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Soliton propagation in nonuniform media

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The inverse-spectral-transform method of solution is shown to be applicable to the physically interesting problem of the nonlinear Schrödinger equation with a general "potential" term, $iq_t+q_{xx}+2[\,|q|^2-F(x)]q=0$. The method determines the class of solutions that are symmetric or antisymmetric in x. This is done with the help of a modification of the Ablowitz-Kaup-Newell-Segur and Zakharov-Shabat (AKNS-ZS) formalism incorporating an x- and t-dependent eigenvalue parameter ζ, together with a transformation of variables. In certain physical applications, $F(x)$ describes the inhomogeneity of the medium in which nonlinear wave propagation occurs. The functions $F(x)$ for which the equation is amenable to solution by our method are shown to fall into two classes, depending on whether or not ζ is explicitly t dependent. If it is, we show that $F(x)$ must be a general quadratic function of x. An explicit solution $q(x,t)$ is written down and interpreted for a parabolic potential barrier. If ζ is independent of t, we find that localized solutions with static envelopes can exist for certain other functional forms of $F(x)$. Finally, we comment on the extension of the analysis to explicitly time-dependent potentials or inhomogeneities $F(x,t)$.

I. INTRODUCTION AND SUMMARY

The method of inverse spectral transforms[1] (IST) has been used frequently in recent years to analyze the initial-value problem for a variety of nonlinear evolution equations. Starting with the general second-order scattering problem of Zakharov and Shabat,[2] Ablowitz, Kaup, Newell, and Segur[3] have developed the AKNS-ZS formalism to generate a class of equations solvable by IST. Such equations are obtained by (i) requiring that the eigenvalue ζ of the scattering problem be independent of x and t, and (ii) making the ansatz that the AKNS coefficients A, B, and C describing the time evolution of the scattering process are polynomials in ζ (or $1/\zeta$). Physical examples of these evolution equations are provided by the propagation of nonlinear waves in uniform media. If the medium is made inhomogeneous, the coefficients in the equations may be expected to become dependent on the function describing the inhomogeneity and on its derivatives. Not much work has been done on such nonlinear evolution equations. We have described elsewhere[4] one such example in magnetism, where the following generalized nonlinear Schrödinger equation (NLSE) occurs in the context of the inhomogeneous, classical Heisenberg chain:

$$iq_t+(fq)_{xx}+2fq\,|q|^2+2q\int_{-\infty}^{x} f_x\,|q|^2\,dx=0\ . \qquad (1.1)$$

The function $f(x)$ arises from the site dependence or inhomogeneity of the coupling between the spins. Setting $f=1$ yields the conventional NLSE corresponding to the homogeneous Heisenberg chain. We have shown[5] that to solve Eq. (1.1) for a general $f(x)$ by means of the IST, one must relax the conventional requirements (i) and (ii) listed above and develop an extension of the AKNS-ZS formalism. The detailed results obtained[5,6] are strongly dependent on the precise structure of Eq. (1.1) and are therefore specific to that equation.

In view of this, a natural and interesting question that has been posed is whether our extended formalism can be used to carry out the inverse-spectral-transform analysis of the following modified NLSE, whose structure lends it physical relevance of considerable generality:

$$iq_t+q_{xx}+2[\,|q|^2-F(x)]q=0\ . \qquad (1.2)$$

Equation (1.2) describes, for instance, the propagation of envelope solitons in inhomogeneous media—an example being that of electromagnetic waves in an inhomogeneous plasma.[7] $F(x)=0$ corresponds to a homogeneous plasma. It is well known[2] that this case supports a soliton traveling with a constant velocity. The presence of $F(x)$ has the effect of introducing a potential barrier (or well) in the path of the soliton. The equation is also relevant in the context of Davydov's alpha-helix solitons[8] which are responsible for energy transport along molecular chains. $F(x)$ would then represent inhomogeneities in the arrangement of molecules along the chain. Finally, of course, there is the literature[9] on the possibility that the NLSE rather than the usual Schrödinger equation is relevant to quantum mechanics. It would then be of obvious interest to study its solution in the presence of the potential $F(x)$. Some work already exists on the solution of Eq. (1.2) under restricted circumstances. Chen and Liu[10] have shown that, if $F(x)$ is a linear function of x, then the IST is applicable and soliton solutions exist. Newell[11] has studied the equation perturbatively, when $F(x)$ has a small quadratic term. A reduction of the equation based on a transformation of variables, once again for the case of a quadratic $F(x)$, has recently been suggested by Herrera.[12]

