Solitons in weakly nonlinear electron-positron plasmas and pulsar microstructures

U. A. Mofiz, U. de Angelis, and A. Forlani
Department of Physics, University of Naples, Naples, Italy
(Received 17 February 1984)

The propagation of electromagnetic waves in weakly nonlinear, relativistic electron-positron plasmas is investigated. It is shown that the resulting waves can have a solitonic envelope, both for linearly and circularly polarized waves. A dispersion relation is derived, and the results for the linearly polarized case are used to check a recent suggestion of possible self-modulational formation of pulsar microstructure.

I. INTRODUCTION

In a recent paper, Chian and Kennel[1] (hereafter referred to as CK) propose an interesting mechanism to explain the ultrashort intensity variations within individual pulses in pulsar radio emission (Cordes;[2] see also the paper by CK for further details).

In the model of CK nonlinearities arising from wave-intensity-induced particle-mass variation may excite a modulational instability of circularly and linearly polarized pulsar radiation: The result is a modulating solitonic envelope on a high-frequency carrier wave and the microstructure analysis shows that the number N of micropulses within an individual pulse and the temporal pulse width τ are within the observed ranges ($N \sim 10^2-10^3$, $\tau \sim 1$ μsec) provided that the emission takes place in regions of "low" particle density (plasma frequency ω_p much less than wave frequency).

The paper by CK is a contribution in favor of "temporal models" of pulse microstructure as opposed to "beaming models." The reader is referred to the paper by CK for a detailed discussion of the arguments above and below; we shall only report, from the paper by CK, what is necessary in order to understand the present paper, which is organized as follows: In Sec. II we show that the calculation by CK is not, in our opinion, fully consistent. In particular, (a) their solitonic solution for circularly polarized waves is not in complete agreement with a similar solution recently proposed by us (Mofiz *et al.*[3]); (b) for linearly polarized waves, in their case of constant streaming velocity, there is *no* solitonic solution.

In Sec. III we show that there is a solitonic solution, in the general case of streaming velocity not constant, *both* for circularly and linearly polarized waves. A dispersion relation for this solution is also derived following the method of Karpman and Krushkal.[4]

In Sec. IV our solution is used for a discussion of pulsar microstructures along the lines of the paper by CK: Our conclusion is that the model is in agreement with observed quantities (N,τ) but that both cases $\omega/\omega_p \gg 1$ and $\omega/\omega_p \sim 1$ are possible and hence no conclusion on the emission region can be reached. Some other aspects of the present theory, its shortcomings, and future developments are discussed in the conclusions.

II. SOLITONS IN e^+e^- PLASMAS: CASE OF CONSTANT STREAMING VELOCITY

Consider an electromagnetic (em) wave propagating in a cold, locally neutral, electron-positron plasma with no ambient magnetic field: The direction of wave propagation is taken along the z axis and it is assumed that all quantities do not depend on x and y but only on z and time t. This system is described by the two-fluid relativistic equations for the plasma, the wave equation for the vector potential $\mathbf{A}$ and the continuity equation:

$$\left[\frac{\partial}{\partial t}+v_{\alpha z}\frac{\partial}{\partial z}\right]\mathbf{U}_\alpha=\frac{q_\alpha}{mc^2}\left[-\frac{\partial \mathbf{A}}{\partial t}+\mathbf{v}_\alpha\times(\nabla\times\mathbf{A})\right], \quad (1)$$

$$\frac{\partial^2\mathbf{A}}{\partial t^2}-c^2\frac{\partial^2\mathbf{A}}{\partial z^2}=4\pi c\sum_\alpha n_\alpha q_\alpha \mathbf{v}_\alpha , \quad (2)$$

$$\frac{\partial n_\alpha}{\partial t}+\frac{\partial}{\partial z}(n_\alpha v_{\alpha z})=0 , \quad (3)$$

where α refers to $p=e^+$ and $e=e^-$, $q_\alpha=\pm e$ is the charge, $\mathbf{v}_\alpha$ the particle velocity, $\mathbf{U}_\alpha=\mathbf{P}_\alpha/mc$ the dimensionless momentum (m is the electron rest mass):

$$\mathbf{U}_\alpha=\gamma_\alpha\frac{\mathbf{v}_\alpha}{c},\ \ \gamma_\alpha=(1+|\mathbf{U}_\alpha|^2)^{1/2} , \quad (4)$$

and n_α is the particle density ($n_p=n_e=n$ for the local neutrality condition).

