Electromagnetic fields in layered-inhomogeneous uniaxial media: Validation criterion and higher-order solutions of the geometrical-optics approximation

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We study the electromagnetic fields in layered-inhomogeneous uniaxial planar structures using the geometrical-optics approximation (GOA). Explicit expressions for the fields in the higher- (first-) order GOA are obtained. We also obtain the validation criterion for the zeroth-order GOA. There results are important for studying the true significance of the geometrical-optics approximation and the extension of its usefulness. The GOA solution shows that when total internal reflection occurs in the medium, the field amplitudes become zero and the angle of refraction becomes $\pi/2$. The azimuthal dependence of the ray direction and total internal reflection is discussed. The results are applied to a wave propagation in a hybrid oriented uniaxial nematic liquid crystal in an external field.

I. INTRODUCTION

Approximate solutions are of significance in the study of wave propagation in inhomogeneous media. They include first of all the geometrical-optics approximation (GOA), and also the phase-integral method and the method of perturbation theory.[1] For slowly varying layered-inhomogeneous isotropic media, GOA and its extension have been shown to be most suitable. GOA has also been widely used in the study of optics of layered-inhomogeneous anisotropic media, such as liquid crystals.[2,3] The wide range of applicability of GOA is due to the fact that the properties of the inhomogeneous medium usually vary only slowly in space. The variation is said to be slow if the dielectric properties of the medium change very little over distances of the order of the wavelength of the wave. Evidently the propagation in any relatively small region of the layer may then be regarded as being the same as in a homogeneous medium, with the corresponding dielectric properties.

Most of the previous study used directly the results of the homogeneous medium. Recently Ong and Meyer presented a general GOA formalism for finding the electromagnetic fields in layered-inhomogeneous uniaxial structures using the GOA.[4] Explicit expressions for the fields in the zeroth-order GOA are obtained. By applying the results to a wave propagation in a periodically bent nematic liquid crystal,[5] excellent agreement between the GOA and exact solution is demonstrated. For example, the difference between the GOA and the exact solutions for the amplitudes of the fields is always less than 2×10^{-5} of the value of the exact solution through all space. The solutions of the fields can be improved by including the higher-order terms. To investigate the true significance of the GOA and to extend its usefulness, one should obtain higher-order correction terms. Only in this way can one place a precise meaning on the term "slowly varying." The first-order term is of particular importance since by comparing the amplitudes of the zeroth-order and first-order solutions, the validation criterion for the zeroth-order GOA can then be obtained. It is the purpose of this paper to present the first-order GOA solutions and the validation criterion of the zeroth-order GOA. The results are then applied to a wave propagation in a hybrid oriented uniaxial nematic liquid crystal in an external field. The results show that GOA is more accurate for the slowly varying medium. The GOA solution shows that when total internal reflection occurs in the medium, the field amplitudes become zero and the angle of refraction becomes $\pi/2$. In general, the ray direction depends on the azimuthal angle of the incidence wave but the total internal reflection is independent of the two azimuthal angles for the case that we are considering.

II. HIGHER-ORDER GEOMETRICAL-OPTICS APPROXIMATION

A. Geometry

Throughout this paper, the geometry and notations will be the same as those defined in Ref. 4. We consider a layered-inhomogeneous uniaxial medium in which the optical axis always lies in the x-z plane of a Cartesian coordinate system and whose components depend only on the z coordinate. It follows that the components of the dielectric tensor $\epsilon(\mathbf{x})$ are functions only of z. The medium is assumed to be nonmagnetic so that the magnetic permeability can be set to unity throughout all space. We let $\theta(z)$ be the angle between the optical axis and the z axis. Then the orientation of the optical axis can be described by the unit vector $\mathbf{n}(z)=(\sin\theta,0,\cos\theta)$. The components of the dielectric tensor can be expressed in terms of $\theta(z)$ through $\epsilon_{ij}(z)=\epsilon_\perp\delta_{ij}+(\epsilon_{\|}-\epsilon_\perp)n_i n_j$, where $\epsilon_\perp=n_o^2$, $\epsilon_{\|}=n_e^2$, n_o and n_e are the ordinary and extraordinary indexes of the refraction. An electromagnetic wave propagating in the z direction is obliquely incident on the medium with polarization parallel to the plane of incidence (p-polarized wave), which is the x-z plane. Consequently, only the extraordinary wave will be excited in the medium. (See Fig. 1.)

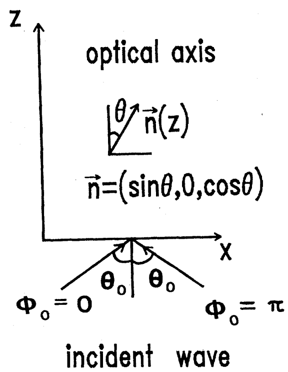

FIG. 1. Geometry of the layered-inhomogeneous uniaxial medium. The optical axis of the uniaxial medium always lies in the x-z plane of a Cartesian coordinate system and varies in the z direction. The orientation of the optical axis is described by the unit vector $\mathbf{n}(z)=(\sin\theta, 0, \cos\theta)$, where $\theta(z)$ is the angle between the optical axis and the z axis. An electromagnetic wave is obliquely incident on the medium with polarization parallel to the plane of incidence, which is the x-z plane. Consequently, only the extraordinary wave is excited in the medium. θ_0 is the angle of incidence and ϕ_0 is the azimuthal angle of the plane of incidence. The azimuthal angle assumes either 0 or π for the case that we are considering.