The approach we adopt in this paper, in contrast, directly considers the case of an arbitrary function $F(x)$ in Eq. (1.2) and proceeds to determine the solutions $q(x,t)$ that are either even or odd in x, by means of the IST—

including, in some detail, the case of a quadratic function $F(x)$. (We shall finally even allow for some explicit *time* dependence in the coefficients of the quadratic.) Applying our extension of the AKNS-ZS formalism to this equation, we show first that the presence of $F(x)$ makes the eigenvalue parameter ζ occurring in the corresponding Zakharov-Shabat (ZS) problem [Eqs. (2.1) below] develop a dependence on both x and t: in fact, it is found that $\zeta(x,t)$ itself satisfies a certain nonlinear evolution equation involving $F(x)$. It turns out that only separable solutions of the latter equation, of the form $\zeta(x,t)=g(x)h(t)$, are relevant because they lead to a solvable Zakharov-Shabat problem in a transformed variable. This solvability criterion (which is crucial in any IST analysis) in turn classifies into two categories the functions $F(x)$ for which Eq. (1.2) is amenable to solution by the method under discussion. These categories correspond respectively to the cases $\zeta_t \neq 0$ and $\zeta_t=0$. The general quadratic form for $F(x)$ is shown to emerge as the sole member of the first of these. An explicit solution for $q(x,t)$ is written down for the case $F(x)=-\frac{1}{2}L^2x^2$. This corresponds to the physically interesting problem of an envelope soliton in a parabolic potential barrier. The solution is found to be an antisymmetric function of x, vanishing as $|x| \to \infty$. It has one maximum and one minimum, and these decrease in magnitude and simultaneously move away from each other as time progresses. In a physical context, these "lumps" can be long-lived entities when the parameters in the problem have appropriate numerical values. Examining the second of the categories listed above, we find that localized solutions with static envelopes can exist for some other functional forms of $F(x)$, provided $F(x)$ satisfies certain asymptotic conditions to be derived in the course of our analysis. We close with some remarks on the extension of our analysis to time-dependent "potentials" $F(x,t)$.

II. EIGENVALUE EVOLUTION

To solve Eq. (1.2) by the IST method, we first reduce it to the usual AKNS-ZS form[2,3]

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_x = \begin{bmatrix} -i\zeta & q \\ -q^* & i\zeta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_t = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{2.1}$$

The conventional formalism assumes $\zeta=$const. Allowing[5] for the possibility $\zeta=\zeta(x,t)$ leads to the constraints

$$\begin{aligned} &A_x - qC - q^*B = -i\zeta_t ,\\ &B_x + 2i\zeta B + 2Aq = q_t ,\\ &C_x - 2i\zeta C + 2Aq^* = -q_t^* \end{aligned} \tag{2.2}$$

on the coefficients A, B, and C. As explained elsewhere,[6,13] the solution of Eqs. (2.2) is facilitated by the introduction of the functions W, Y, and Z according to

$$\begin{aligned} &A = i|q|^2 - iF(x) - 2i\zeta^2 + W ,\\ &B = iq_x + 2q\zeta + Y ,\\ &C = iq_x^* - 2q^*\zeta + Z . \end{aligned} \tag{2.3}$$