The "streaming velocity" is the z component of the particle velocity and, in the above model, it can be shown to be the same for electrons and positrons. Under the assumption of constant streaming velocity $\beta=v_z/c=\beta_0$, the system (1)–(3) has been solved, for the case of a circularly polarized wave

$$\mathbf{A}(z,t)=\alpha(z,t)(\hat{x}\cos\omega t+\hat{y}\sin\omega t) \quad (5)$$

by CK and by the authors (Mofiz *et al.*,[3] hereafter referred to as I), with the only "initial" difference that CK work in the particle rest frame ($\beta_0=0$) while we stay in the laboratory frame.

Under the common assumptions of weak nonlinearity ($|\alpha|^2 \lesssim 1$) and slowly time varying modulating amplitude ($\partial^2\alpha/\partial t^2 \sim 0$), the nonlinear Schrödinger equation (NLS) is shown to govern the amplitude evolution: The soliton is a well-known solution of this equation (see, e.g., Whitham[5]).

We refer the reader to the paper by CK and to I for all the details of the calculation: In the present paper we wish to comment on the differences in the two results and, in particular, on the solution of CK for the linearly polarized case that is relevant to pulsar radiation.

First of all, a general remark should be made: The NLS is not Lorentz invariant and therefore we think that it is not correct to transform the rest-frame solutions to the laboratory frame. In order to make final comparisons with observational data in our opinion it is necessary to work directly in the laboratory frame in this case. A second point is that, as shown in I, for $\beta_0=0$ (or in the rest frame) the equation of motion implies that $|\alpha|^2$ is only a function of time, not of z: In this case $\alpha(z,t)$ cannot be a solution of NLS.

The first conclusion is that, under the stated assumptions, there is a solitonic solution for circularly polarized waves for any constant value (except zero) of the streaming velocity and it has to be worked out in the laboratory frame. In I it is also shown that this soliton is superluminous, its phase velocity being $V=c/\beta_0$: This is another reason why β_0 cannot be zero.

For linearly polarized waves the procedure in CK is essentially the following: A nonlinear dispersion relation for an electron plasma, as given by Chian and Clemmow,[6] is generalized to the case of an electron-positron plasma and the governing equation for the wave amplitude is derived using the (inverse) Karpman-Krushkal method. The result is NLS again, hence the conclusion that a solitonic solution exists also for the linearly polarized case.

The fault in this procedure, in our opinion, is that when the Karpman-Krushkal method is used and an equation derived from a dispersion relation, it is also necessary to check consistency with the remaining equations or assumptions of the model: In fact, we think that for a linearly polarized wave, in the case of constant streaming velocity (whether it is zero or not), there is no solitonic solution since the NLS in this case is not consistent with the equation of motion. To show this we recall here (from I) some of the steps in the solution of Eqs. (1)—(3); these will also be necessary for the general solution in the next section.

The equation of motion can be separated into a longitudinal (z) and a transverse (x,y) equation and the last one can be immediately integrated to

$$\mathbf{U}_{\alpha\perp}=\hat{x}U_{\alpha x}+\hat{y}U_{\alpha y}=-\frac{q_\alpha}{mc^2}\mathbf{A} \tag{6}$$

and this gives

$$\gamma_\alpha=(1+|\mathbf{U}_\alpha|^2)^{1/2}=\frac{(1+\lambda|\mathbf{A}|^2)^{1/2}}{(1-\beta^2)^{1/2}}\equiv\gamma, \quad \left[\lambda\equiv\frac{e^2}{m^2c^4}\right] \tag{7}$$

and

$$\mathbf{v}_\alpha=(\mathbf{v}_{\alpha\perp},c\beta), \quad \mathbf{v}_{\alpha\perp}=-\frac{q_\alpha}{mc}\frac{(1-\beta^2)^{1/2}}{(1+\lambda|\mathbf{A}|^2)^{1/2}}\mathbf{A}\,. \tag{8}$$

The transverse particle motion is therefore solved (in general) in terms of streaming motion (β) and vector potential.

