Inverse-scattering theory for the non-spherically-symmetric three-dimensional plasma wave equation

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The three-dimensional inverse problem for the scalar time-domain plasma wave equation is discussed using causality and time-reversal invariance. As shown by Balanis in one dimension and by Morawetz and others in three dimensions, the reconstruction of the potential requires only the discontinuity of the wave front at the characteristic surface. A complete solution that can be made rigorous is given for near-field inverse scattering for smooth compact potentials. Two formal methods for far-field inverse scattering from non-spherically-symmetric potentials are presented.

I. INTRODUCTION

Direct and inverse problems are pairs of problems where the solution of one problem depends on all or part of the solution to the other.[1–4] In direct scattering theory, the equations of motion, the interaction potential, and the boundary conditions are given. The problem is to calculate the far-field scattering amplitude. In inverse-scattering theory, the equations of motion, boundary conditions, and the far-field scattering amplitude are given. The inverse problem is to calculate the interaction potential and exact wave. In addition, *a priori* information may be given. An inverse-scattering problem is called a pure inverse problem if complete, noise-free scattering data are given and is called an applied inverse problem if the data are band-limited, corrupted by noise or sampled or any combination thereof. The interpretation of any scattering experiment requires the solution of an applied inverse problem. The presence of noise in these problems requires the formulation of a statistical measurement model, which is an ill-posed problem but one that is regularized. Its solutions are nonunique, even when the corresponding pure inverse problem has a unique solution; *a priori* knowledge is needed to determine the "best" solution. There is always a minimum spatial resolution to this best solution. The present study will be primarily restricted to the pure inverse problem although a comment on the sensitivity of the inverse methods to noise will be made.

In models of electromagnetic waves propagating in the ionosphere,[5–7] when electron collisions and static magnetic fields are negligible, the relative permittivity can be written as

$$\frac{\epsilon(\mathbf{k},\mathbf{x})}{\epsilon_0}=1-\frac{1}{k^2}V(\mathbf{x}) , \qquad (1.1)$$

where ϵ_0 is the permittivity of free space, $k=2\pi/\lambda$ is the magnitude of wave number, $\epsilon(\mathbf{k},\mathbf{x})$ is the variable permittivity, and $V(\mathbf{x})$ is an "interaction potential." Also, $k=\omega/c$ where ω is the angular frequency and c is the phase velocity of light in free space. Then the model consists of the wave equation

$$\left[\Delta u-\frac{1}{c^2}u_{tt}-V(\mathbf{x})u\right](x,t)=0 \qquad (1.2)$$

together with initial conditions and boundary conditions. Here u is the amplitude of the scalar wave, Δ is the three-dimensional (3D) Laplacian, and $u_{tt}=\partial^2u/\partial t^2$ in the center-of-momentum Lorentz frame of the (massless) wave and the medium. The D'Alembertian in (1.2) is Lorentz invariant. The potential function V will be assumed to be real and continuous, repulsive [i.e., $V(x)\geq 0$, everywhere], compact, and to possess a smooth boundary. The coordinates (x,t) are in $R^3\times[-T,T]$. This model is applicable in the rest frame of the atmospheric electrons comprising the potential. The Lorentz transformation to other inertial frames would transform the potential into $V=V(\mathbf{x}',t')$ and the simplicity of the model would disappear.

Inverse scattering with spherical symmetry has been established for some time[8,9,1–4] and most of these studies were carried out in the frequency domain. Bleistein[10] formulated a version of inverse scattering for general three-dimension convex scatters in the time domain, based upon the Kirchoff approximation. Rose and Richardson[11] have presented a time-domain Born-Neumann approximation. Morawetz[12] has written down exact formal solutions to several three-dimensional inverse problems. For the time-independent Schrödinger equation, Moses[13] and Newton[14] have written two series of papers which add a good deal to this problem. The plasma wave equation in Eq. (1.2) reduces to Schrödinger's equation for an initially quiescent, monochromatic wave. This feature has been exploited in a separate study.[15]

The goal of this study is to provide a clear physical formulation of the time-domain inverse-scattering theory in the plasma wave equation. Intuitive clarity rather than great generality or rigor is intended. The causal structure of solutions to Eq. (1.2) play a key role in the time domain making it simpler than the frequency domain in many respects. Hyperbolic partial differential equations such as Eq. (1.2) propagate their singularities without smoothing or attenuation. Balanis[6] used this structure in his development of a one-dimensional inverse theory for the wave equation. He first derived the fundamental identity in one dimension (1D) and correctly used the impulse response function. Morawetz[12] and independently DeFacio[16] have used this fact to derive a useful three-dimensional identity. As this identity is clearly based on the sharpness of a singular surface in the time domain, it will be spread over a wide band in the frequency domain. Consequently, the frequency-domain discussions of inverse scattering are somewhat nonintuitive in this respect. The method discussed here has an additional simplification. Instead of the five-variable data function required by Newton[14] and others, a three-variable data function suffices for the present study because of the restrictions on the potential. Thus, the data function depends upon the same number of variables as the potential. As a consequence, the "miracle condition" of Newton appears constructively during the reconstruction, rather than as a cumbersome consistency check which must be applied after the process is completed. See also Cheney[17] for additional discussion of the "miracle."