Here $W=W(x,t,\zeta)$, and likewise for Y and Z. Using Eqs. (2.3) in Eqs. (2.2), we find

$$\begin{aligned} &W_x - qZ - q^*Y = 0 ,\\ &Y_x + 2i\zeta Y = -2q(W+\zeta_x) ,\\ &Z_x - 2i\zeta Z = -2q^*(W-\zeta_x) \end{aligned} \tag{2.4}$$

and also a nonlinear evolution equation for the "eigenvalue" $\zeta(x,t)$ itself:

$$\zeta_t = 2(\zeta^2)_x + F_x . \tag{2.5}$$

We see at once that $\zeta_x=0$ implies $F_x=0$ or $F_x=$const, i.e., the case in which $F(x)$ is either a constant or else a linear function of x. These are the only cases which can be handled directly without modifying the conventional AKNS-ZS formalism.[10] For any other $F(x)$, Eq. (2.5) must be dealt with. However, owing to the nonlinearity of that equation, it turns out[6,13] that only separable solutions of the form

$$\zeta(x,t) = g(x)h(t) \tag{2.6}$$

render the corresponding ZS problem [Eqs. (2.1)] directly solvable, so that we may henceforth restrict our attention to these.

We substitute Eq. (2.6) in the first of Eqs. (2.1) and make a transformation of independent variables from (t,x) to $[t,y=\int g(x)dx]$, and of the dependent variable from $q(x,t)$ to $Q(y,t)=q/g$. This yields a ZS problem in the variable y, with a time-dependent eigenvalue $h(t)$:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_y = \begin{bmatrix} -ih(t) & Q(y,t) \\ -Q^*(y,t) & ih(t) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{2.7}$$

Further, if $|y| \to \infty$ as $|x| \to \infty$, the necessary condition[2,3] $\lim_{|y|\to\infty} Q(y,t)=0$ of the ZS problem will be satisfied because we are, in any case, concerned only with solutions $q(x,t)$ that vanish sufficiently rapidly as $|x| \to \infty$ so as to make the AKNS-ZS formalism applicable. The transformed version of the second of Eqs. (2.1) is

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_t = \begin{bmatrix} \tilde{A}(y,t,h) & \tilde{B}(y,t,h) \\ \tilde{C}(y,t,h) & -\tilde{A}(y,t,h) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{2.8}$$

where $A(x) \to \tilde{A}(y)$, etc.

The IST method must now be applied to Eqs. (2.7) and (2.8) to determine the unknown potential $Q(y,t)$. The required solution $q(x,t)$ of Eq. (1.2) is then given by

$$q(x,t) = g(x)Q\left[\int g(x)dx, t\right]. \tag{2.9}$$

It is evident that we must first determine $g(x)$ and $h(t)$. Using Eq. (2.6) and Eq. (2.5), we get

$$h_t = 2(g^2)_x h^2/g + F_x/g . \tag{2.10}$$

We may distinguish between two possibilities.

Case (a). If $h(t) \neq$const, then we *must* have

$$(g^2)_x/g=\lambda, \quad F_x/g=\mu \ , \tag{2.11}$$

where λ and μ are arbitrary constants. The first of these yields

$$g(x)=\tfrac{1}{2}\lambda x+\lambda_0 \ , \tag{2.12}$$

where λ_0 is a constant. Hence $F(x)$ must necessarily be a quadratic function

$$F(x)=\tfrac{1}{4}\mu\lambda x^2+\mu\lambda_0 x+\mu_0 \ , \tag{2.13}$$

where μ_0 is a constant. Moreover, Eq. (2.10) gives in this case

$$h_t=2\lambda h^2+\mu \ , \tag{2.14}$$

with the solution (for λ, $\mu \neq 0$)

$$h(t)=\alpha[h(0)+\alpha\tanh(rt)]/[\alpha+h(0)\tanh(rt)] \ , \tag{2.15}$$

where $\alpha=(-\mu/2\lambda)^{1/2}$ and $r=(-2\mu\lambda)^{1/2}$. Equations (2.12) and (2.15) express the x and t dependence of $\zeta(x,t)$ in this case. For the subsequent analysis in terms of the variable y, it is convenient to note that

$$g^2=\lambda y+\lambda_0^2, \quad F=\mu y+\mu_0 \ . \tag{2.16}$$

We mention in passing that the linear inhomogeneity studied by Chen and Liu[10] is recovered easily as the special case $\lambda=0$, so that [from Eq. (2.14)] $h(t)=\mu t+h(0)$. We do not repeat here the analysis of this special case in which solitons moving with a constant acceleration can be shown to exist.

