Stability of magnetically insulated electron flow

C. W. Mendel, Jr., J. A. Swegle, and D. B. Seidel
Sandia National Laboratories, Albuquerque, New Mexico 87185
(Received 19 February 1985)

The stability of magnetically insulated electron flow is studied for wavelengths longer than gyrolengths and frequencies lower than gyrofrequencies. This is done by using a general theory that allows for a distribution of electrons of different canonical momentum and total energy. The stability theory leads to a set of integro-differential equations for the eigenfunctions of the potentials. The integral portion of the equations relates positions across the flow that are connected by the finite size of the electron orbits. A general stability criterion is found. The stability of the flow is determined by the distribution of the electrons in canonical-momentum—total-energy space. The criterion can be stated roughly as depending upon whether the distribution function increases or decreases with drift velocity. If it increases, the flow is unstable.

INTRODUCTION

In the effort to achieve inertial confinement fusion with light-ion beams, magnetic insulation is crucial to the transport of megavolt- and megampere-level power pulses along transmission lines as well as to the efficient production of multiterawatt ion beams. In this context, magnetic insulation is the use of a magnetic field to prevent field-emitted electrons from the cathode of a voltage-stressed gap from crossing to the anode. This magnetic field can either be externally applied or can result self-consistently from the flow of the current in the insulated device.

The theoretical analysis of magnetically insulated systems dates back to the study of nonrelativistic magnetrons.[1] Developments in the field of pulsed power, however, have necessitated the treatment of relativistic, strongly diamagnetic systems. Until recently, equilibrium theories of these flows have been either laminar,[1–6] in which all electrons move in straight lines parallel to the electrodes, or quasilaminar,[7] in which the electrons all have a single, cycloidlike orbit.[8,9] A more recent theory allows a continuum of orbits.[10,11] The study of the linear stability of these systems is far from complete. The simplicity of the electron trajectories in laminar systems has allowed the stability of these equilibria to be studied in some detail.[5–6,12–16]

We present here a linear stability theory for arbitrary distribution of electron orbits based upon the linearization of the general, three-dimensional (3D), time-dependent theory of Ref. 11. This theory assumes that wavelengths are long compared to electron gyrolengths, and frequencies are low compared to gyrofrequencies.

Particle-in-cell simulations at Sandia National Laboratories[17] using the MAGIC code,[18] as well as other laboratories using the MASK code,[19] and earlier work using static codes,[20] have indicated that these flows tend to be nearly laminar, although not the Brillouin flow used in early laminar theories of magnetic insulation where all electrons have the same total energy and canonical momentum. The final distribution of electrons in these simulations is determined by fluctuations in the electric and magnetic fields. The stability theory to be presented indicates that the tendency toward laminar (but not Brillouin) flow may be due to the system's efforts to reach a stable configuration.

THEORY

We will consider the 2D, time-dependent case here. It is convenient to use normalized variables. Time will be multiplied by the speed of light so that it has dimensions of meters, and the potentials are given by

$$\phi=\frac{eV}{mc^2},\quad \alpha=\frac{eA}{mc}\ .$$

In the above, V is in the scalar potential; A is the x component of the vector potential, where x is the dimension along the direction of power and electron flow; m and e are the electron mass and the magnitude of the electron charge, respectively; and c is the speed of light. The dimension across the flow is y, and the magnetic field is in the z direction (Fig. 1).

The relevant equations[11] are the equations for the scalar and vector potentials in terms of the charge and current density (Lorentz gauge). The current and charge densities involve integrals over the distribution function J

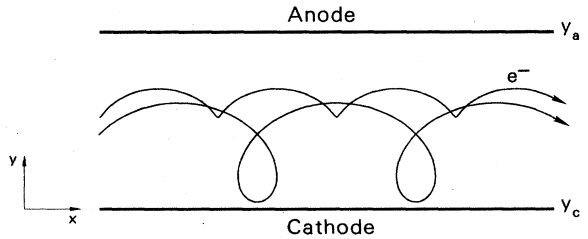

FIG. 1. Magnetically insulated flow. For this problem there is no z dependence, but the magnetic field is in the z direction. The shape of the particular orbits is determined by the values of P, W. The distribution function $J(P,W)$ determines how many electrons are in any interval of P, W space, and also determines the equilibrium potentials $\alpha(y), \phi(y)$.

