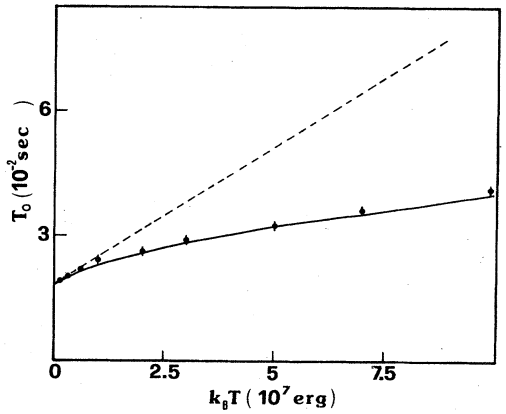

FIG. 4. Relaxation rate of the normalized stationary correlation function $\Phi_x(t)$ vs the thermal average energy k_BT. The parameters of Eq. (3.15) are $\gamma=400$ sec^{-1}, $\omega_0'=460$ sec^{-1}, and $\lambda=29.4$ (sec$\times$cm^2)$^{-1}$. These values correspond to a fully overdamped regime even at $T=0$. The dashed line corresponds to the mean-field prediction [Eq. (3.22a)], while dots correspond to the result obtained by using 15 Mori states. The vertical error bars indicate the estimated error of convergence of the continued fraction (5% in the high-temperature regime). The solid line corresponds to the predictions of Eq. (4.9).

lowing section [see Eq. (4.7)]. This calculation is based on a diffusion equation for the slow variable x which will be obtained in Sec. IV. The good agreement between the dots and the solid line in Fig. 4 can be, therefore, considered as a check on the adiabatic elimination procedure of Sec. IV.

C. Analog experiment

We have simulated Eqs. (2.12) by means of two electric oscillators and multiplier devices.[26] Figure 5 illustrates the corresponding electric configuration. A nonlinear oscillator (x) is coupled via two multipliers (M_1,M_2) to an overdamped oscillator (y). The strength of the coupling is given a proper value so as to result in the effective harmonic potential of Eq. (2.18). Each of the two oscillators is coupled to a white and Gaussian noise via independent lines. The amplitudes of the two noises are regulated in such a way as to ensure the physical condition described in Sec. II, i.e., the canonical equilibrium. Note that switching off this coupling does not change this condition. Details on noise generators and nonlinear oscillators can be found in paper I. The x output was connected to the input of an autocorrelator for the evaluation of the stationary correlation function. The value of the thermal energy k_BT is obtained by means of a direct measurement of $\langle v^2\rangle_{eq}=k_BT$: indeed the output corresponding to v was sent to an analog multiplier (M_3) and then to a low-pass filter.

The results of this experiment are illustrated by Fig. 6. A good agreement with the theoretical predictions is obtained. Note that the experimental curve bends down slightly below the theoretical curve resulting from the approximation $\gamma_{eff}=\gamma+\lambda\langle x^2\rangle_{eq}$, in agreement with the results of the computer calculation of the preceding subsection (Fig. 3). For technical reasons we could not explore via analog simulation the overdamped regime involved in Fig. 4.

IV. DIFFUSION EQUATION FOR THE VARIABLE X ($T_1=T_2$)

In this section we shall focus our attention on the physical condition:

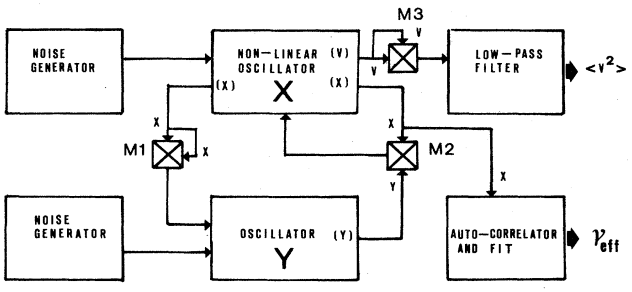

FIG. 5. Scheme of the analog circuit used to simulate Eq. (2.12).