Case (b). $h(t)=\text{const}=h(0)$, which happens when the two terms on the right-hand side of Eq. (2.10) cancel each other, i.e., $g(x)$ is determined for a given $F(x)$ from the equation

$$F(x)=c_0-2h^2(0)g^2(x) \ , \tag{2.17}$$

where c_0 is an arbitrary constant. We must of course ensure that $\int g(x)dx$ becomes unbounded as $|x| \to \infty$, and also satisfy the (rather weak) condition $q/g(x) \to 0$ asymptotically. These serve as restrictions on $F(x)$. Illustrative examples are $g(x)=\tanh kx$, $g(x)=(kx)^{2n+1}$ ($n=1,2,\ldots$), etc. The former corresponds to a localized inhomogeneity $F(x)$ that is relevant in physical problems.

An important point that should be noted with regard to the solution in case (b) is the following. The quantity $h(0)$ is actually an eigenvalue of a certain (differential) operator in the variable $y=\int g(x)dx$, in the ZS problem of Eq. (2.7). Now, y itself cannot be dependent on $h(0)$—in other words, one cannot redefine the scale of $g(x)$ by absorbing part of $h(0)$ in it for more than *one* such $h(0)$. On the other hand, if we *specify* $F(x)$ [as we indeed do in writing down Eq. (1.2)], then, because $h(0)$ occurs in the relation (2.17) connecting $g(x)$ and $F(x)$, one must choose once and for all a *single* value from the eigenvalue spectrum $\{h(0)\}$ and determine $g(x)$ (and thence y) according to

$$g^2(x)=[F(x)-c_0]/2h^2(0) \ . \tag{2.18}$$

The ZS problem will, in general, have a set of discrete eigenvalues as well as a continuous spectrum. The latter can be eliminated by choosing initial conditions in such a way that the potential $Q(y,0)$ is reflectionless. If we further select a potential that supports only a single pair of complex-conjugate eigenvalues [pure imaginary ones, in fact, because $F(x)$ must be real—see Eq. (2.17)], the corresponding eigenvalue $h(0)$ is of course the appropriate quantity to use in Eq. (2.18), and the rest of the IST method goes through in the new variables y and Q, as we shall see. The price we pay is that in this case (b), our extended AKNS-ZS method enables us to discuss only the *single* soliton solutions in the variable y. The individual lumps of a multisoliton solution correspond to distinct bound states [or eigenvalues $h(0)$], and our transformation method cannot handle these as it stands.

III. TIME EVOLUTION OF THE SCATTERING DATA

We consider next the determination of the time development of the scattering data using Eqs. (2.7) and (2.8). As is customary, one begins by defining w as a Jost solution of Eq. (2.7) with boundary conditions[3]

$$w \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \chi(t)\exp(-ihy), \quad \text{as } y \to -\infty \ , \tag{3.1}$$

$$w \sim \begin{bmatrix} a(h,t)\exp(-ihy) \\ b(h,t)\exp(ihy) \end{bmatrix} \chi(t), \quad \text{as } y \to +\infty \tag{3.2}$$

for real h. (The analyticity of a and b required for subsequent continuation to complex h can be proven.[3]) The function $\chi(t)$ is found by considering the asymptotic behavior of Eq. (2.8), for which we require that of $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ as $|y| \to \infty$. Written in terms of the variable y, Eqs. (2.3) involve functions $\widetilde{W}$, $\widetilde{Y}$, and $\widetilde{Z}$, which can be determined by dividing both sides of Eq. (2.4) by $g(x)$ to obtain