$$\nabla^2\alpha-\frac{\partial^2\alpha}{\partial t^2}=-\int dP\int dW\,U(Q)\frac{\partial Q^{1/2}}{\partial P}J(x,t,P,W)$$
$$=-\frac{\partial G}{\partial\alpha}\,,$$
$$\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}=\int dP\int dW\,U(Q)\frac{\partial Q^{1/2}}{\partial W}J(x,t,P,W)$$
$$=\frac{\partial G}{\partial\phi}\,, \tag{1a}$$

where U is the unit step function [$U(x)=1$ for $x>0$, $U(x)=0$ for $x<0$] and G is the pressure integral

$$G(x,y,t)=\int dP\int dW\,U(Q)Q^{1/2}J(x,t,P,W) \tag{1b}$$

and Q is defined by

$$Q(x,y,t,P,W)=[1+\phi(x,y,t)+W]^2$$
$$-1-[\alpha(x,y,t)+P]^2\,.$$

In the above, P,W are the canonical momentum (along the flow) and total energy of the electrons. When $Q>0$, $Q=(\gamma u_y)^2$, where γ is the usual relativistic factor and u_y is the y velocity component, and those values of y where $Q<0$ represent regions of the transmission line which are inaccessible to electrons with that P,W. The equation governing the distribution function is

$$\frac{\partial H}{\partial x}\frac{\partial J}{\partial P}-\frac{\partial H}{\partial P}\frac{\partial J}{\partial x}-\frac{\partial H}{\partial t}\frac{\partial J}{\partial W}+\frac{\partial H}{\partial W}\frac{\partial J}{\partial t}=0\,, \tag{2a}$$

where the function H is an adiabatic constant of motion defined by

$$H(x,t,P,W)=\int dy\,U(Q)Q^{1/2}\,. \tag{2b}$$

In Ref. 11, the conservation of H is proven only to second order in the ratio of the gyrolength to scale length of changes along x, and in the ratio of gyroperiod to the scale of temporal changes. On the left-hand side of (1a) we have kept changes to all order, and second-order changes will be important. For the unperturbed solution, H gives the exact gyroperiod ($2\partial H/\partial W$), gyrolength ($-2\partial H/\partial P$), change in total energy over one gyration ($-2\partial H/\partial t=0$), and change in canonical momentum over one gyration ($2\partial H/\partial x=0$). Since the gradients (along the flow or in time) are first order, the dynamics are correct to second order, so we have been consistent. In addition, it is often true that action integrals such as (2b) are conserved for faster changes than has been proven. Therefore, while the result is rigorously correct for

$$\left|k\frac{\partial H}{\partial P}\right|\ll 1,\quad \left|\Omega\frac{\partial H}{\partial W}\right|\ll 1\,,$$

where k and Ω are the wave number and frequency, respectively, it may well be true that the result will be valid when these numbers are of order one.

The potentials, the distribution function, and the adiabatic constant are now linearized about a 1D, time-independent state:

$$\phi\to\phi(y)+\delta\phi(y,t)e^{ikx}\,,$$
$$\alpha\to\alpha(y)+\delta\alpha(y,t)e^{ikx}\,,$$
$$J\to J(P,W)+\delta J(t,P,W)e^{ikx}\,,$$
$$H\to H(P,W)+\delta H(t,P,W)e^{ikx}\,.$$

These are now put into the governing equations. In the potential equations (1a), the linearization of the left-hand side is straightforward. The source terms on the right-hand side contain two parts: those due to expanding $Q^{1/2}$ in the perturbed potentials, and those due to the perturbed distribution function J. The expansion in the potentials may be done by writing the source as a function of the potentials and expanding. In taking derivatives one must be careful about moving them from outside to inside the G and H integrals. The higher order derivatives will have integrals with infinite parts which are canceled by the derivatives of the endpoints. When done as a limiting process, all derivatives exist, however, and the first derivatives used here have no such problem in any case. The equations for the one-dimensional, time-independent system are

$$\alpha''=-\frac{\partial G}{\partial\alpha}\,,$$
$$\phi''=\frac{\partial G}{\partial\phi}\,, \tag{3}$$

where a prime denotes derivatives with respect to y, and the G integral (1b) involves only the zero-order potentials and distribution function from here on. The first-order equations become

$$\delta\alpha''+\frac{\partial^2 G}{\partial\alpha^2}\delta\alpha+\frac{\partial^2 G}{\partial\phi\,\partial\alpha}\delta\phi-k^2\delta\alpha-\frac{\partial^2\delta\alpha}{\partial t^2}$$
$$=-\int dP\int dW\,U(Q)\frac{\partial Q^{1/2}}{\partial P}\delta J(P,W)\,, \tag{4a}$$

$$\delta\phi''-\frac{\partial^2 G}{\partial\phi^2}\delta\phi-\frac{\partial^2 G}{\partial\alpha\,\partial\phi}\delta\alpha-k^2\delta\phi-\frac{\partial^2\delta\phi}{\partial t^2}$$
$$=\int dP\int dW\,U(Q)\frac{\partial Q^{1/2}}{\partial W}\delta J(P,W)\,,$$

and

$$ik\,\delta J\frac{\partial H}{\partial P}-\frac{\partial\delta J}{\partial t}\frac{\partial H}{\partial W}=ik\,\delta H\frac{\partial J}{\partial P}-\frac{\partial\delta H}{\partial t}\frac{\partial J}{\partial W}\,. \tag{4b}$$