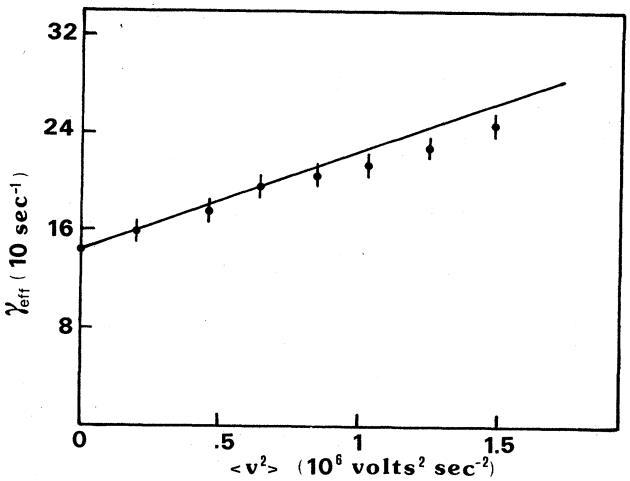

FIG. 6. Effective friction coefficient (γ_{eff}) as a function of the thermal energy k_BT in the case $T_1=T_2\equiv T$. The experimental results (circles) are compared to the theoretical prediction of the mean-field approximation [Eq. (3.19)]. The parameters of the system are $\omega_0'=465$ sec^{-1}, $\omega_0=10\omega_0'$, $\gamma=0.316\omega_0'$, and $\Lambda=3.16\omega_0$.

$$T=T_1=T_2\ . \tag{4.1}$$

Case (b): $1/\omega_0' \gg 1/\gamma \gg 1/\Gamma$ can be dealt with starting from Eq. (3.11) itself. The corresponding operator $\mathscr{L}_{eff}$ can be divided into unperturbed and perturbation part as follows

$$\mathscr{L}_0\equiv\gamma\left[\frac{\partial}{\partial v}v+k_BT\frac{\partial^2}{\partial v^2}\right], \tag{4.2}$$

$$\mathscr{L}_1\equiv\mathscr{L}-\mathscr{L}_0\ .$$

A further possible choice is

$$\mathscr{L}_0\equiv\gamma\left[\frac{\partial}{\partial v}v+k_BT\frac{\partial^2}{\partial v^2}\right]+\frac{\langle y^2\rangle_{eq}}{\Gamma}\psi'(x)^2\frac{\partial}{\partial v}\left[\frac{\partial}{\partial v}+\frac{v}{k_BT}\right], \tag{4.3}$$

$$\mathscr{L}_1\equiv\mathscr{L}-\mathscr{L}_0\ . \tag{4.3'}$$

Although $\mathscr{L}_0$ of the latter choice depends also on x, the corresponding equilibrium distribution of the "irrelevant" variable would be the same as that of the former choice. In other words, in both cases the projection operator P reads

$$P\sigma(x,v;t)\equiv\sigma_{eq}(v)\varphi(x;t)\equiv\sigma_{eq}(v)\int dv\,\sigma(x,v;t)\ , \tag{4.4}$$

where

$$\sigma_{eq}(v)\propto\exp(-v^2/2k_BT)\ . \tag{4.5}$$

The fact that $\mathscr{L}_0$ of Eq. (4.3) depends also on x only implies some minor technical difficulties which, however, when limiting ourselves to a second-order calculation, can be straightforwardly overcome, and do not deserve to be illustrated.

The choice of Eq. (4.2) results in

$$\frac{\partial}{\partial t}\varphi(x;t)=\frac{\partial}{\partial x}\left[\frac{k_BT}{\gamma}-\frac{k_BT}{\omega_0^2\Gamma\gamma^2}[\psi'(x)]^2\right]\times\left[\frac{\partial}{\partial x}+\frac{\Phi'_{\rm eff}}{k_BT}\right]\varphi(x;t)\ . \tag{4.6}$$

The choice of Eq. (4.3) leads us to

$$\frac{\partial}{\partial t}\varphi(x;t)=\frac{\partial}{\partial x}\frac{k_BT}{\gamma+[\psi'(x)]^2/\omega_0^2\Gamma}\left[\frac{\partial}{\partial x}+\frac{\Phi'_{\rm eff}}{k_BT}\right]\varphi(x;t)\ . \tag{4.7}$$

The direct comparison between Eqs. (4.6) and (4.7) immediately leads us to some interesting conclusions.