$$\begin{aligned} &\widetilde{W}_y=Q(y,t)\widetilde{Z}+Q^*(y,t)\widetilde{Y} \ , \\ &\widetilde{Y}_y+2ih(t)\widetilde{Y}=-2Q(y,t)[\widetilde{W}+h(t)\widetilde{g}\widetilde{g}_y] \ , \\ &\widetilde{Z}_y-2ih(t)\widetilde{Z}=-2Q^*(y,t)[\widetilde{W}-h(t)\widetilde{g}\widetilde{g}_y] \ . \end{aligned} \tag{3.3}$$

What is required in the IST method is the asymptotic behavior of $\widetilde{W}$, $\widetilde{Y}$, and $\widetilde{Z}$ as $y \to \pm\infty$. It turns out that if $\widetilde{Y},\widetilde{Z} \to 0$ as $y \to \pm\infty$, all the subsequent steps in the AKNS analysis can be carried out to determine $Q(y,t)$, thereby rendering Eq. (1.2) fully integrable. A proof of this is straightforward if $h(t)$ is complex ($\text{Im}h \neq 0$): For $\widetilde{Y}$ and $\widetilde{Z}$ are then automatically damped out at one of the limits, while the boundary condition at the other limit can be chosen to be zero.

However, when h is real, $\widetilde{Y}$ and $\widetilde{Z}$ display an oscillatory behavior as $y \to \pm\infty$. The coefficients of these oscillatory terms can be made to vanish for certain special forms of Q [ensuring the integrability of Eq. (1.2) for quadratic $F(x)$ in the present approach], and the general conditions under which this happens can be found in principle as follows. Combining Eqs. (3.3) to obtain a third-order differential equation for $\widetilde{W}$, one may show that $\widetilde{Y},\widetilde{Z} \to 0$ as $y \to \pm\infty$ provided the integrals $\int_{-\infty}^{\infty} Q\widetilde{W}\exp(2ihy)dy$ and $\int_{-\infty}^{\infty} Q\widetilde{g}\,\widetilde{g}_y\exp(2ihy)dy$ vanish. It is evident that these conditions are not of much use in practice. We therefore determine *sufficiency* conditions on $q(x,t)$ and $F(x)$ which lead to the integrability of Eq. (1.2). For this purpose, it is convenient to return to the original set of equations (2.4). We ask for the conditions under which these equations support solutions W, Y, and Z *of definite parity* in x. The conclusion is that it is possible to have solutions satisfying $W(x)=W(-x)$, $Y(x)=\pm Y(-x)$, and $Z(x)=\pm Z(-x)$ when $q(x,t)=\mp q(-x,t)$ along with $g(x)=-g(-x)$. Thus, if we choose one set of boundary conditions such that $W,Y,Z\to 0$ as $x\to+\infty$, they vanish as $x\to-\infty$ as well. Given this result, we may return to the behavior of $\tilde{W}$, $\tilde{Y}$, and $\tilde{Z}$. Since $y=\int g\,dx$ is now an even function of x, the solutions of Eq. (2.4) in the full space $-\infty<x<+\infty$ determine the above functions only in the half-space $0\le y<+\infty$. Further, since $|y|\to\infty$ as $|x|\to\infty$ (this requirement was imposed on the ZS problem [Eq. (2.7)] to obtain localized solutions $q(x,t)$ asymptotically), and the functions W, Y, and Z have been shown to vanish as $|x|\to\infty$, the functions $\tilde{W},\tilde{Y},\tilde{Z}\to 0$ as $y\to+\infty$.

It remains to determine $\tilde{W}$, $\tilde{Y}$, and $\tilde{Z}$ in the half-space $-\infty<y\le 0$. This may be done by solving Eqs. (3.3) together with the choice of boundary condition $\lim_{y\to-\infty}\tilde{W},\tilde{Y},\tilde{Z}=0$. We have thus shown that these functions vanish as $|y|\to\infty$ for both real and complex values of $h(t)$, *provided q has a* definite parity in x, and g is an odd function of x. The latter condition, in turn, implies that the inhomogeneity function $F(x)$ must be *even* in x [cf., respectively, Eqs. (2.11) and (2.17)]. In the rest of this paper we shall assume that these conditions are satisfied.