In Sec. II causality will be discussed and the key identity will be reviewed. In Sec. III an exact (aptly rigorous[15,17]) near-field inverse method is given. In Sec. IV the far-field inverse problem will be formulated using the physical concept of causality—as carried in the fundamental identity—together with the time-reversal invariance. The far-field approach will be expressed in terms of both nonlinear differential and integral equations. The problems of existence and uniqueness of solutions to these equations will not be addressed here as the goal of this paper is to provide a physically motivated approach. For this reason the discussion of far-field inverse scattering is limited to a new approach rather than a complete solution. The paper is concluded with a discussion section.

II. TIME-DOMAIN SCATTERING FOR THE PLASMA WAVE EQUATION

The causal structure of hyperbolic equations in odd-space dimensions is well known.[18–20] Initial data on a plane at $t=0$ as shown in Fig. 1 fill the cone of influence K which is bounded by D_0 and D_1. The solution identically vanishes, $u(\mathbf{x},t)\equiv 0$, outside of K. This vanishing is a consequence of causality. Usually causality is used to provide analyticity in $k^2=\omega^2/c^2$ in the reduced Helmholtz equation. The main point to the present paper is to show that the inverse-scattering problem for Eq. (1.2) is actually *simpler* in the time domain. In particular, the leading singularity at the edge of the advancing light cone determines the potential.

The experiments described by both the direct and inverse-scattering theories for Eq. (1.2) begin with an initial wave $u_0^+(\mathbf{x},t)$ at large negative time which is sharp in time and is incident along the unit vector $\hat{e}_i$:

$$u(\mathbf{x},t)\to u_0^+(\mathbf{x},t)=\delta(ct-\hat{e}_i\cdot\mathbf{x}) \tag{2.1a}$$

and

$$\left[\frac{\partial}{\partial t}u\right](\mathbf{x},t)\to\left[\frac{\partial}{\partial t}u_0^+\right](\mathbf{x},t)\ . \tag{2.1b}$$

The initial conditions in Eqs. (2.1a) and (2.1b) are distributional and require smearing over a suitable test-function space such as $\mathscr{S}$, the functions of fast decrease at infinity, or $\mathscr{D}$, the functions with compact support. These additional complications are justified for the present application because these pulses are perfectly sharp. From now on the pulses (waves or fields) are to be interpreted in this sense. During the propagation of this pulse through the support of the potential the interaction $V(\mathbf{x})$ scatters and disperses the incident pulse which is now a complicated wave $u(\mathbf{x},t)$.

After leaving the support of the potential, the wave propagates as a free wave to the receiver as $t\to+\infty$. The scattered field $u_{\rm sc}(\mathbf{x},t)$ is defined through

$$u_{\rm sc}(\mathbf{x},t)=u(\mathbf{x},t)-u_0^+(\mathbf{x},t)\ , \tag{2.2}$$

which is well defined at all $\mathbf{x}$ and t. Of course, it may be identically zero in some regions. The quantity measured in far-field direct scattering studies is the impulse response $R(\hat{e}_i,\hat{e}_s,\tau)$ which is defined as the limit

$$R(\hat{e}_i,\hat{e}_s,\tau)=\lim_{t,\,|\mathbf{x}|\to\infty}\{\,|\mathbf{x}|\,[u(\mathbf{x},t)-u_0^+(\mathbf{x},t)]\}\ ,$$
$$\tau=ct-|\mathbf{x}|\ . \tag{2.3}$$

Here $\hat{e}_s=\mathbf{x}/|\mathbf{x}|$ is a unit vector in the scattering direction and all other quantities are already defined. Observe that the impulse response plays the same role in the time domain as the scattering amplitude in the frequency domain. Additionally for classical waves, the phase of the wave can in principle be measured. The entire system being modeled is invariant under time-reversal transformations.