First of all we see that Eq. (4.6) exhibits a diffusion coefficient which seems to be obtained from the second one via an expansion into a Taylor series of $[1+[\psi'(x)]^2/\omega_0^2\gamma\Gamma]^{-1}$. This touches again the problem mentioned in Sec. II. Lugiato,[16] for instance, developed an approach, alternative to the projection methods, the aim of which was to solve the problem of resummating over infinite terms. Our point of view is that the solution to this intriguing problem can also be found within the context of the projection methods themselves, provided that a clear understanding of the physics behind the system under study is reached. The inclusion of the second term on the rhs of Eq. (4.3) into the perturbation part is quite natural from a purely mathematical point of view, because this term involves both the variable x and the variable v (this is precisely what an interaction term is formally meant to be). However, as widely discussed in the preceding section, this term plays the major role of a standard diffusion operator for a variable velocity, the friction of which depends on x. When the temperature is sufficiently large this term cannot be regarded as a small perturbation term and must be included into the unperturbed part.

Now we are also in a position to settle a question left not completely answered by the results of the preceding section, i.e., how to account for the deviations from the mean-field approximation leading to the effective damping $\gamma_{\rm eff}$ defined by

$$\gamma_{\rm eff}\equiv\gamma+\lambda\langle x^2\rangle_{\rm eq}\ . \tag{4.8}$$

In the region $\gamma_{\rm eff}>2\omega_0'$, to which Eq. (4.7) refers, the same mean-field criterion as that leading from Eq. (3.11) to Eq. (4.8), when applied to Eq. (4.7) would produce the relaxation rate

$$k_1=\left\langle\frac{(\omega_0')^2}{\gamma+\lambda x^2}\right\rangle_{\rm eq}\ , \tag{4.9}$$

whereas Eq. (4.8) would predict the rate

$$k_2=(\omega_0')^2/(\gamma+\lambda\langle x^2\rangle_{\rm eq})\ . \tag{4.10}$$

Let us consider the case $\gamma\gg2\omega_0'$. When this condition applies, we can study cases characterized by a temperature T so small as to make it legitimate to expand both Eqs. (4.9) and (4.10) into a Taylor series. We thus obtain

$$k_1=\frac{(\omega_0')^2}{\gamma}\left[1-\frac{\lambda}{\gamma}\langle x^2\rangle_{\rm eq}+\frac{\lambda^2}{\gamma^2}\langle x^4\rangle_{\rm eq}+\cdots\right]\ , \tag{4.11}$$

$$k_2=\frac{(\omega_0')^2}{\gamma}\left[1-\frac{\lambda}{\gamma}\langle x^2\rangle_{\rm eq}+\frac{\lambda^2}{\gamma^2}\langle x^2\rangle_{\rm eq}^2+\cdots\right]\ . \tag{4.12}$$

In the case $\Phi'_{\rm eff}=\frac{1}{2}(\omega_0')^2x^2$, to which the computer calculation of the preceding section refers, the canonical equilibrium distribution is Gaussian, thereby resulting in $\langle x^4\rangle_{\rm eq}=3\langle x^2\rangle_{\rm eq}^2$. This means that both k_1 and k_2 decrease upon the temperature, and the decrease of k_1 is slower than that of k_2. Notice that the exact calculation of the relaxation time $(1/k_1)$ as given by Eq. (4.9) allows us to obtain the solid line in Fig. 4. A virtually perfect agreement with the relaxation rate calculated by using the CFP of Sec. III B is obtained (dots in Fig. 4). This definitely settles the question left unanswered by the preceding section.

Let us address now case (c): $1/\omega_0'\gg1/\Gamma\gg1/\gamma$. We are thus allowed to replace Eq. (2.13) [supplemented by Eq. (2.15)] with

$$\dot{x}=-\frac{\Phi'(x)}{\gamma}-\frac{\psi'(x)}{\gamma}y+\frac{f(t)}{\gamma}\ , \tag{4.13}$$

$$\dot{y}=-\frac{\psi(x)}{\Lambda}-\frac{\omega_0^2}{\Lambda}y+\frac{f'(t)}{\Lambda}\ , \tag{4.13'}$$

while keeping y as being much faster than x. After writing the Fokker-Planck equation corresponding to this set of Langevin equations, we use the same change of variables as that of Eq. (3.8). The unperturbed operator $\mathscr{L}_0$ is then the same as that of Eq. (3.10). Straightforward use of the AEP of Refs. 12–14 leads again to Eq. (4.6). This interesting result answers the question (i) of the Introduction: The diffusion equation for the variable x is characterized by a unique form, regardless of whether this is obtained by assuming v to be much faster than y, or vice versa. Paper I shows that this is no longer true when the Brownian particle is kept far from canonical equilibrium precisely by the interaction itself with the fast variable y and no back-reaction term is present.