The derivatives of G in (4a) are evaluated at the zero-order solution $\alpha(y),\phi(y)$. The perturbed Hamiltonian is given by

$$\delta H=\int_{y_c}^{y_a}dy\,U(Q)\left[\frac{\partial Q^{1/2}}{\partial W}\delta\phi+\frac{\partial Q^{1/2}}{\partial P}\delta\alpha\right]. \tag{4c}$$

Laplace transforming (4a), (4b), and (4c) in time, using

$$\overline{\delta\phi}(y,s)=\int_0^\infty dt\,\delta\phi(y,t)e^{-st}\,,$$

etc., we find

$$\overline{\delta\alpha''}+\frac{\partial^2 G}{\partial\alpha^2}\overline{\delta\alpha}+\frac{\partial^2 G}{\partial\phi\partial\alpha}\overline{\delta\phi}+\int_{y_c}^{y_a} d\tilde{y}[K_{PW}(y,\tilde{y})\overline{\delta\phi}(\tilde{y})+K_{PP}(y,\tilde{y})\overline{\delta\alpha}(\tilde{y})]=(k^2+s^2)\overline{\delta\alpha}+S_P\ ,$$
$$\overline{\delta\phi''}-\frac{\partial^2 G}{\partial\phi^2}\overline{\delta\phi}-\frac{\partial^2 G}{\partial\alpha\partial\phi}\overline{\delta\alpha}-\int_{y_c}^{y_a} d\tilde{y}[K_{WP}(y,\tilde{y})\overline{\delta\alpha}(\tilde{y})+K_{WW}(y,\tilde{y})\overline{\delta\phi}(\tilde{y})]=(k^2+s^2)\overline{\delta\phi}-S_W\ , \tag{5}$$

$$K_{AB}(y,\tilde{y})=\int dP\int dW\frac{ik\partial J/\partial P-s\partial J/\partial W}{ik\partial H/\partial P-s\partial H/\partial W}U(Q(y))U(Q(\tilde{y}))\frac{\partial Q^{1/2}(y)}{\partial A}\frac{\partial Q^{1/2}(\tilde{y})}{\partial B}\ , \tag{6}$$

where the bar denotes the Laplace-transformed perturbations, and A,B in the expression for the kernel K are P or W. The S terms are the source terms at $t=0$, which have been lumped together. These terms arise due to the Laplace transformation.

The kernel (6) of the eigenvalue equation shows resonances between electron drift velocity and wave phase velocity. If s in the Laplace transform of (4b) is replaced by $-i\Omega$, it is found that $\overline{\delta J}$ is given by

$$\overline{\delta J}=\frac{k\partial J/\partial P+\Omega\partial J/\partial W}{k\partial H/\partial P+\Omega\partial H/\partial W}\overline{\delta H}\ . \tag{7}$$

The derivative of H with respect to momentum P and energy W are minus one-half the gyrolength, and one-half the gyroperiod, respectively. Since the ratio of the gyrolength to the gyroperiod is the electron drift velocity, the denominator will have a zero when the phase velocity Ω/k equals the drift velocity. There will thus be a singularity in the integrands involving $\overline{\delta J}$ for these electrons. Moreover, it is clear from (5) that when the contours of equal adiabatic constant, H, and equal density, J, are parallel, there is no imaginary part to $k^2-\Omega^2$ because the kernels in (6) will be real. It is convenient to change the coordinates of the distribution function from P,W to θ,H, where H is as defined in (2b), and θ lines are normal to H contours in P,W space (see Fig. 2). H obeys the differential equation

$$dH=\frac{\partial H}{\partial P}dP+\frac{\partial H}{\partial W}dW\ , \tag{8a}$$

of course, so θ can be defined as

$$d\theta=-\frac{\partial H}{\partial W}dP+\frac{\partial H}{\partial P}dW\ . \tag{8b}$$

The parameter θ is given by (8b), but has a rough relationship with the average distance of a particle orbit from the cathode. Since θ can be found on the stable laminar line L_s, and each point on L_s corresponds to the y of the particular laminar orbit, there is a simple correspondence with y. It can be shown that this y is between y_- and y_+ for particles of the same θ, even when those particles are away from the laminar line.

The Jacobian of the transformation from P,W to θ,H gives us

$$\int dP\,dW\rightarrow\int d\theta\,dH\,h^{-2},\quad h^2=(\partial H/\partial P)^2+(\partial H/\partial W)^2. \tag{9}$$

Changing the partial derivatives of J with respect to P,W to partial derivatives with respect to θ,H then yields

$$\frac{ik\partial J/\partial P-s\partial J/\partial W}{ik\partial H/\partial P-s\partial H/\partial W}=\frac{\partial J}{\partial H}-\frac{ik\partial H/\partial W+s\partial H/\partial P}{ik\partial H/\partial P-s\partial H/\partial W}\frac{\partial J}{\partial\theta}\ .$$

It is thus clear that any J which is independent of θ (i.e., isocontours of H and J are parallel) has a real K_{AB}. Moreover, this suggests that the sign of $\partial J/\partial\theta$ determines whether the wave is growing or damped.