The Green's functions $G_0^{\pm}(\mathbf{x},t,\mathbf{x}',t')$ for advanced $(-)$ and retarded $(+)$ propagation for the free-space time-domain wave equation is given (in noncovariant form) by[19]

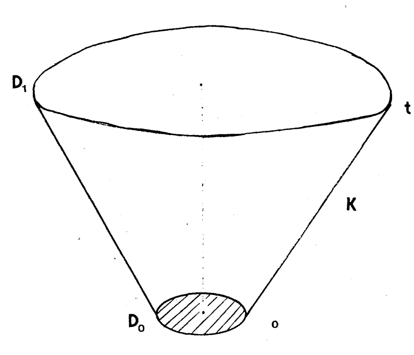

FIG. 1. Cone of influence for a circular disk of initial data of radius R called D_0 at $t=0$. The phase velocity has constant magnitude c. At t the disk has spread to D_1 which has the radius $R+ct$. Causality requires that $u(\mathbf{x},t)\equiv 0$ for all $(\mathbf{x},t)$ not in the cone of influence. The construction of spherical surfaces from each point is the Huygen's construction for data D_0.

$$G_0^{\pm}(\mathbf{x},t,\mathbf{x}',t')=\frac{-\delta(|\mathbf{x}-\mathbf{x}'|\mp c(t-t'))}{4\pi|\mathbf{x}-\mathbf{x}'|}\ . \tag{2.4}$$

The time-reversal invariance of the model gives two representations for the same physical wave, $u(\mathbf{x},t)$. One corresponds to the wave evolving to the space-time point $(\mathbf{x},t)$ from $t\to-\infty$, the other evolves from $t\to+\infty$ to the same space-time point $(\mathbf{x},t)$. A wave with initial condition $u_0^+(\mathbf{x},t)$ which scatters from V is given by

$$u(\mathbf{x},t)=u_0^+(\mathbf{x},t)+\int d^3x'\int dt' V(\mathbf{x}')G_0^+(\mathbf{x},t,\mathbf{x}',t')u(\mathbf{x}',t')\ . \tag{2.5}$$

The same wave at the same point evolves from the final condition $u_0^-(\mathbf{x},t)$ according to the time-reversed process

$$u(\mathbf{x},t)=u_0^-(\mathbf{x},t)+\int d^3x'\int dt' V(\mathbf{x}')G_0^-(\mathbf{x},t,\mathbf{x}',t')u(\mathbf{x}',t')\ . \tag{2.6}$$

Information on the final field is carried in the incoming free-space wave $u_0^-(\mathbf{x},t)$ which is obtained from the far-field data by propagating backwards in time using G_0^- and the representation theorem. The time-reversed wave equation [Eq. (2.6)] is fundamental to the formal far-field inverse method given in Sec. IV. Since the wave equation is somewhat unfamiliar in this form, a brief derivative is given in the Appendix. Explicitly

$$u_0^-(\mathbf{x},t)=\int_{\partial S}dA'\int c\,dt'\left[u(\mathbf{x}',t')\frac{\partial G_0^-}{\partial n'}(\mathbf{x},t,\mathbf{x}',t')-\frac{\partial u}{\partial n'}(\mathbf{x}',t')G_0^-(\mathbf{x},t,\mathbf{x}',t')\right]\ , \tag{2.7}$$

where ∂S is the surface of a sphere S of arbitrarily large radius centered about and containing the potential, $\partial/\partial n'$ is the normal derivative to ∂S, and the integration over dA' is over the surface of the sphere. Evaluation of Eq. (2.7) in the far-field ($|x'|\to\infty$) yields $u_0^-(\mathbf{x},t)$ in terms of the impulse-response function

$$u_0^-(\mathbf{x},t)=\delta(ct-\hat{e}_i\cdot\mathbf{x})-\frac{1}{2\pi c}\int_{S^2}d^2\hat{e}_s\frac{d}{dt}R\left[\hat{e}_i,\hat{e}_s,t-\frac{\hat{e}_s\cdot\mathbf{x}}{c}\right] \tag{2.8}$$

for each fixed $\hat{e}_i$.

For the well-behaved potentials assumed, the causality and repulsive potential imply that

$$u(\mathbf{x},t)\equiv 0 \tag{2.9}$$

for all $t<\hat{e}_i\cdot\mathbf{x}/c$. For times approaching $\hat{e}\cdot\mathbf{x}/c$ from above, the wave is inside the cone of influence K shown in Fig. 1 and is nonzero. In this case, the most singular terms[21] are

$$u(\mathbf{x},t)=\lim_{t\to\frac{(\hat{e}_i\cdot\mathbf{x})}{c}}[\delta(ct-\hat{e}_i\cdot\mathbf{x})+A(\mathbf{x})\Theta(ct-\hat{e}_i\cdot\mathbf{x})]\ , \tag{2.10}$$

where $\delta(\ )$ is the Dirac delta distribution and $\Theta(\ )$ is the Heaviside step function