B. Geometrical-optics formalism and the zeroth-order solution

For the obliquely incident extraordinary wave, the fields in the medium can be written as

$$E(\mathbf{x},t)=(F_1(z),0,F_2(z))e^{ikpx-i\omega t}\ , \tag{2.1}$$

and

$$H(\mathbf{x},t)=(0,dF_1/dz-ikpF_2,0)e^{ikpx-i\omega t}\ , \tag{2.2}$$

where $k=\omega/c$, and $p=(\epsilon_0)^{1/2}\sin\theta_0\cos\phi_0$, depending on the angle of incidence θ_0, the azimuthal angle of the plane of incidence ϕ_0, and the dielectric constant ϵ_0 for the medium from which the wave is incident. Since the wave polarized in the plane containing the optical axis, the azimuthal angle can have one of the two possible values 0 or π for the case that we are considering.

In the following discussion, we define a two-dimensional reduced field $F(z)=(F_1(z),F_2(z))$. In the GOA, the solution of F is expressed in the form

$$F(z)=e^{ik\phi(z)}\sum_{n=0}^{n=\infty}k^{-n}\mathbf{F}^{(n)}(z)\ , \tag{2.3}$$

in which the amplitudes $\mathbf{F}^{(n)}(z)$ are slowly varying functions of z. The phase $k\phi(z)$ is almost linear in z and varies much more rapidly than the amplitudes. It has been shown by Luneberg and by Kline using Duhamel's principle that this form of solution is an asymptotic solution of Maxwell's equations.[6,7] We refer to $e^{ik\phi(z)}k^{-n}\mathbf{F}^{(n)}(z)$ as the nth-order GOA solutions. By substituting Eq. (2.3) into the wave equation obtained from Maxwell's equations, and equating to zero the terms in each power of k as we formulated in Ref. 4, we have

$$\hat{a}\cdot\mathbf{F}^{(0)}=\mathbf{0}\ , \tag{2.4}$$

$$\hat{a}\cdot\mathbf{F}^{(1)}=i\mathbf{b}^{(0)}\ , \tag{2.5}$$

and

$$\hat{a}\cdot\mathbf{F}^{(2)}=i\mathbf{b}^{(1)}+\mathbf{h}^{(0)},\ \ldots\ , \tag{2.6}$$

where

$$\hat{a}=\begin{bmatrix}\epsilon_{11}-\left[\dfrac{d\phi}{dz}\right]^2 & \epsilon_{13}+p\dfrac{d\phi}{dz}\\ \epsilon_{13}+p\dfrac{d\phi}{dz} & \epsilon_{33}-p^2\end{bmatrix}. \tag{2.7}$$

The components of $\mathbf{b}^{(n)}$ are

$$b_1^{(n)}=-\frac{d^2\phi}{dz^2}F_1^{(n)}-2\frac{d\phi}{dz}\frac{dF_1^{(n)}}{dz}+p\frac{dF_2^{(n)}}{dz}$$

and

$$b_2^{(n)}=p\frac{dF_1^{(n)}}{dz}\ . \tag{2.8}$$

The components of $\mathbf{h}^{(n)}$ are

$$h_1^{(n)}=-\frac{d^2F_1^{(n)}}{dz^2}$$

and

$$h_2^{(n)}=0\ . \tag{2.9}$$

The solution of the reduced electromagnetic fields in the zeroth-order GOA are[4]

$$F_1^{(0)}=(\epsilon_{33}-p^2)^{1/4}\ , \tag{2.10}$$

and

$$F_2^{(0)}=RF_1^{(0)}\ , \tag{2.11}$$

where $F_1^{(0)}$ is defined up to a constant and the phase is $k\phi(z)$ with $\phi(z)$ given by

$$\phi=\pm\int_{z_0}^{z}\frac{n_on_e(\epsilon_{33}-p^2)^{1/2}\mp p\epsilon_{13}}{\epsilon_{33}}dz\ , \tag{2.12}$$

and R is given by

$$R=-\frac{\epsilon_{13}\pm pn_on_e/(\epsilon_{33}-p^2)^{1/2}}{\epsilon_{33}}\ . \tag{2.13}$$

In Eqs. (2.12) and (2.13), z_0 is some constant and the upper sign is for wave propagating in the positive-z direction and the lower sign is for wave propagating in the negative-z direction. Evidently, $F_1^{(0)}$ is independent of the two azimuthal angles but the phase $k\phi(z)$ and R and hence $F_1^{(0)}$ are dependents of the azimuthal angle.

C. Higher- (first-) order solution

A necessary condition for a nontrivial solution to $F^{(0)}$ is that the determinant $|\hat{a}|$ should vanish,

$$|\hat{a}|=0\ . \tag{2.14}$$

The relation between $\mathbf{b}^{(1)}$ and $\mathbf{h}^{(0)}$ is given by, according to Eqs. (2.14) and (2.6),

$$a_{11}b_2^{(1)}-a_{12}b_1^{(1)}+ia_{12}h_1^{(0)}=0 . \quad (2.15)$$

By Eq. (2.5), $F_2^{(1)}$ can be expressed in terms of $F_1^{(1)}$ as follows:

$$F_2^{(1)}=ip\frac{1}{a_{22}}\frac{dF_1^{(0)}}{dz}-\frac{a_{12}}{a_{22}}F_1^{(1)} . \quad (2.16)$$