The kernel (6) can be rewritten

$$K_{AB}(y,\tilde{y})=\int d\theta\int dH[h(H,\theta)]^{-2}\left[\frac{\partial J}{\partial H}-\frac{ik\partial H/\partial W+s\partial H/\partial P}{ik\partial H/\partial P-s\partial H/\partial W}\frac{\partial J}{\partial\theta}\right]_{H,\theta}\left[\frac{R_A\tilde{R}_B}{Q^{1/2}\tilde{Q}^{1/2}}\right]_{H,\theta}=K_{BA}(\tilde{y},y)\ , \tag{10}$$

$$R_A=-\frac{U(Q(y))}{2}\frac{\partial Q(y)}{\partial A},\quad \tilde{R}_A=-\frac{U(Q(\tilde{y}))}{2}\frac{\partial Q(\tilde{y})}{\partial A}\ , \tag{11}$$

where the subscripts on the parentheses indicate that $P(H,\theta)$, $W(H,\theta)$ are to be used for P,W. It is easily shown that R_A and h^{-2} are well behaved, and not singular. The integrand of (10) is singular when $\tilde{Q}$ or Q are zero.

The function Q is dependent on H,θ, and y, i.e.,

$$Q=Q\{y,P[H,\theta],W[H,\theta]\}\ .$$

There are zeros of Q at $y_+(H,\theta)$ and $y_-(H,\theta)$, i.e., $Q(y_\pm,H,\theta)=0$. If y_+ and y_- are plotted versus H (holding θ constant) they would look something like Fig. 3. If the H integral were done before the θ integral to obtain $K_{AB}(y,\tilde{y})$ for the case shown in Fig. 3, the first unit step function would become one at H_1, and the second at H_2, so the integrand would be zero for $H<H_2$. At $H=H_2$, the denominator term $[Q(y,H)Q(\tilde{y},H)]^{1/2}$ would vary as $(H-H_2)^{1/2}$ for $H>H_2$, and the integral would exist. When y and $\tilde{y}$ are such that $H_1=H_2$, on the other hand, that term varies as $[(H-H_1)(H-H_2)]^{1/2}$

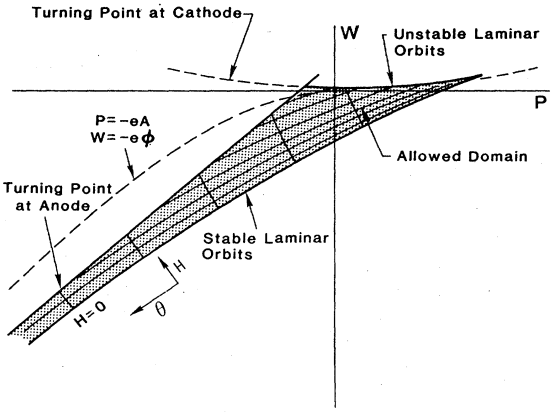

FIG. 2. The allowed region of P,W space. Electrons that have P,W above the cathode (anode) line will hit the cathode (anode) in less than one gyration. Those on the laminar line will be in straight-line orbits. The unstable laminar line refers to the stability of the orbit, not the entire flow. It is impossible for an electron to be below the stable laminar line as it would have negative $(\gamma u_y)^2$. Several contours of equal H (more or less parallel to the stable laminar line which is also the $H=0$ line) and equal-θ contours (normal to the H contours) are shown.

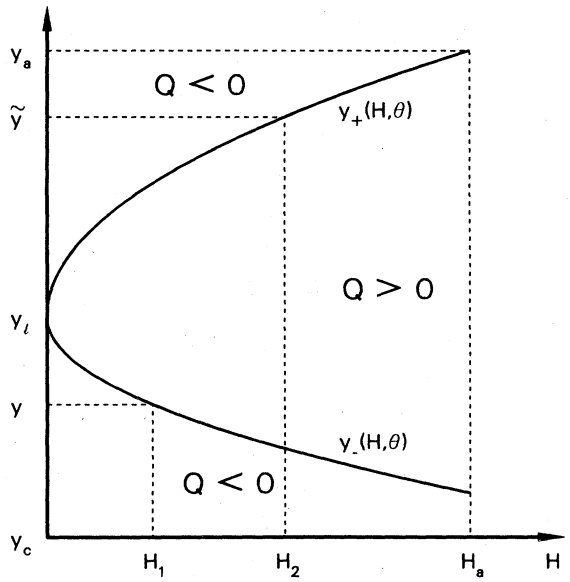

FIG. 3. A plot of y_+ and y_- versus H for constant θ. It is easily shown that $\partial y_+/\partial H>0$ and $\partial y_-/\partial H<0$. For a larger θ both of the lines would move upwards. The function Q is positive inside the two curves, and therefore when integrating over H one of the unit step functions would be zero until H_2 was reached. The denominator of the last term in brackets of (10) would have zeros at H_1 and H_2.