A further check on this interesting result is obtained by studying case (d): $1/\omega_0'\gg1/\Gamma\sim1/\gamma$. In this case we must have direct recourse to Eq. (2.13) supplemented by Eq. (2.15). Then we use again the change of variables of Eqs. (3.8) and the unperturbed operator of Eq. (3.10). Unfortunately, to obtain results comparable with the preceding ones we are now obliged to take into account both second- and fourth-order perturbations terms (in the corresponding interaction $\mathscr{L}_1$), whereas the preceding results relied on a simpler second-order calculation. Although tedious, this calculation does not involve any difficulty and leads us to

$$\frac{\partial}{\partial t}\varphi(x;t)=\frac{\partial}{\partial x}\left[\frac{k_BT}{\gamma}-\frac{k_BT}{\omega_0^2\gamma^2\Gamma}[\psi'(x)]^2\right]\times\left[\frac{\partial}{\partial x}+\frac{\Phi'_{\rm eff}}{k_BT}\right]\varphi(x;t)+\frac{k_BT}{\gamma^3}\Phi''_{\rm eff}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}+\frac{\Phi'_{\rm eff}}{k_BT}\right]\varphi(x;t)\ . \tag{4.14}$$

The only remarkable difference with respect to Eq. (4.6) is the appearance of the second term on the rhs of Eq. (4.14), i.e., a contribution of higher order in the perturbation parameter $1/\gamma$ which has nothing to do with the corrections coming from the interaction between x and y.

V. ON THE APPEARANCE OF NONCANONICAL EFFECTS ($T_2 > T_1$)

In a preceding short paper[27] it has been shown that Eq. (3.11) results in a sort of noise-induced phase transition, which was checked via analog experiment. This phenomenon is as follows: at $T_1=0$ the slow oscillator is characterized by $\langle x^2\rangle_{SS}=0$ ($\langle x^2\rangle_{SS}$ denotes the steady-state mean value of x^2) for an interval of the temperature T_2 ranging from $T_2=0$ to $T_2=T_c$. At a threshold value T_c a new regime appears where $\langle x^2\rangle_{SS}$ increases as a linear function of T_2.

The major conclusion of Ref. 26 was that this phenomenon is related in some way with those explored in paper I. In this section we shall support via theoretical arguments this statement.

When applying the same calculation as that concerning case (d) of the preceding section to the case $T_2 > T_1$, we obtain

$$\frac{\partial}{\partial t}\varphi(x;t)=\left[\frac{k_BT_1}{\gamma}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+\frac{\Phi'_{\text{eff}}}{k_BT_1}\right)-\frac{k_BT_1}{\omega_0^2\gamma^2\Gamma}\frac{\partial}{\partial x}\psi'^2\left(\frac{\partial}{\partial x}+\frac{\Phi'_{\text{eff}}}{k_BT_1}\right)+\frac{k_BT_1}{\gamma^3}\Phi''_{\text{eff}}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+\frac{\Phi'_{\text{eff}}}{k_BT_1}\right)\right]\varphi(x;t)$$
$$+\frac{k_BT_2}{\omega_0^2}\left[\frac{1}{\gamma\Gamma(\gamma+\Gamma)}\frac{\partial}{\partial x}\psi'(x)\frac{\partial}{\partial x}\psi'(x)+\frac{1}{\gamma^2(\gamma+\Gamma)}\frac{\partial^2}{\partial x^2}[\psi'(x)]^2\right]\left[1-\frac{T_1}{T_2}\right]\varphi(x;t)\,. \tag{5.1}$$