By direct substitution of the components of $\mathbf{b}^{(1)}$ and with the use of conditions (2.14) and (2.16), we obtain

$$a_{11}b_2^{(1)}-a_{12}b_1^{(1)}=\left[\frac{a_{12}}{a_{22}^2}\right]\left[\left[a_{22}^2\frac{d^2\phi}{dz^2}+pa_{22}\frac{da_{12}}{dz}-pa_{12}\frac{da_{22}}{dz}\right]F_1^{(1)}+\left[2a_{22}^2\frac{d\phi}{dz}+2pa_{12}a_{22}\right]\frac{dF_1^{(1)}}{dz}-ip^2a_{22}\frac{d^2F_1^{(0)}}{dz^2}+ip^2\frac{da_{22}}{dz}\frac{dF_1^{(0)}}{dz}\right] . \quad (2.17)$$

On making the substitution $h_1^{(0)}=-d^2F_1^{(0)}/dz^2$ in Eq. (2.15), and with the use of Eq. (2.17), we have

$$2\left[\frac{d\phi}{dz}-pR\right]\frac{dF_1^{(1)}}{dz}+\left[\frac{d^2\phi}{dz^2}-p\frac{dR}{dz}\right]F_1^{(1)}=i\left[1+\frac{p^2}{a_{22}}\right]\frac{d^2F_1^{(0)}}{dz^2}-i\frac{p^2}{a_{22}^2}\frac{da_{22}}{dz}\frac{dF_1^{(0)}}{dz} . \quad (2.18)$$

Since $d\phi/dz-pR=\pm n_on_e/(\epsilon_{33}-p^2)^{1/2}$ and $a_{22}=\epsilon_{33}-p^2$, Eq. (2.18) can be written as

$$2\frac{dF_1^{(1)}}{dz}\left[\frac{n_on_e}{\epsilon_{33}-p^2}\right]+F_1^{(1)}\frac{d}{dz}\left[\frac{n_on_e}{\epsilon_{33}-p^2}\right]=\pm i\left[\frac{d^2F_1^{(0)}}{dz^2}\left[\frac{\epsilon_{33}}{\epsilon_{33}-p^2}\right]-\frac{p^2}{(\epsilon_{33}-p^2)^2}\frac{dF_1^{(0)}}{dz}\frac{d\epsilon_{33}}{dz}\right] . \quad (2.19)$$

To simplify the discussion, we define two new variables G and M as follows:

$$G=\frac{1}{(\epsilon_{33}-p^2)^{1/2}} \quad (2.20)$$

and

$$M=\pm i\frac{1}{2n_on_e}\left[\frac{d^2F_1^{(0)}}{dz^2}\left[\frac{\epsilon_{33}}{\epsilon_{33}-p^2}\right]-\frac{p^2}{(\epsilon_{33}-p^2)^2}\frac{dF_1^{(0)}}{dz}\frac{d\epsilon_{33}}{dz}\right] . \quad (2.21)$$

Then by Eq. (2.19), $F_1^{(1)}$ satisfies the following first-order differential equation with coefficients containing derivatives of $F_1^{(0)}$:

$$\frac{dF_1^{(1)}}{dz}+\frac{1}{2}\frac{d\ln G}{dz}F_1^{(1)}=\frac{M}{G} . \quad (2.22)$$

Equation (2.22) has the solution of the form

$$F_1^{(1)}=(\epsilon_{33}-p^2)^{1/4}\int_{z_0}^{z}M(\epsilon_{33}-p^2)^{1/4}dz . \quad (2.23)$$

Consequently by Eq. (2.16), the solution of $F_2^{(1)}$ is

$$F_2^{(1)}=R(z)(pBF_1^{(0)}+F_1^{(1)}) , \quad (2.24)$$

where

$$B=i\frac{1}{4R(z)(\epsilon_{33}-p^2)^2}\frac{d\epsilon_{33}}{dz} . \quad (2.25)$$

Therefore the approximate solutions for the electromagnetic fields up to and including first-order GOA are

$$E_x(\mathbf{x},t)=A\left[F_1^{(0)}+\frac{1}{k}F_1^{(1)}\right]\exp[ik\phi(z)+ikpx-i\omega t] , \quad (2.26)$$

$$E_z(\mathbf{x},t)=AR(z)\left[F_1^{(0)}+\frac{1}{k}(pBF_1^{(0)}+F_1^{(1)})\right]\times\exp[ik\phi(z)+ikpx-i\omega t] , \quad (2.27)$$

and

$$H_y(\mathbf{x},t)=A\frac{n_on_e}{(\epsilon_{33}-p^2)^{1/2}}\left[F_1^{(0)}+\frac{1}{k}(F_1^{(1)}+QF_1^{(0)})\right]\times\exp[ik\phi(z)+ikpx-i\omega t] . \quad (2.28)$$

We now discuss the following two important cases:

(i) Homogeneous anisotropic media. For homogeneous media, $d\epsilon_{33}/dz=dF_1^{(0)}/dz=0$. Thus $F^{(1)}=0$ and the only nonvanishing term in the solution of $\mathbf{F}$ is the zeroth-order term $F^{(0)}e^{ik\phi}$. Therefore Eqs. (2.26)–(2.28) reduced to the exact solution described by $F^{(0)}e^{ik\phi}$ in the homogeneous anisotropic media.