$=|H-H_1|$ and the integral diverges logarithmically. This can happen either when $y=\tilde{y}$ which will give a $\ln|y-\tilde{y}|$ term to the kernel, or it can happen when either y_+ or y_- is greater than y_l and the other is less than y_l, as the case in Fig. 3 shows. This will give another logarithmic term. If the curves y_+ and y_- were symmetric about $y=y_l$ the term would look like $\ln|(y+\tilde{y})/2-y_l|$, for example. These two terms have simple physical significance. The electron charge density per unit H, per unit θ is infinite at $y_\pm$, which is to say at the two extremes in y of the particular orbit. If that electron is perturbed at one of these extremes, say y_+, it has a great effect at both y_+ and y_-. When the θ integral is done following the H integral, the y_+ and y_- will not occur at the same places for different θ, and the singularity due to y on one branch and $\tilde{y}$ on the other will integrate out. The singularity when both are on the same branch can be moved outside of the θ integral after the H integral has been performed, and consequently will remain. The kernel will always have a $\ln|y-\tilde{y}|$ singularity, as might be expected for a two-dimensional problem. The singularity will integrate out when the y integrals are done.

In this paper we wish to find a stability criterion for the distribution function $J(\theta,H)$. To do this we will assume small growth rates. The drift velocity of an electron with P,W is given by[11]

$$v_d=\frac{-\partial H/\partial P}{\partial H/\partial W}\ . \tag{12}$$

Using this, and defining $s=-i\Omega=-i(\omega+i\nu)$, the kernel (10) can be written

$$K_{AB}(y,\tilde{y})=\int dH\int d\theta[h(H,\theta)]^{-2}\left[\frac{\partial J}{\partial H}+\frac{[1+(\omega/k)v_d]+i(\nu/k)v_d}{(v_d-\omega/k)-i(\nu/k)}\frac{\partial J}{\partial\theta}\right]_{H,\theta}\left[\frac{R_A\tilde{R}_B}{Q^{1/2}\tilde{Q}^{1/2}}\right]_{H,\theta}$$

$$=\int dH\int d\theta[h(H,\theta)]^{-2}\left[\frac{\partial J}{\partial H}+\frac{[1+(\omega/k)v_d](v_d-\omega/k)-(\nu/k)^2v_d}{(v_d-\omega/k)^2+(\nu/k)^2}\frac{\partial J}{\partial\theta}\right.$$

$$\left.+i(1+v_d^2)\frac{\nu/k}{(v_d-\omega/k)^2+(\nu/k)^2}\frac{\partial J}{\partial\theta}\right]_{H,\theta}\left[\frac{R_A\tilde{R}_B}{Q^{1/2}\tilde{Q}^{1/2}}\right]_{H,\theta}. \tag{13}$$

Since we assume $|\nu|\ll|\omega|$, we would like $K_{AB}(y,\tilde{y})$ in the limit of $\nu\to0$. Taking the limit of (13) as $\nu\to0$, we obtain

$$\lim_{\nu\to 0} K_{AB}(y,\tilde{y}) = \int dH \int d\theta [h(H,\theta)]^{-2} \frac{\partial J}{\partial H} \left[\frac{R_A \tilde{R}_B}{Q^{1/2}\tilde{Q}^{1/2}} \right]_{H,\theta}$$
$$+ \int dH \, \mathcal{C} d\theta [h(H,\theta)]^{-2} \frac{1+(\omega/k)v_d}{v_d-\omega/k} \frac{\partial J}{\partial \theta} \left[\frac{R_A \tilde{R}_B}{Q^{1/2}\tilde{Q}^{1/2}} \right]_{H,\theta}$$
$$\pm i\pi \int dH [h(H,\theta)]^{-2}(1+v_d^2) \left[\frac{\partial v_d}{\partial \theta} \right]^{-1} \frac{\partial J}{\partial \theta} \left[\frac{R_A \tilde{R}_B}{Q^{1/2}\tilde{Q}^{1/2}} \right]_{\theta_r(H)}, \tag{14}$$

where $\mathcal{C}$ indicates the Cauchy principal-value integral, and the sign of the last term depends upon whether ν has approached zero from the positive or negative side. The function $\theta_r(H)$ is the value of θ at which the electrons with H resonate with the wave, i.e.,

$$v_d(\theta_r(H),H) = \frac{\omega}{k} . \tag{15}$$

The side from which ν approaches zero is determined by the requirement that the contour of s integration for the inverse Laplace transform pass to the right of any singularities in the integrand. This requires that the real part of s (which equals ν), approach zero from the positive side. Therefore the plus sign applies in this case. Since we will only need the imaginary part of K_{AB} in what follows, we define k_{AB} by

$$k_{AB}(y,\tilde{y}) = \lim_{\nu\to 0+} \mathrm{Im}[K_{AB}(y,\tilde{y})]$$
$$= \pi \int dH \left[h^{-2}(1+v_d^2) \left[\frac{\partial v_d}{\partial \theta} \right]^{-1} \frac{\partial J}{\partial \theta} \frac{R_A \tilde{R}_B}{Q^{1/2}\tilde{Q}^{1/2}} \right]_{\theta_r(H)} . \tag{16}$$