It is worth remarking that the condition $T_2 > T_1$ produces a correction term which is precisely the same diffusion term as that describing the coupling between x and y when no back-reaction term is present (see paper I). As pointed out in I, the form of this correction term changes from the Itô to the Stratonovich one with increasing the parameter $R=\gamma/\Gamma$ from $R=0$ to $R=\infty$. The remarks made in the preceding section on how to make a complete resummation on the "canonical" perturbation series leads us to replace Eq. (5.1) with

$$\frac{\partial}{\partial t}\varphi(x;t)=\left[\frac{\partial}{\partial x}\frac{k_BT_1}{\gamma+[\psi'(x)]^2/\omega_0^2\gamma\Gamma}\left(\frac{\partial}{\partial x}+\frac{\Phi'_{\text{eff}}}{k_BT_1}\right)\right]\varphi(x;t)$$
$$+\left[\frac{k_BT_2}{\omega_0^2}\left[\frac{1}{\gamma\Gamma(\gamma+\Gamma)}\frac{\partial}{\partial x}\psi'(x)\frac{\partial}{\partial x}\psi'(x)+\frac{1}{\gamma^2(\gamma+\Gamma)}\frac{\partial^2}{\partial x^2}[\psi'(x)]^2\right]\left[1-\frac{T_1}{T_2}\right]\right]\varphi(x;t) \tag{5.2}$$

[the third term within the first pair of large square brackets of Eq. (5.1) has been neglected]. This is a quite general diffusion equation consistently of both a canonical [first term on the rhs of Eq. (5.1)] and noncanonical [second term on the rhs of Eq. (5.2)] contribution.

The reliability of the first term is supported by the computer calculation and analog experiment of the present paper. On the other hand, paper I was devoted to check a noncanonical term with the same form as the second term on the rhs of Eq. (5.1). It was shown that the validity of this description is limited to the low-temperature region. With increasing the temperature, a transition from the overamped to the inertial regime may indeed take place, which completely invalidates the assumptions of the AEP leading to this equation. This transition is now in part counterbalanced by the canonical term.

To make this aspect clear, let us consider the case when Φ_{eff} is given the same double-well potential shape as that of the bare potential of paper I ($s=1$), and $\psi(x)$ is given by Eq. (2.16). Note that the breakdown of the AEP takes place for values of the temperature of the fast oscillator so strong as to oblige the Brownian particle to explore the regions of extremely high potential energy and, therefore, characterized by extremely large effective frequencies. When this condition is attained by the system of Eq. (5.2), however, the canonical term produces a relaxation rate, the intensity of which being proportional to

$$\langle\{\gamma+[\psi'(x)]^2/\omega_0^2\gamma\Gamma\}^{-1}\rangle\,,$$

should become increasingly weaker with increasing T_2-T_1. This, of course, should not prevent the breakdown of the AEP from ultimately taking place.

In the low-temperature region of the fast oscillator, on the contrary, paper I shows that the steady-state distribution is characterized by two Gaussian-like curves, originally centered at the bottom of the two wells, which, with increasing $\langle y^2\rangle_{\text{eq}}$ [see Eq. (1.1)] shift towards the top of the barrier and, at a certain threshold, merge into a distribution with a single peak at the top of the barrier itself. Throughout this process the relaxation time of x was shown to increase with increasing $\langle y^2\rangle_{\text{eq}}$. Even this process is now counterbalanced by an effect of opposite nature resulting from the canonical contribution (first term on the rhs of Eq. (5.2). Indeed, when the two Gaussian-like distributions shift towards the top of the barrier the weight of

$$\{\gamma+[\psi'(x)]^2/\omega_0^2\gamma\Gamma\}^{-1}$$

should increase. For extremely large values of γ, however, the net effect should be the same as that detected in paper I.

VI. CONCLUDING REMARKS

As to problem (i) in the Introduction, we proved that the canonical constraint restricts the wide range of dif-

fusional equations (from the Itô to the Stratonovich form) to a unique kind of equation.

As to problem (ii) raised there, we showed that the canonical constraint makes it possible to produce a phenomenon of nature opposite to that monitored in paper I: rather than a transition from the overdamped to the inertial regime, with increasing the thermal energy of the fast variable y a transition from the inertial to the overdamped case is produced [in the case where $\Phi'_{\text{eff}}(x)$ is linear].

This extends the former results of Lindenberg and Seshadri[7] who limited their investigation to the inertial case. The overdamped regime, including that induced by the coupling between the Brownian particle and y, was explored analytically and accounted for completely via the AEP.

We also showed that the problem of resummating infinite perturbation contributions can be solved within the context of projection procedures, provided that physical rather than mathematical arguments are used.

ACKNOWLEDGMENTS

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