(ii) Total internal reflection. The amplitude of $F^{(0)}$ $[=(\epsilon_{33}-p^2)^{1/4}]$ and the phase $k\phi(z)$ are real only if $\epsilon_{33}-p^2>0$. At the point z' satisfying $\epsilon_{33}(z')=p^2$, the field amplitudes become zero and the phase is still real. However, for z having $\epsilon_{33}(z)<p^2$, both the field amplitudes and the phase become complex and the wave is attenuated in the medium. Since the components of the

dielectric tensor are real, absorption is absent and energy is conserved in the medium.[8] The damping of the wave without damping signifies that the average energy flux from the incident wave to the medium having $\epsilon_{33}(z)<p^2$ is zero. Therefore the energy of the incident wave is transformed to the reflected wave and total internal reflection occurs at the point z' satisfying $\epsilon_{33}(z')=p^2$. We will show in Sec. IV that when total internal reflection occurs, the angle of refraction becomes $\pi/2$ and the ray direction becomes parallel to the x axis. Since total internal reflection depends only on $|p|$, it is independent of the two azimuthal angles (0 and π).

III. GEOMETRICAL-OPTICS APPROXIMATION VALIDATION CRITERION

The solutions we have obtained in Sec. II allow us to obtain the limits of applicability of GOA. Clearly, GOA may be considered a sufficiently good approximation to reality if the higher-order waves can be neglected in comparison with the primary wave. Since the higher-order approximations are not used, we shall refer the condition for the zeroth-order GOA to be valid as that for the GOA to be valid.

From the form of solution Eq. (2.3), the approximation of GOA is valid only if the first term need to be taken in Eq. (2.13). For definiteness we will take the component E_x in the following discussions. Then by assuming that all second- and higher-order terms are less than the first-order term and comparing the amplitudes of the zeroth- and first-order solution, we obtain that GOA is valid if

$$\frac{1}{k}\left|\frac{F_1^{(1)}}{F_1^{(0)}}\right| \ll 1 . \tag{3.1}$$

Since the amplitudes of the field become imaginary and the wave will be attenuated if $\epsilon_{33}(z)<p^2$, in the following discussion we consider only that $\epsilon_{33}\geq p^2$. Using the solutions (2.10) and (2.23), we obtain the following integral condition for the GOA to be valid:

$$\frac{1}{8kn_o n_e}\left|\int_{z_0}^{z}\left\{\frac{1}{(\epsilon_{33}-p^2)^{1/2}}\frac{d^2\epsilon_{33}}{dz^2}+\frac{p^2}{(\epsilon_{33}-p^2)^{3/2}}\frac{d^2\epsilon_{33}}{dz^2}-\frac{3}{4}\frac{1}{(\epsilon_{33}-p^2)^{3/2}}\left[\frac{d\epsilon_{33}}{dz}\right]^2 -\frac{p^2}{4}\frac{1}{(\epsilon_{33}-p^2)^{5/2}}\left[\frac{d\epsilon_{33}}{dz}\right]^2\right\}dz\right| \ll 1 . \tag{3.2}$$

We now consider the case in which the function ϵ_{33} is monotonic in the range (z_0,z). Using integration by parts, we have

$$\left|\frac{F_1^{(1)}}{F_1^{(0)}}\right| = \frac{1}{8n_o n_e}\left|\frac{1}{(\epsilon_{33}-p^2)^{1/2}}\frac{d\epsilon_{33}}{dz}+\frac{p^2}{(\epsilon_{33}-p^2)^{3/2}}\frac{d\epsilon_{33}}{dz}-\frac{1}{4}\int_{z_0}^{z}\frac{1}{(\epsilon_{33}-p^2)^{3/2}}\left[\frac{d\epsilon_{33}}{dz}\right]^2 dz +\frac{5p^2}{4}\int_{z_0}^{z}\frac{1}{(\epsilon_{33}-p^2)^{5/2}}\left[\frac{d\epsilon_{33}}{dz}\right]^2 dz\right| . \tag{3.3}$$

In the expression (3.3), the factors in the integrals are either both positive or both negative. For $z\geq z_0$, the integral is positive so that

$$\int_{z_0}^{z}\frac{1}{(\epsilon_{33}-p^2)^{m/2}}\left[\frac{d\epsilon_{33}}{dz}\right]^2 dz = \int_{z_0}^{z}\frac{1}{(\epsilon_{33}-p^2)^{m/2-1}}\frac{d\epsilon_{33}}{dz}d\ln(\epsilon_{33}-p^2) \leq \left[\frac{1}{(\epsilon_{33}-p^2)^{m/2}}\frac{d\epsilon_{33}}{dz}\right]_{\max}\left|\ln\left[\frac{\epsilon_{33}(z)-p^2}{\epsilon_{33}(z_0)-p^2}\right]\right| , \tag{3.4}$$

where $m=3,5$ and the subscript max signifies the maximum value in the range (z_0,z). Thus we see that the inequality $|F_1^{(1)}/(kF_1^{(0)})| \ll 1$ is satisfied if throughout the range we have

$$\frac{1}{8kn_o n_e}\left|\frac{1}{(\epsilon_{33}-p^2)^{1/2}}\frac{d\epsilon_{33}}{dz}+\frac{p^2}{(\epsilon_{33}-p^2)^{3/2}}\frac{d\epsilon_{33}}{dz}\right| = \frac{\epsilon_{33}}{4kn_o n_e}\left|\frac{d}{dz}\frac{1}{(\epsilon_{33}-p^2)^{1/2}}\right| \ll 1 , \tag{3.5}$$