We wish to find the growth rates for the normal modes of (5). Rewriting (5) and dropping the source terms

$$\xi_P'' + \left[\Omega^2 - k^2 + \frac{\partial^2 G}{\partial \alpha^2} \right] \xi_P + \frac{\partial^2 G}{\partial \phi \, \partial \alpha} \xi_W + \int_{y_c}^{y_a} d\tilde{y} [K_{PW}(y,\tilde{y})\xi_W(\tilde{y}) + K_{PP}(y,\tilde{y})\xi_P(\tilde{y})] = 0 , \tag{17a}$$

$$\xi_W'' + \left[\Omega^2 - k^2 - \frac{\partial^2 G}{\partial \phi^2} \right] \xi_W - \frac{\partial^2 G}{\partial \alpha \, \partial \phi} \xi_P - \int_{y_c}^{y_a} d\tilde{y} [K_{WP}(y,\tilde{y})\xi_P(\tilde{y}) + K_{WW}(y,\tilde{y})\xi_W(\tilde{y})] = 0 , \tag{17b}$$

where ξ_W are the normal modes of $\overline{\delta\phi}$, and ξ_P are the normal modes of $\overline{\delta\alpha}$. Multiplying (17a) by ξ_P^*, and taking the complex conjugate of (17b) and multiplying by ξ_W, we get

$$\xi_P^* \xi_P'' + \left[\Omega^2 - k^2 + \frac{\partial^2 G}{\partial \alpha^2} \right] \xi_P^* \xi_P + \frac{\partial^2 G}{\partial \phi \, \partial \alpha} \xi_P^* \xi_W + \xi_P^* \int_{y_c}^{y_a} d\tilde{y} [K_{PW}(y,\tilde{y})\xi_W(\tilde{y}) + K_{PP}(y,\tilde{y})\xi_P(\tilde{y})] = 0 ,$$
$$\xi_W \xi_W^{*''} + \left[\Omega^{*2} - k^2 - \frac{\partial^2 G}{\partial \phi^2} \right] \xi_W \xi_W^* - \frac{\partial^2 G}{\partial \alpha \, \partial \phi} \xi_W \xi_P^* - \xi_W \int_{y_c}^{y_a} d\tilde{y} [K_{WP}^*(y,\tilde{y})\xi_P^*(\tilde{y}) + K_{WW}^*(y,\tilde{y})\xi_W^*(\tilde{y})] = 0 , \tag{18}$$

which can be added and integrated over $y = y_c$ to y_a to yield

$$\int_{y_c}^{y_a} dy \left[-\xi_P^{*\prime}\xi_P' - \xi_W'\xi_W^{*\prime} + \left[\Omega^2 - k^2 - \frac{\partial^2 G}{\partial \alpha^2} \right] \xi_P^* \xi_P + \left[\Omega^{*2} - k^2 + \frac{\partial^2 G}{\partial \phi^2} \right] \xi_W \xi_W^* \right]$$
$$- \int_{y_c}^{y_a} dy \int_{y_c}^{y_a} d\tilde{y} [K_{WW}^*(y,\tilde{y})\xi_W \tilde{\xi}_W^* + K_{WP}^*(y,\tilde{y})\xi_W \tilde{\xi}_P^* - K_{PW}(y,\tilde{y})\xi_P^* \tilde{\xi}_W - K_{PP}(y,\tilde{y})\xi_P^* \tilde{\xi}_P]$$
$$+ (\xi_P^* \xi_P' + \xi_W \xi_W^{*\prime})_{y_c}^{y_a} = 0 , \tag{19}$$

where we have integrated by parts to get the first and last terms. By subtracting (19) from the complex conjugate of (19), and repeatedly using $K_{AB}(y,\tilde{y}) = K_{BA}(\tilde{y},y)$ from (10), and the fact that y and $\tilde{y}$ can be switched in any term of the double integrals, we get the expression

$$-2\omega\nu \int_{y_c}^{y_a} dy [\xi_W^* \xi_W - \xi_P^* \xi_P] + \mathrm{Im}(\xi_P^* \xi_P' + \xi_W \xi_W^{*\prime})_{y_c}^{y_a} + \int_{y_c}^{y_a} dy \int_{y_c}^{y_a} d\tilde{y} \{ \mathrm{Im}[K_{WW}(y,\tilde{y})]\xi_W \tilde{\xi}_W^* + \mathrm{Im}[K_{WP}(y,\tilde{y})]\xi_W \tilde{\xi}_P^*$$
$$+ \mathrm{Im}[K_{PW}(y,\tilde{y})]\xi_P \tilde{\xi}_W^* + \mathrm{Im}[K_{PP}(y,\tilde{y})]\xi_P \tilde{\xi}_P^* \} = 0 .$$