The longitudinal component of the equation of motion can be written in the form

$$\left[\frac{\partial}{\partial t}+c\beta\frac{\partial}{\partial z}\right]\beta+(1-\beta^2)\left[\beta\frac{\partial F}{\partial t}+c\frac{\partial F}{\partial z}\right]=0\,, \qquad F=\ln(1+\lambda|\mathbf{A}|^2)^{1/2}\,. \tag{9}$$

For a linearly polarized wave $\mathbf{A}=(A,0,0)$ in the case of $\beta=\beta_0$, Eq. (9) implies

$$A=A\left[z-\frac{c}{\beta_0}t\right] \quad (\text{for } \beta_0\neq 0)\,, \qquad A=A(t) \quad (\text{for } \beta_0=0 \text{ or in the rest frame})\,. \tag{10}$$

Also for $\beta=\beta_0$ we have $n=n_0=$const from the equation of continuity and a choice of initial value $n(t=0)=n_0$ (see I), $\gamma=\gamma(A^2)$ from (7) and $\mathbf{v}=\mathbf{v}(\mathbf{A})$ from (8).

The consequences are as follows.

(i) The wave can no longer be written as a modulating amplitude on a carrier high-frequency wave (ω).

(ii) The wave equation can be shown to have periodic solutions (elliptic functions for $\beta_0\neq 0$, plane waves for $\beta_0=0$) but no localized solutions.

(iii) For $\beta_0=0$ (or in the rest frame), $A=A(t)$: This could still be written in the form $\alpha(t)\cos\omega t$ but of course the time-dependent amplitude would not satisfy a NLS.

We can then draw our second conclusion: There is no solitonic solution, in the stated assumptions, for linearly polarized waves when the particle streaming velocity is a constant. In the next section we show that such a solution exists when the $\beta=\beta_0$ assumption is removed.

III. SOLITONS IN e^+e^- PLASMAS: GENERAL CASE

In this section we give a solution for the system (1)—(3) describing an em wave propagating in a neutral electron-positron plasma. The transverse particle motion has already been solved in Eqs. (6)—(8). It is possible to have the same analytical description for the case of a linearly polarized wave (LP) and of a circularly polarized wave (CP) introducing the real (complex) dimensionless quantity

$$A=\begin{cases}A_x\lambda^{1/2} & \text{for LP}\\ (A_x+iA_y)\lambda^{1/2} & \text{for CP}\,,\end{cases} \qquad |A|^2=\lambda|\mathbf{A}|^2=\begin{cases}\lambda A_x^2 & \text{for LP}\\ \lambda(A_x^2+A_y^2) & \text{for CP}\,.\end{cases} \tag{11}$$

Taking into account the transverse solution (7) and (8) to express the current $\mathbf{J}$, the wave equation in terms of A takes the form

$$\frac{\partial^2 A}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}=\frac{\omega_p^2}{c^2}\frac{n}{n_0\gamma}A\,, \tag{12}$$

where n_0 is a constant (background) density and

$$\omega_p=\left[\frac{8\pi e^2}{m}n_0\right]^{1/2} \tag{13}$$

(m is the rest mass) is the plasma frequency for an e^+e^- plasma.

The (real or complex) wave equation (12) has to be solved with the remaining equations: Continuity,

$$\frac{\partial}{\partial t}\left[\frac{n}{n_0}\right]+c\frac{\partial}{\partial z}\left[\beta\frac{n}{n_0}\right]=0 , \tag{14}$$

and longitudinal motion that can be written as

$$\begin{aligned}\left[\frac{\partial}{\partial t}+c\beta\frac{\partial}{\partial z}\right]\gamma&=\frac{1}{2\gamma}\frac{\partial |A|^2}{\partial t} ,\\ \left[\frac{\partial}{\partial t}+c\beta\frac{\partial}{\partial z}\right](\gamma\beta)&=-\frac{c}{2\gamma}\frac{\partial|A|^2}{\partial z} .\end{aligned} \tag{15}$$

We look for a solution in the form of a space- and time-dependent amplitude modulation on a (high-frequency ω) carrier plane wave:

$$A(z,t)=\alpha(z,t)e^{-i\omega t} , \tag{16}$$

where the real part of the solution will be taken for the case of linear polarization.