and the value of

$$\left| \ln \left[\frac{\epsilon_{33}(z)-p^2}{\epsilon_{33}(z_0)-p^2} \right] \right|$$

does not become very large. The latter requirement is usually quite unimportant [the requirement $|\ln(a)| < 10$ corresponds to $10^{-4} < a < 10^4$] so that (3.5) can be taken to be the sufficient condition for the GOA. Therefore the sufficient (but not necessary) condition for the GOA to be valid when $\epsilon_{33}(z)$ is monotonic in the range (z_0, z) is

$$\frac{\epsilon_{33}}{4kn_o n_e} \left| \frac{d}{dz} \frac{1}{(\epsilon_{33}-p^2)^{1/2}} \right| \ll 1 . \tag{3.6}$$

Similarly one can show that Eq. (3.6) is also the sufficient condition for the GOA to be valid when $\epsilon_{33}(z)$ is monotonic in the range (z, z_0) with $z_0 > z$.

The sufficient condition (3.6) for the GOA to be valid is violated in three cases: (a) if the gradient of $\epsilon_{33}(z)$ is sufficiently steep (i.e., if the derivative $d\epsilon_{33}/dz$ is large); (b) if ϵ_{33} is sufficiently small; (c) if $\epsilon_{33}(z)$ is not monotonic. Since $\min(\epsilon_\perp, \epsilon_\parallel) \le \epsilon_{33} \le \max(\epsilon_\perp, \epsilon_\parallel)$ and for a typical uniaxial medium, $\epsilon_\perp$ and $\epsilon_\parallel$ are greater than 2, ϵ_{33} is usually sufficiently large. The criterion (3.6) will not be valid if the properties of the medium vary periodically in the z direction. It could happen that even small amplitude waves reflected from the various layers would add in phase, thereby creating a resultant reflected wave, comparable in amplitude with the incident wave. This is the so called Bragg reflection.

When condition (3.6) holds, the two waves propagating in the opposite z directions are independent of each other. If there is a sharp boundary where condition (3.6) fails, the two waves are no longer independent: One transforms into the other and reflection occurs.

IV. DISCUSSIONS: HYBRID ORIENTED NEMATIC LIQUID CRYSTAL IN AN EXTERNAL FIELD

The wave propagation in the optically uniaxial liquid crystals has been studied extensively, but the problem is extremely complicated and the exact solution for the fields has only been found for the fields in a cholesteric liquid crystal[9] and in a periodic bent nematic liquid crystal.[5] As an example of the application of the GOA, we have considered in Ref. 4 the electromagnetic wave propagation in a periodic bent nematic liquid crystal. The results showed excellent agreement between the GOA and exact solutions: The difference between the GOA and the exact solutions for the amplitudes of the fields is always less than 2×10^{-5} of the value of the exact solution for all z. Whereas the difference between the GOA and the exact solutions for the tangent to the extraordinary ray as well as for the path of the extraordinary ray are always less than 8×10^{-6} of the value of the respective exact solution for all z.

As a second example of the application of the GOA, we consider the electromagnetic wave propagation in a hybrid oriented nematic liquid crystal[3,10] (HNLC) in an external electric field. Figure 2 shows the spatial orientation of the optical axis in the HNLC cell. The optical axis of the

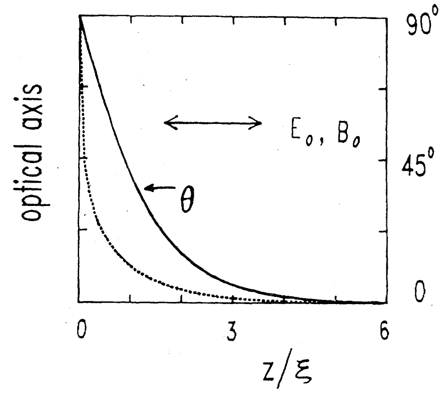

FIG. 2. Spatial orientation of the optical axis in the hybrid oriented nemetic liquid crystal (HNLC). Solid line describes the orientation of the optical axis. Dashed line describes the angle between the optical axis and the z axis normalized by the coherence length ξ, $\theta(z)$. An external orientating field (electric field E_0 or magnetic field B_0) is applied along the z direction. The orientating effect of the applied field conflicts with the orientation imposed by the surface boundary condition at $z=0$.

nematic liquid crystal (NLC) is oriented along the long axis of the elongated molecule. The space between $0 \le z \le d$ is filled with a HNLC in which the optical axis always lies in the x-z plane which is also the plane of incidence. An orientating external field (electric or magnetic field) is applied along the z direction. For electric field, we require that the NLC has positive dielectric anisotropic at the applied electric field frequency. Then the orientating effect of the applied field conflicts with the orientation imposed by the surface boundary condition at $z=0$. Molecules far from the surface align with the field. Because of the surface orientating condition, there is a transition region in which the molecular orientation changes from parallel to the x axis at $z=0$ to parallel to the z axis at $z \gg 1$, as shown in Fig. 2. The depth of this transition region is measured by a coherence length ξ, which is called the electric or magnetic coherence length, depending on whether the external field is an electric or magnetic field.[3] For an applied electric field E_0,