Taking the $\nu \to 0^+$ limit and using (16),

$$2\omega\nu - \left[\int_{y_c}^{y_a} dy[\xi_W^*\xi_W - \xi_P^*\xi_P]\right]^{-1} \mathrm{Im}(\xi_P^*\xi_P' + \xi_W\xi_W'^*)_{y_c}^{y_a}$$
$$= \pi \int dH \left[h^{-2}(1+v_d^2)\left[\frac{\partial v_d}{\partial\theta}\right]^{-1}\frac{\partial J}{\partial\theta}(F_W F_W^* + F_W F_P^* + F_P F_W^* + F_P F_P^*)\right]_{\theta_r(H)}, \quad (20)$$

where

$$F_A(P,W) = \left[\int_{y_c}^{y_a} dy(\xi_W^*\xi_W - \xi_P^*\xi_P)\right]^{-1/2} \times \int_{y_c}^{y_a} dy\, \xi_A R_A Q^{-1/2} \quad (21)$$

and we assume for the moment that the integral term in the parentheses is positive.

The second term on the left-hand side of (20) is zero for the following reason. The potentials ξ_P and ξ_W are zero at the cathode because that was assumed in the definitions of P and W. The tangential electric field, ϵ_x, at the anode is given by

$$\epsilon_x = -i(k\xi_W - \Omega\xi_P) \quad (22a)$$

and is zero since the anode is metal. The Lorentz condition is given by

$$i(k\xi_P - \Omega\xi_W) + \xi_{Py}' = 0\,, \quad (22b)$$

where ξ_{Py} is the y component of the vector potential (recall that ξ_P is the x component). At the anode ξ_{Py}' is zero since it is zero inside the anode, and integrating the y component of Ampere's law

$$\xi_{Py}'' - (k^2-\Omega^2)\xi_{Py} = j_y \quad (22c)$$

tells us that it is also zero at the anode surface. We therefore know from (22a) and (22b) that the potentials are both zero at the anode. These are the same boundary conditions that are found for all slow (sub-speed-of-light), or non-TEM (transverse-electromagnetic mode) modes in transmission line theory. Using these boundary conditions the second term on the left-hand side of (20) is zero and consequently,

$$2\omega\nu = \pi \int dH \left[h^{-2}(1+v_d^2)\left[\frac{\partial v_d}{\partial\theta}\right]^{-1}\frac{\partial J}{\partial\theta} \times (F_W+F_P)(F_W+F_P)^*\right]_{\theta_r(H)}$$
$$= \pi \int dH \left[\left[\frac{\partial H}{\partial W}\right]^{-2}\left[\frac{\partial v_d}{\partial\theta}\right]^{-1}\frac{\partial J}{\partial\theta} \times |F_W+F_P|^2\right]_{\theta_r(H)} \quad (23)$$

Finally we must argue the assumption that the integral

$$\int_{y_c}^{y_a} dy(\xi_W^*\xi_W - \xi_P^*\xi_P) \quad (24)$$

is positive. Equations (17) are merely the expressions

$$\xi_P'' + (k^2-\Omega^2)\xi_P = j\,,$$
$$\xi_W'' + (k^2-\Omega^2)\xi_W = \rho\,,$$

where j and ρ are the current and charge perturbations, respectively, and $(k^2-\Omega^2)$ is positive for these slow (phase velocity less than the speed of light) waves. All zero order terms in the continuity equation are zero. The first order continuity equation is

$$ikj - i\Omega\rho + j_y' = 0 \quad \text{or} \quad j = \frac{\Omega}{k}\rho - \frac{j_y'}{ik}\,. \quad (25)$$

Clearly, if j_y were zero, we would get that $\xi_P = (\Omega/k)\xi_W$ and since $|\Omega| < |k|$, (24) would be positive. For this case we can make an even stronger statement, however. The perturbed currents in the x direction (j) and the y direction (j_y) are of the same magnitude. This must be so since the x and y velocities are similar, so the magnetic forces, which are comparable to the electric forces, must be about the same. But $j_y' \sim j_y/(y_a - y_c)$, and since we assume that $k(y_a - y_c) \ll 1$, the two terms on the right-hand side of (25) must be much larger than j. If this is so, then $|j| \ll |(\Omega/k)\rho| < |\rho|$ and $|\xi_P| \ll |\xi_W|$. Therefore, (24) is positive.

The $\partial v_d/\partial\theta$ term in (23) is always positive for flows with only electrons, so the sign of ν is the same as that of $\partial J/\partial\theta$. The criterion derived is global in the sense that it depends upon all of the electrons that drift with the phase velocity. There is no *a priori* reason for $\partial J/\partial\theta$ to have the same sign over the entire H integral along the $\theta_r(H)$ path. For a case where the sign does change, the positive part of the integral must dominate the negative part for instability to occur.