Using Eqs. (2.14) and (2.16) in Eq. (2.3) yields

$$\tilde{A}\sim -ih_t y-2i\lambda_0 h^2-i\mu_0,\quad B\sim C\sim 0 \tag{3.4}$$

[for case (a)]. Similarly, using Eq. (2.18) in (2.3) gives

$$\tilde{A}\sim -ic_0,\quad B\sim C\sim 0 \tag{3.5}$$

[for case (b)]. In both cases, Eq. (2.8) leads to the asymptotic equation

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_t \sim \begin{bmatrix} \tilde{A} & 0 \\ 0 & -\tilde{A} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix},\quad \text{as } |y|\to\infty . \tag{3.6}$$

Making use of Eqs. (3.1) and (3.2) for the asymptotic form of w in the above, we get the following results, respectively, in the two cases (a) and (b).

Case (a).

$$\begin{aligned} &\chi(t)=\chi(0)\exp\left[-\mu_0 t-2\lambda_0\int_0^t h^2(t')dt'\right], \\ &a(h,t)=a[h(0),0], \\ &b(h,t)=b[h(0),0]\exp\left[2i\mu_0 t+4i\lambda 0\int_0^t h^2(t')dt'\right], \end{aligned} \tag{3.7}$$

$h(t)$ being given by Eq. (2.15).

Case (b).

$$\begin{aligned} &\chi(t)=\chi(0)\exp(-ic_0 t), \\ &a(h,t)=a[h(0),0], \\ &b(h,t)=b[h(0),0]\exp(-2ic_0 t). \end{aligned} \tag{3.8}$$

For a given initial potential $Q(y,0)$, $h(0)$ is an eigenvalue of the corresponding ZS problem, Eq. (2.7). Given the eigenvalue spectrum $\{h(0)\}$, the time evolution of the scattering data is given by Eqs. (3.7) and (3.8), respectively, in the two cases (a) and (b). These represent the scattering data for the new potential $Q(y,t)$. Following conventional procedure, it can be proved[3] that the zeros of $a(h)$ in the upper half plane [denoted by $h_k(t)$, $k=1,2,\ldots,N$] correspond to the (discrete) bound-state spectrum, while the real eigenvalues ω represent the scattering states. The (time-dependent) scattering data is given by the set[3]

$$S(t)=\left\{\left[h_k,\frac{b(h_k)}{a'(h_k)}\right],\left[\bar{h}_j,\frac{\bar{b}(\bar{h}_j)}{\bar{a}'(\bar{h}_j)}\right],\frac{b(\omega)}{a(\omega)},\frac{\bar{b}(\omega)}{\bar{a}(\omega)}\right\}, \tag{3.9}$$

where $j=1,2,\ldots,N$ and the barred quantities correspond to the boundary conditions

$$\begin{aligned} &w\sim\begin{bmatrix} 0 \\ -1 \end{bmatrix}\chi(t)\exp(ihy),\quad \text{as } y\to\infty, \\ &w\sim\begin{bmatrix} b(h,t)\exp(-ihy) \\ -a(h,t)\exp(ihy) \end{bmatrix}\chi(t),\quad \text{as } y\to+\infty . \end{aligned} \tag{3.10}$$