$$\xi = \left[\frac{4\pi k_{22}}{\epsilon_a^e} \right]^{1/2} \frac{1}{E_0} ,$$

where k_{22} is the bend elastic constant. ϵ_a^e is the dielectric anisotropic defined by $\epsilon_a^e = \epsilon_\parallel^e - \epsilon_\perp^e > 0$, where $\epsilon_\perp^e$ and $\epsilon_\parallel^e$ are the dielectric constants perpendicular and parallel to the director at the applied electric field frequency. The magnetic coherence length is given by

$$\xi = \left[\frac{k_{22}}{\chi_a} \right]^{1/2} \frac{1}{B_0} ,$$

where $\chi_a > 0$ is the magnetic susceptibility and B_0 is the applied magnetic field. The angle made by the optical axis with the z axis, θ, is given by $\tan(\theta/2) = e^{-z/\xi}$. The orientation of the optical axis can be described by $-\ln[\tanh(z/2\xi)]$, as illustrated in Fig. 2. We consider the HNLC layer is sufficiently thick (say $d > 10\xi$) so that at $z=d$ the NLC is orienated parallel to the field. The orientation of the NLC molecules can be described by the unit vector $n=(\sin\theta, 0, \cos\theta)=(\mathrm{sech}(z/\xi), 0, \tanh(z/\xi))$. Using $\epsilon_{ij}=\epsilon_\perp\delta_{ij}+(\epsilon_{||}-\epsilon_\perp)n_i n_j$, the dielectric tensor of the HNLC medium is given by

$$\hat{\epsilon}=\begin{bmatrix}\epsilon[1-\delta+2\delta\tanh^2(z/\xi)] & 0 & 2\epsilon\delta\,\mathrm{sech}(z/\xi)\tanh(z/\xi)\\ 0 & \epsilon_\perp & 0\\ 2\epsilon\delta\,\mathrm{sech}(z/\xi)\tanh(z/\xi) & 0 & \epsilon[1+\delta-2\delta\tanh^2(z/\xi)]\end{bmatrix}, \quad (4.1)$$

where $\epsilon=(\epsilon_\perp+\epsilon_{||})/2=(n_o^2+n_e^2)/2$, $\delta=(\epsilon_\perp-\epsilon_{||})/(\epsilon_\perp+\epsilon_{||})=(n_o^2-n_e^2)/(n_o^2+n_e^2)$. Since $n_o<n_e$ for all NLC's, we have $\delta<0$.

We consider a wave incident from $z>d$ with the polarization parallel to the plane of incidence which is the x-z plane. The exact solution for the fields are complicated and the solutions have only been found for the case of normal incidence. However, the zeroth-order GOA approximate solutions for the fields propagating in the negative-z direction in the HNLC for normal incidence as well as oblique incidence can easily be found from Eqs. (2.10)—(2.13) and (2.26)—(2.28) as follows:

$$E_x(\mathbf{x},t)=A\{\epsilon[1+\delta-2\delta\tanh^2(z/\xi)]-p^2\}^{1/4}\exp[ik\phi(z)+ikpx-i\omega t] , \quad (4.2)$$

$$E_z(\mathbf{x},t)=R(z)E_x(\mathbf{x},t) , \quad (4.3)$$

and

$$H_y(\mathbf{x},t)=\frac{n_o n_e}{\{\epsilon[1+\delta-2\delta\tanh^2(z/\xi)]-p^2\}^{1/2}}E_x(\mathbf{x},t) , \quad (4.4)$$

where $\phi(z)$ and $R(z)$ are given by

$$\phi=-\int_d^z \frac{n_o n_e\{\epsilon[1+\delta-2\delta\tanh^2(z/\xi)]-p^2\}^{1/2}+2p\epsilon\delta\,\mathrm{sech}(z/\xi)\tanh(z/\xi)}{\epsilon[1+\delta-2\delta\tanh^2(z/\xi)]}dz , \quad (4.5)$$

and

$$R(z)=-\frac{2\epsilon\delta\,\mathrm{sech}(z/\xi)\tanh(z/\xi)-pn_o n_e\{\epsilon[1+\delta-2\delta\tanh^2(z/\xi)]-p^2\}^{-1/2}}{\epsilon[1+\delta-2\delta\tanh^2(z/\xi)]} . \quad (4.6)$$

As we did in Ref. 4, we choose the liquid-crystal material 4-n-penthyl-4′-cyanobiphenyl (PCB, also referred to as 5CB,K15) as an example. PCB is a nematic liquid crystal between 22 and 35 °C with $n_o=1.562$, and $n_e=1.806$. Figure 3 shows the field amplitudes as a function of z for $\epsilon_0=1$, $\theta_0=\pi/6$, $\phi_0=0$, and $\phi=\pi$. The amplitudes of the fields are normalized so that $|E_x(z=d)|=1$. The results show that $|E_x(z)|$ and $|H_y(z)|$ are independent of the two values of the azimuthal angle but $|E_z(z)|$ depends strongly on the azimuthal angle. Figures 2 and 3 show that as $z\gg\xi$, the molecules align along the orientating field direction so that $\theta=0$ and the field amplitudes become constants.