Under the assumption $(1/\omega)\alpha_{tt}\sim 0$ the wave equation gives

$$i\frac{2\delta}{\omega_p}\alpha_t+\frac{c^2}{\omega_p^2}\alpha_{zz}+\left[\delta^2-\frac{n}{n_0\gamma}\right]\alpha=0 \quad \left[\delta=\frac{\omega}{\omega_p}\right] . \tag{17}$$

Writing down

$$\alpha(z,t)=\sigma(z,t)e^{i\phi(z,t)}, \quad \sigma,\phi \text{ real} \tag{18}$$

from (17) we have the two equations:

$$\frac{\partial}{\partial t}\sigma^2+\frac{c^2}{\omega}\frac{\partial}{\partial z}(\phi_z\sigma^2)=0 , \tag{19a}$$

$$\frac{2\delta}{\omega_p}\sigma\phi_t-\frac{c^2}{\omega_p^2}\sigma_{zz}+\left[\frac{c^2\phi_z^2}{\omega_p^2}-\delta^2\right]\sigma+\frac{n}{n_0\gamma}\sigma=0 . \tag{19b}$$

In the linear-phase approximation (see, e.g., Yu *et al.*[7]):

$$\phi(z,t)=\psi(z)+\Phi(t), \quad \phi_z,\phi_t \text{ constants} . \tag{20}$$

Eq. (19a) gives the condition

$$|\alpha|^2=\sigma^2=f(\xi), \quad \xi=z-V_0t, \quad V_0=\frac{c^2\phi_z}{\omega} . \tag{21}$$

Then from Eqs. (19b) and (7) also γ, β, and n are functions of ξ: The continuity equation can then be integrated to give

$$\frac{n(\xi)}{n_0}=\frac{\mu}{V-\beta(\xi)} , \tag{22}$$

where μ is a constant and $V\equiv V_0/c$. A constant of motion can also immediately be found in this case from the equation of motion (15): This is integrated once to give

$$\gamma(\xi)[V\beta(\xi)-1]=\Gamma, \quad \Gamma \text{ constant} . \tag{23}$$

Since we are looking for localized solutions we can evaluate the constants Γ, μ making use of the conditions

$$|\alpha|^2(\xi\rightarrow\pm\infty)=0, \quad \beta(\xi\rightarrow\pm\infty)=\beta_0, \quad n(\xi\rightarrow\pm\infty)=n_0 ,$$

where β_0, n_0 are to be taken as background (initial) constant values; then we have

$$\gamma(\xi\rightarrow\pm\infty)=\frac{1}{(1-\beta_0^2)^{1/2}}=\gamma_0 . \tag{24}$$

Taking the limits $\xi\rightarrow\pm\infty$ in Eqs. (22) and (23) we have

$$\mu=V-\beta_0, \quad \Gamma=\gamma_0(V\beta_0-1) . \tag{25}$$

Eliminating $\beta(\xi)$ between (7) and (23) we can write γ as a function of the amplitude:

$$\gamma=\frac{\Gamma+V(\Gamma^2+V^2-1)^{1/2}[1+(V^2-1)/(\Gamma^2+V^2-1)|\alpha|^2]^{1/2}}{V^2-1} , \tag{26}$$

where the sign has been chosen for the case of a superluminous ($V>1$) solution.

Making use of (25) and expanding the square root for the weakly nonlinear case $|\alpha|^2\lesssim 1$ we have

$$\gamma=\gamma_0+\frac{V}{V-\beta_0}\frac{|\alpha|^2}{2\gamma_0} \tag{27}$$

and

$$\beta=\frac{1}{V}\left[1+\frac{\gamma_0(V\beta_0-1)}{\gamma_0+\dfrac{V}{V-\beta_0}(|\alpha|^2/2\gamma_0)}\right] , \tag{28}$$

$$\frac{n}{n_0}=(V-\beta_0)\left[\left[V-\frac{1}{V}\right]-\frac{\gamma_0(V\beta_0-1)}{V\gamma_0+[V^2/(V-\beta_0)](|\alpha|^2/2\gamma_0)}\right]^{-1} . \tag{29}$$

We can now evaluate the term $(n/n_0\gamma)$ from Eqs. (27)–(29) and substitute in Eq. (17) for the complex amplitude α to get

$$i\alpha_t+\frac{c^2}{2\omega}\alpha_{zz}+\frac{\omega_p}{2\delta}\left[\delta^2-\frac{1}{\gamma_0}+\frac{V^2-1}{\gamma_0^2(V-\beta_0)^2}\frac{|\alpha|^2}{2\gamma_0}\right]\alpha=0 .$$