In many flows the drift velocity is dependent mostly upon θ. For these flows the ratio of the partial derivatives of J and v_d looks like $\partial J/\partial v_d$, and so the question is approximately whether electron density in P,W space increases with drift velocity. If so the flow is unstable. It is important to look at the density gradient in P,W space and not x,y space. For example, for Brillouin flow[1-6] the charge density and drift velocity both increase with distance from the cathode, so that dn_e/dv_d is positive. However, in P,W space the distribution function is nonzero only at the point $P=0$, $W=0$, and the criterion for stability is met. This is a moot point in this particular case, however, because of the highly-degenerate nature of the Brillouin flow solution. For these flows, it can be shown that all of the electrons lie at the intersection between the stable and unstable laminar lines (Fig. 2), and all electrons at that point can be shown to have infinite gyrolength and gyroperiod. Therefore, the assumptions of the theory are not met. For the case of the quasilaminar flows[8,9] the distribution function is also infinite at $P=0$, $W=0$ and zero elsewhere. Unlike the Brillouin case all

electron orbits have the same shape. For this case the gyrolength and gyrofrequency are finite and the theory predicts stability.

The growth or damping of the waves predicted by (23) is very similar to the growth or damping of Landau waves.[21] In either case, the wave tends to slow the electrons moving slightly faster than the wave and accelerate those moving slower than the wave. If there are more electrons moving faster than the wave, there will be a net transfer of particle energy to the wave and it will grow.

CONCLUSION

We have found a general criterion for instability of magnetically insulated electron flow for long wavelengths and low frequencies compared to the gyrolengths and gyrofrequencies of the electrons involved. The growth rate is determined by (23). For many situations the stability criterion can be stated roughly as the following: Flows with electron populations in P, W space increasing with drift velocity are unstable. This criterion has been helpful in understanding two-dimensional, time-dependent simulations.[22]

ACKNOWLEDGMENTS

The authors would like to thank S. E. Rosenthal and J. P. Quintenz for helpful discussions on this theory. They would also like to thank S. A. Slutz for useful suggestions on the manuscript.

[1]L. Brillouin, Phys. Rev. **67**, 260 (1945).
[2]O. Buneman, Proc. Cambridge Philos. Soc. **50**, 77 (1954).
[3]J. M. Creedon, J. Appl. Phys. **46**, 2946 (1975); **48**, 1070 (1977).
[4]M. Y. Wang, Appl. Phys. Lett. **33**, 284 (1978).
[5]J. Swegle, Phys. Fluids **26**, 1670 (1983).
[6]R. C. Davidson, K. T. Tsang, and J. A. Swegle, Phys. Fluids **27**, 2332 (1984).
[7]P. T. Kirstein, G. S. Kino, and W. E. Waters, *Space Charge Flow* (McGraw-Hill, New York, 1967).
[8]A. Ron, A. A. Mondelli, and N. Rostoker, IEEE Trans. Plasma Sci. **PS-1**, 85 (1973).
[9]R. V. Lovelace and E. Ott, Phys. Fluids **17**, 1263 (1974).
[10]C. W. Mendel, Jr., J. Appl. Phys. **50**, 3830 (1979).
[11]C. W. Mendel, Jr., D. B. Seidel, and S. A. Slutz, Phys. Fluids **26**, 3629 (1983).
[12]G. G. MacFarlane and H. G. Hay, Proc. Phys. Soc. London Ser. **63**, 409 (1949).
[13]O. Buneman, R. H. Levy, and L. M. Linson, J. Appl. Phys. **37**, 3203 (1966).
[14]J. Swegle and E. Ott, Phys. Fluids **24**, 1821 (1981).
[15]A. V. Gordeev, A. V. Gulin, and N. P. Savenkova, Fiz. Plazmy **7**, 930 (1981) [Sov. J. Plasma Phys. **7**, 512 (1981)].
[16]R. C. Davidson and K. T. Tsang, Phys. Rev. A **30**, 488 (1984).
[17]D. B. Seidel and C. W. Mendel, Jr., IEEE International Conference on Plasma Science, St. Louis, Mo., 1984 (unpublished).
[18]B. Goplen, R. E. Clark, J. McDonald, and W. M. Bollen, Mission Research Corportation Report No. MRC/WDC-R-068, 1983 (unpublished).
[19]A. Palevsky and A. Drobot, in Proceedings of the Ninth Conference on the Numerical Simulation of Plasmas, Evanston, Illinois, 1980 (unpublished).
[20]O. Buneman, IEEE International Conference on Plasma Science, Santa Fe, New Mexico, 1981 (unpublished).
[21]D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York, 1964), Appendix B.
[22]D. B. Seidel, S. E. Rosenthal, and C. W. Mendel, Jr., Bull. Am. Phys. Soc. **29**, 1276 (1984).