Let us now consider the applicability of GOA. Since $\epsilon_{33}(z)=\epsilon[1+\delta-2\delta\tanh^2(z/\xi)]$ is monotonic in the range $(0,\infty)$, by Eq. (3.6), the sufficient condition for the GOA to be valid is

$$L=\left[\frac{\epsilon^2|\delta|[1+\delta-2\delta\tanh^2(z/\xi)]\tanh(z/\xi)\mathrm{sech}^2(z/\xi)}{2k\xi n_o n_e\{\epsilon[1+\delta-2\delta\tanh^2(z/\xi)]-p^2\}^{3/2}}\right]_{\max}\ll 1 . \quad (4.7)$$

The estimate of the validity of GOA, L, depends mainly on two terms: (i) The first main factor is $k\xi$ $(=2\pi\xi/\lambda)$ which describes the rate of the spatial variation of the dielectric properties (ξ) with respect to the wavelength of the light ($k=2\pi/\lambda$). (ii) The second important factor is $(\epsilon_{33}-p^2)^{3/2}$ $[=(\epsilon_{33}-\epsilon_0\sin^2\theta_0)^{3/2}]$ which relates the spatial variation of the dielectric properties (ϵ_{33}) with the angle of incidence (θ_0) and the dielectric properties of the medium from which the light is incident (ϵ_0).

Figure 4 shows the criterion L as a function of the product of coherence length and wave vector, ξk, for PCB with $\epsilon_0=1$, $\theta_0=\pi/3$, and $\theta_0=\pi/6$. For $\xi=1$ μm, $\xi k\sim 10$—20 for visible light but $L<0.003$. Thus GOA is still valid for $\xi\sim 1$ μm. When the coherence length is comparable with the wavelength of the light, $\xi k\sim 2\pi$, $L\sim 0.005$. This shows that when the wave is incident from air into the HNLC, GOA is a good approximation even when dielectric properties change quite rapidly as compared to the wavelength of the light. However, when ϵ_0 is of value comparable to or greater than $\epsilon_\perp$ or $\epsilon_{||}$, L depends strongly on the angle of incidence. The dependence of L as a function of the reduced wave vector $|p|$

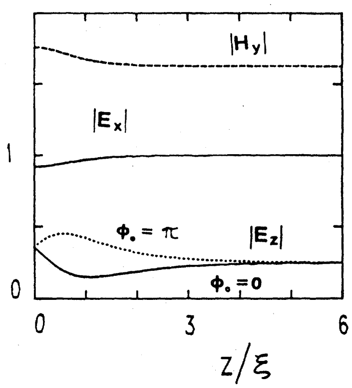

FIG. 3. Amplitudes of the fields in the zeroth-order GOA as a function of z for a HNLC with PCB as the material. In the calculation, we put $n_o=1.562$, $n_e=1.806$, $\epsilon_0=1$, $\theta_0=\pi/6$, $\phi_0=0$, and $\phi_0=\pi$. The amplitudes of the fields are normalized so that $|E_x(z=d)|=1$. The amplitudes of E_x and H_y are independent of the two azimuthal angles ϕ_0 (0 and π), but the amplitude of E_z depends on the azimuthal angle.

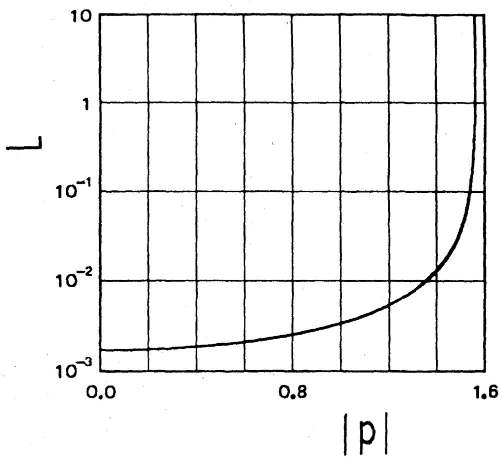

FIG. 5. Estimate of the validity of GOA, L, as a function of the reduced wave vector $|p|=(\epsilon_0)^{1/2}\sin\theta_0$ for PCB (4-n-pentyl-4′-cyanobiphenyl) at $\xi=1$ μm. GOA becomes inappropriate for $|p|>1.45$.

$[=(\epsilon_0)^{1/2}\sin\theta_0]$ with $\xi=1$ μm is shown in Fig. 5. Clearly L increases as p is increased. GOA becomes inappropriate for $|p|>1.45$. For wave incident from air, $\epsilon_0=1$, we have $|p|\leq 1$ and $L<0.004$. This shows that GOA is a good approximation for all angles of incidence. For $|p|>1.45$, $L>0.02$ and GOA becomes inappropriate. At that region, strong reflection in the layer occurs. Although GOA is not appropriate for the case where strong reflection occurs, as will be shown below, GOA does consistently predict that when the field amplitudes become zero, the ray direction becomes parallel to the x axis. We consider a wave incident from a medium with index of refraction equal to 1.7, i.e., $\epsilon_0=3.89$. Then $\theta_0=30°$ corresponds to $|p|=0.85$ and $\theta_0=70°$ corresponds to $|p|=1.60$. The amplitude of the x component of the electric field as a function of z for $\theta_0=30°$ and $\theta_0=70°$ is shown in Figs. 6 and 7, respectively. The field amplitude decreases to zero in the layer for $\theta_0=70°$, i.e., for $|p|=1.60$. Therefore the energy of the incident wave is transformed into the reflected wave and total reflection occurs in the medium. The rays turn around in the layer at the point z' satisfying $\epsilon_{33}(z')=p^2$, i.e., at the point

$$z'=\xi\tanh^{-1}\left[\left[\frac{p^2-\epsilon(1+\delta)}{-2\epsilon\delta}\right]^{1/2}\right]. \tag{4.8}$$