If the linear term is removed with the substitution

$$\alpha=\bar{\alpha}e^{iRt}, \quad R=\frac{\omega_p}{2\delta}\left[\delta^2-\frac{1}{\gamma_0}\right] \tag{30}$$

the equation for $\bar{\alpha}$ is the nonlinear Schrödinger equation

$$i\bar{\alpha}_t+P\bar{\alpha}_{zz}+Q\bar{\alpha}\,|\,\bar{\alpha}\,|^2=0 \tag{31}$$

with

$$P=\frac{c^2}{2\omega}, \quad Q=\frac{\omega_p^2}{2\omega}\frac{V^2-1}{2\gamma_0^3(V-\beta_0)^2} . \tag{32}$$

A well-known solution of (31) is the soliton (Whitham[5]):

$$\bar{\alpha}(z,t)=\alpha_0 \operatorname{sech}\left[\left|\frac{Q}{2P}\right|^{1/2}\alpha_0(z-V_0t)\right]e^{i(k_0z-\omega_0t)} \tag{33}$$

with

$$k_0=\frac{\omega V}{c}, \quad \omega_0=\tfrac{1}{2}\omega V^2-\frac{\omega_p^2}{4\omega}\frac{V^2-1}{2\gamma_0^3(V-\beta_0)^2}\alpha_0^2 . \tag{34}$$

The solution for the wave is then

$$\lambda^{1/2}\mathbf{A}(z,t)=\nu_0 \operatorname{sech}\left[\left|\frac{Q}{2P}\right|^{1/2}\nu_0(z-V_0t)\right]\begin{cases}\hat{x}\cos\Theta(z,t) & \text{for LP}\\ [\hat{x}\cos\Theta(z,t)+\hat{y}\sin\Theta(z,t)] & \text{for CP}\end{cases} , \tag{35}$$

where $\nu_0=\lambda^{1/2}A_0$ is the dimensionless amplitude and the amplitude-dependent phase $\Theta(z,t)$ is given by

$$\Theta(z,t)=\frac{\omega V}{c}z-\left[\tfrac{1}{2}(1+V^2)\omega+\frac{\omega_p^2}{2\omega}\frac{1}{\gamma_0}-\frac{\omega_p^2}{4\omega}\frac{V^2-1}{2\gamma_0^3(V-\beta_0)^2}\nu_0^2\right]t . \tag{36}$$

These solutions have wave envelopes in the form of wave trains of solitons, both for circularly and linearly polarized waves, and this is the first requirement to account for the structure of pulsar radiation.

A dispersion relation for the present weakly nonlinear case can be obtained with the Karpman-Krushkal[4] method:

$$\omega=\omega(k,\nu_0^2), \quad P=\frac{1}{2}\frac{\partial^2\omega}{\partial k^2}, \quad Q=-\frac{\partial\omega}{\partial\nu_0^2} , \tag{37}$$

where P and Q are given by Eq. (32). The result is

$$\omega^2=c^2k^2+\omega_p^2-\omega_p^2\frac{V^2-1}{2\gamma_0^3(V-\beta_0)^2}\nu_0^2 \tag{38}$$

both for circularly and linearly polarized waves: A result of the present analysis is that there is no difference in the dispersion relation for the two cases.

IV. PULSAR MICROSTRUCTURE

The requirement for self-modulational formation of microstructures is occurrence of modulational instability: This can be seen to occur for $PQ>0$ (see the paper by CK) and in our case this condition is satisfied for the chosen superluminous solution ($V>1$).

Now we can estimate the number N of micropulses within a single pulse and the pulse temporal width τ: these are given by

$$N=\left[\frac{2P}{Q}\right]^{1/2}\frac{1}{\nu_0}\frac{k}{2\pi}, \quad \tau=\left[\frac{2P}{Q}\right]^{1/2}\frac{1}{\nu_0}\Big/V_0 . \tag{39}$$

Taking P and Q from Eq. (32) and the wave number k from Eq. (38) we have

$$N=\frac{1}{\pi\nu_0}(V-\beta_0)\left[\frac{4\gamma_0^3}{V^2-1}\right]^{1/2}\left[\frac{\omega^2}{\omega_p^2}-1+\frac{V^2-1}{2\gamma_0^3(V-\beta_0)^2}\nu_0^2\right]^{1/2} , \tag{40}$$

$$\tau=\frac{2}{\omega_p V\nu_0}(V-\beta_0)\left[\frac{\gamma_0^3}{V^2-1}\right]^{1/2} . \tag{41}$$