In Eq. (3.9), $a'(h_k)$ stands for $da(h(0))/dh(0)$ evaluated at the eigenvalue $h(0)=h_k(0)$; similarly for $a'(h_j)$. [In the latter case, the bound states correspond to the zeros of $a(h)$ in the lower half plane.] The structure of the scattering problem here is such that $\bar{N}=N$, $\bar{h}_k=h_k^*$, $\bar{a}(h)=a^*(h^*)$, and $\bar{b}(h)=b^*(h^*)$. Defining

$$T(s,t)=-i\sum_k\frac{b(h_k)}{a'(h_k)}\exp(ish_k)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(\omega)}{a(\omega)}\exp(is\omega)d\omega \tag{3.11}$$

and $\bar{T}(s,t)=T^*(s,t)$, the Gelfand-Levitan-Marchenko equations for the column matrices K and $\bar{K}$ are given by[1,3]

$$\begin{aligned} &\bar{K}(z,y,t)+\begin{bmatrix} 0 \\ 1 \end{bmatrix}T(z+y,t)+\int_z^{\infty}K(z,s,t)T(s+y,t)ds=0, \\ &K(z,y,t)-\begin{bmatrix} 1 \\ 0 \end{bmatrix}\bar{T}(z+y,t)-\int_z^{\infty}\bar{K}(z,s,t)\bar{T}(s+y,t)ds=0, \end{aligned} \tag{3.12}$$

for $y>z$. The solution required [the potential $Q(y,t)$] is obtained after solving Eqs. (3.12) from

$$Q(y,t)=-2K_1(y,y,t), \tag{3.13}$$

where the subscript 1 refers to the upper element of the column matrix K.

IV. THE SOLUTION $q(x,t)$

As is customary,[3] let $Q(y,0)$ correspond to a reflectionless potential with a single pair of eigenvalues

$\{h_1(0),h_1^*(0)\}$, where $h_1(0)=\xi(0)+i\eta(0)$, with $\eta(0)>0$. [We recall, too, that the potential must be such that $q(x,0)$ is either even or odd in x.] The expression for $T(s,t)$ given in Eq. (3.11) then simplifies to

$$T(s,t)=-ic\exp\left[ish_1(t)+2i\mu_0 t+4i\lambda_0\int_0^t h_1^2(t')dt'\right] \quad (4.1)$$

for case (a), and

$$T(s,t)=-ic\exp[ish_1(0)-2ic_0 t] \quad (4.2)$$

for case (b), where the constant c is given by

$$c=b[h_1(0),0]/a'[h_1(0)] . \quad (4.3)$$

Substituting these in Eqs. (3.12), we can readily solve for $K(y,y,t)$. Using Eq. (3.13), we find that the general solution in both cases has the structure

$$Q(y,t)=ic^*\operatorname{sech}\theta(y,t)\exp[-i\varphi(y,t)] , \quad (4.4)$$

where $\theta(y,t)$ and $\varphi(y,t)$ are given below.

Case (a). This corresponds to a quadratic $F(x)$, cf. Eq. (2.13). We also recall from Eqs. (2.12) and (2.15) that $g(x)=\frac{1}{2}\lambda x+\lambda_0$, $y=\frac{1}{4}\lambda x^2+\lambda_0 x$, and

$$h_1(t)=\alpha[h_1(0)+\alpha\tanh rt]/[\alpha+h_1(0)\tanh rt]$$

in this case, with $\alpha=(-\mu/2\lambda)^{1/2}$ and $r=(-2\mu\lambda)^{1/2}$. Setting $h_1(0)=\xi(0)+i\eta(0)$, we easily obtain $\xi(t)$ and $\eta(t)$, where $h_1(t)=\xi(t)+i\eta(t)$. We find as solutions the expressions

$$\theta(y,t)=16\lambda_0\int_0^t\xi(t')\eta(t')dt'-2y\eta(t)+\ln[\,|c|\,/2\eta(t)] \quad (4.5)$$

and

$$\varphi(y,t)=8\lambda_0\int_0^t[\xi^2(t')-\eta^2(t')]dt'+4\mu_0 t+2y\xi(t)-i\ln[\,|c|\,/2\eta(t)] . \quad (4.6)$$

Once $Q(y,t)$ is found, the solution $q(x,t)$ to Eq. (1.2) is obtained from it by using Eq. (2.9).