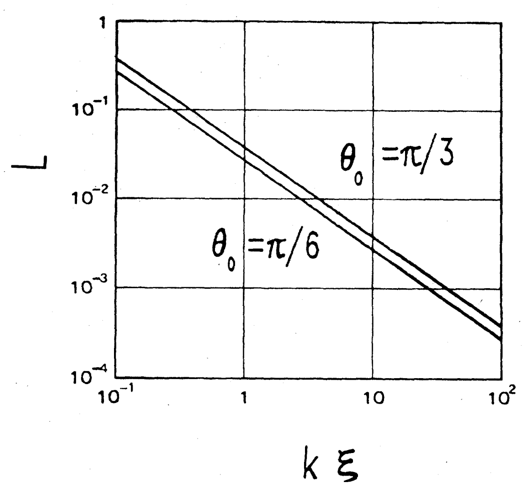

FIG. 4. Estimate of the validity of GOA, L, as a function of the coherence length and wave vector, ξk $(=2\pi\xi/\lambda)$ for PCB (4-n-penyl-4′-cyanobiphenyl) with $\epsilon_0=1$, $\theta_0=\pi/3$, and $\theta_0=\pi/6$. For $\xi=1$ μm, $\xi k\sim 10$–20 for visible light. When the coherence length is comparable with the wavelength of the incident light, $\xi k\sim 2\pi$ and $L\sim 0.005$. GOA is still a good approximation even when $\xi\sim\lambda$.

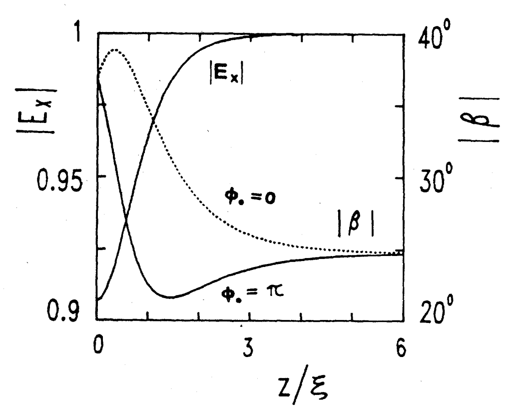

FIG. 6. Amplitudes of the electric fields and the angle of refraction as a function of z for a wave incident from a medium with $\epsilon_0=2.89$, and $\theta_0=30°$. The amplitude of the field is normalized so that $|E_x(z=d)|=1$. In general, the ray direction depends on the two azimuthal angles.

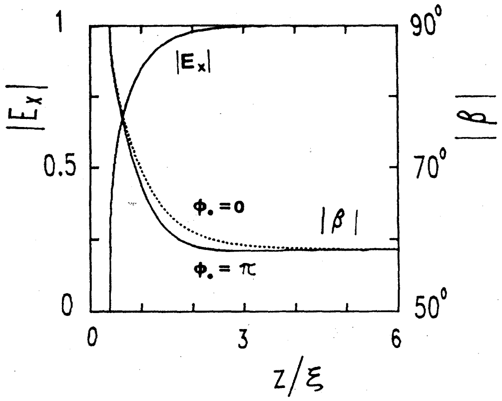

FIG. 7. Amplitudes of the electric fields and the angle of refraction as a function of z for a wave incident from a medium with $\epsilon_0=2.89$ and $\theta_0=70°$. The amplitude of the field is normalized so that $|E_x(z=d)|=1$. When the amplitude of E_x becomes zero, the angle of refraction becomes $\pi/2$ and the ray direction is parallel to the x axis. In general, the ray direction depends on the two azimuthal angles, but total internal reflection depends only on ϵ_0 and θ_0, and is independent of the two azimuthal angles ϕ_0 (0 and π).

To complete the description, we will discuss now the direction of the energy flow. The energy flux vector in the medium is described by the time average of the Poynting vector $\mathbf{S}=c\,\mathrm{Re}(\mathbf{E})\times\mathrm{Re}(\mathbf{H})/4\pi$. By Eqs. (4.2)–(4.4), we have

$$\mathbf{S}(z)=\frac{cn_o n_e\,|A|^2}{8\pi}(-R,0,1)\ . \qquad (4.9)$$

The direction of the ray is given by $S_x/S_z=-R$, thus we find the angle between the ray vector and the z axis to be

$$\beta=-\tan^{-1}(R)$$
$$=\tan^{-1}\left[\frac{\epsilon_{13}-pn_o n_e(\epsilon_{33}-p^2)^{-1/2}}{\epsilon_{33}}\right]\ . \qquad (4.10)$$

In Figs. 6 and 7, the angle of refraction β is shown as a function of z for $\theta_0=30°$ and $\theta_0=70°$. The results show that as the field amplitude becomes zero, the angle of refraction becomes $\pi/2$ and the ray direction becomes parallel to the x axis. Thus total internal reflection occurs in the layer. Equation (4.7) also indicates that L depends only on $|p|$, i.e., L depends only on the absolute value of p and is independent of the two azimuthal angles ϕ_0 (0 and π). This is also evident from Figs. 6 and 7 that in general, the ray direction depends on the azimuthal angle but total internal reflection is independent of the two azimuthal angles. For a wave incident from $z<0$, because $\epsilon_{33}-p^2$ increases as z is increased, there will be no total internal reflection in the slowly varying layer. However, if $|p|>n_o$, then the wave will be total reflected at the boundary $z=0$, but not in the layer.

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