Two cases are possible (for $V\sim1$): (a) Large initial (background) particle energy $\gamma_0 \gg 1 (\beta_0\sim1)$:

$$N\sim\frac{\gamma_0^{3/2}}{\nu_0}\frac{\omega}{\omega_p}, \quad \tau\sim\frac{\gamma_0^{3/2}}{\omega_p\nu_0} . \tag{42}$$

(b) Low initial particle energy $\gamma_0\sim1$ ($\beta_0\ll1$):

$$N\sim\frac{\omega}{\omega_p\nu_0}, \quad \tau\sim(\omega_p\nu_0)^{-1} . \tag{43}$$

If the emission takes place in regions where (see CK) $\gamma_0\approx10^2$, $\omega\sim\omega_p\sim10^8\ \text{sec}^{-1}$, the first case would give

$$N\sim10^3\nu_0^{-1}, \tau\sim10^{-5}\nu_0^{-1}\ \text{sec} \tag{44}$$

which are too high even for $\nu_0\sim1$.

In the second case, however, the values are in the acceptable range both for $\omega/\omega_p\gg1$ ($\nu_0\sim10^{-1}-10^{-2}$) and $\omega/\omega_p\sim1$ ($\nu_0\sim10^{-3}-10^{-2}$) which are still reasonable values in the pulsar magnetosphere. The conclusion of CK is therefore correct (emission in low-density regions) for moderately large amplitudes but the opposite conclusion can also be valid for weak amplitudes. In any case the initial (background) particle energy should be small for the model to agree with observed pulse characteristics. Another conclusion or prediction of the present theory, in agreement with CK, is that micropulses of higher intensity have narrower pulse widths [see Eq. (43)] as suggested by Ferguson.[8]

V. CONCLUSIONS

We have shown that self-modulational formation of pulse microstructures, as suggested by Chian and Kennel,[1] is possible in electron-positron plasmas in pulsar magnetospheres: A consistent calculation of circularly and linearly polarized electromagnetic waves propagating in a neutral e^+e^- plasma, taking into account the nonlinear effects of particle-mass variation, results into a wave structure in the form of a solitonic envelope on a high-frequency carrier wave. The resulting pulse structure, number of micropulses within one single pulse, and temporal pulse width, is in agreement with observational data on pulsar radiation.

A difference with the calculation by CK and Mofiz *et al.*[3] is that here the particle density and streaming velocity are no longer constants: A propagating density hole is formed and the particle streaming velocity is increased in regions of higher field intensity [Eqs. (28) and (29)].

The main fault of the present calculation, in its application to pulsar radiation, is the absence of an ambient magnetic field. In pulsar magnetospheres the dipole field is estimated in the vicinity of 10^{12} G and a more realistic check of self-modulational formation of pulse microstructures should of course take the magnetic field into account.

ACKNOWLEDGMENTS

One of us (U.A.M.) thanks the International Center of Theoretical Physics (ICTP, Trieste) for a grant under the ICTP Programma for training and research in Italian laboratories.

[1]A. C. L. Chian and C. F. Kennel, Astrophys. Space Sci. **97**, 9 (1983).

[2]J. M. Cordes, Space Sci. Rev. **24**, 567 (1979).

[3]U. A. Mofiz, U. de Angelis, and A. Forlani, Plasma Phys. Control. Fusion **26**, 1099 (1984).

[4]V. I. Karpman and E. M. Krushkal, Zh. Eksp. Teor. Fiz. **55**, 530 (1968) [Sov. Phys.—JETP **28**, 277 (1969)].

[5]G. B. Whitham, *Linear and Non Linear Waves* (Wiley, New York, 1974), p. 601.

[6]A. C. L. Chian and P. C. Clemmow, J. Plasma Phys. **14**, 505 (1975).

[7]M. Y. Yu, P. K. Shukla, and N. L. Tsintsadze, Phys. Fluids **25**, 1049 (1982).

[8]D. C. Ferguson, in *Pulsars,* edited by W. Sieber and R. Wielebinski (Reidel, Dordrecht, 1981).