Considerable simplification occurs if $\lambda_0=\mu_0=0$, which corresponds to a parabolic potential barrier $F(x)=-\frac{1}{2}L^2x^2$ where $L^2=-\frac{1}{2}\mu\lambda>0$. The final answer in this case is

$$q(x,t)=(c^*/c)^{1/2}x\eta(t)\exp[-\tfrac{1}{2}i\lambda x^2\xi(t)]\times\operatorname{sech}[\tfrac{1}{2}\lambda x^2\eta(t)+\ln 2\eta(t)-\ln|c|\,] . \quad (4.7)$$

Two arbitrary constants, λ and L, occur in this solution (on eliminating μ in favor of L, we have $\alpha=L/\lambda$ and $r=2L$). L is specified, of course, by the given potential $F(x)$. The constant λ is fixed by the initial datum $q(x,0)$. A concrete example is provided by the initial potential $Q(y,0)=-2\operatorname{sech}2y$, which has[14] a single pair of eigenvalues $[h(0),h^*(0)]=(i,-i)$. (It is easily verified that the corresponding $q(x,0)$ is an odd function of x, fulfilling the requirement that it have a definite parity.) We now have $\xi(0)=0$, $\eta(0)=1$, and

$$\xi(t)=\left[\frac{L}{\lambda}\right]\frac{(L^2+\lambda^2)\tanh(2Lt)}{L^2+\lambda^2\tanh^2(2Lt)} ,\qquad \eta(t)=\frac{L^2\operatorname{sech}^2(2Lt)}{L^2+\lambda^2\tanh^2(2Lt)} . \quad (4.8)$$

Inserted in Eq. (4.7), we have the solution $q(x,t)$ explicitly. In general, the envelope of the solution (4.7) is an antisymmetric function of x composed of two lumps which move away from each other and dissipate as time progresses. For sufficiently small values of L, the lifetime of such an entity could be large enough to be of physical relevance.

Case (b). This corresponds to a time-independent eigenvalue $\zeta(x)=h(0)g(x)$. The form of the solution $Q(y,t)$ is again as in Eq. (4.4), where one now has

$$\theta(y,t)=-2\eta(0)y+\ln[\,|c|\,/2\eta(0)] \quad (4.9)$$

and

$$\varphi(y,t)=-2c_0t+2\xi(0)y-i\ln[\,|c|\,/2\eta(0)] . \quad (4.10)$$

As before, $q(x,t)$ is found from $Q(y,t)$ with the help of Eq. (2.9).

Consider the illustrative example $F(x)=c_0-2h^2(0)\times\tanh^2 kx$, in which case $g(x)=\tanh kx$ and $y=\ln\cosh(kx)$. Taking the initial potential to be $Q(y,0)=-2\operatorname{sech}2y$ once again, with eigenvalues $(i,-i)$, we find the solution

$$q(x,t)=2\tanh(kx)\exp(-2ic_0t)\times\operatorname{sech}[2\ln\cosh(kx)+\text{const}] . \quad (4.11)$$

The solution thus has a static envelope that is a localized, antisymmetric function of x. The static pattern arises essentially because we have taken c_0 to be real, since $F(x)$ must be real in physical problems.

Finally, we note that the analysis in case (a) can be extended to *time-dependent* quadratic functions $F(x,t)$ of the form

$$F(x,t)=\mu(t)(\tfrac{1}{4}\lambda x^2+\lambda_0 x)+\mu_0(t) . \quad (4.12)$$

The formal solution for $Q(y,t)$ continues to be that given by Eqs. (4.4)–(4.6). However, Eq. (2.15) for $h(t)$ is no longer valid. This quantity must now be found by solving the Riccati equation $h_t=2\lambda h^2+\mu(t)$. [For certain functions $\mu(t)$, simple solutions for $h(t)$ exist.] The rest of the development then goes through as before, except that the term $\mu_0(t)$ in Eqs. (3.7), (4.1), and (4.6) is replaced by $\int_0^t\mu_0(t')dt'$.

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