$$\Theta(r)=\begin{cases}1, & r>0\\ 0, & r<0\ .\end{cases}$$

Substituting Eq. (2.5) into the wave equation, the coefficients of the first and second derivatives δ' and δ'' vanish. The coefficient of δ is

$$V(\mathbf{x})=-2(\hat{e}_i\cdot\nabla)A(\mathbf{x}) \tag{2.11}$$

where ∇ is the gradient operator. Using the Heaviside function explicitly and substituting Eq. (2.10) into (2.11) yields

$$V(\mathbf{x})=-2(\hat{e}_i\cdot\nabla)\left[\lim_{t\to\frac{(\hat{e}_i\cdot\mathbf{x})}{c}}[u(\mathbf{x},t)-\delta(ct-\hat{e}_i\cdot\mathbf{x})]\right]\ . \tag{2.12}$$

This simple equation is called the *key identity* or *fundamental identity* from now on. This is the most striking result in Refs. 6, 12, and 16 and is a direct consequence of the propagation of singularities of the hyperbolic equation. The primary consequence is that the potential can easily be determined if $u(\mathbf{x},t)$ is known in the neighborhood of the characteristic surface.

III. EXACT NEAR-FIELD INVERSION

A near-field inverse method is introduced in this section. The utility of this method for solving the applied atmospheric problem of inverting for the electron density is unknown. Consider the classical *Gedankenexperiment* schematically shown in Fig. 2. A unit Dirac delta pulse

$$u_0^+(\mathbf{x},t)=\delta(ct-\hat{e}_i\cdot\mathbf{x}) \tag{3.1}$$

is launched along unit vector $\hat{e}_i$ at large negative time. Rather than measuring the far-field scattering amplitude everywhere on a sphere, consider a series of observations of the wave on a plane perpendicular to $\hat{e}_i$. Let z^+ be a coordinate on the axis of incidence past the support of the potential. As the incident wave propagates through the potential a jump appears at the wave front ($\hat{e}\cdot\mathbf{x}=ct^-$) as indicated by Eq. (2.10) and as illustrated in Fig. 2. The height of the jump is given by integrating the fundamental identity along the direction of incidence, say z,

$$A(x,y,z^+)=-\tfrac{1}{2}\int_{-\infty}^{z^+}V(\mathbf{r})dz\ . \tag{3.2}$$

Here (x,y) are Cartesian coordinates normal to the direc-

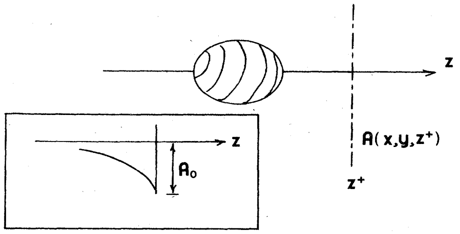

FIG. 2. Geometry for the line integral in Eq. (3.5) of the text. The inset is a schematic of the wave amplitude near the characteristic surface. The coefficient A_0 is the height of the step function.

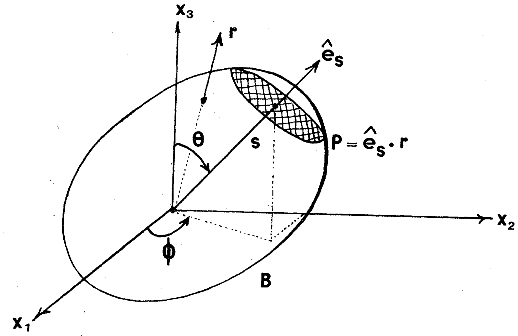

FIG. 3. Geometry of the three-dimensional Radon transform. The support of V is denoted as B; $\hat{e}_s=\hat{e}_s(\theta,\phi)$ is the unit vector in the scattered direction.

tion of incidence. The compact support of the potential [i.e., $V(x,y,z)=0,\ z>z^+$] allows the upper limit of the integral to be extended to infinity

$$A(x,y,z^+)=-\tfrac{1}{2}\int_{-\infty}^{\infty}V(\mathbf{r})dz\ . \qquad (3.3)$$

Thus, Eq. (3.2) expresses $A(x,y,z^+)$ as a set of line integrals of V along z at a given value of x and y. Given a complete set of noise-free line integrals, the three-dimensional Radon transform[22–24] yields a unique reconstruction of the potential.[25] If $f(\mathbf{r})$ is a good function with compact support, its projections (onto planes whose directed normal vectors from the origin are $\mathbf{s}=s\hat{e}_s$) are

$$\hat{f}(s,\hat{e}_s)=\int\delta(s-\hat{e}_s\cdot\mathbf{r})f(\mathbf{r})d^3r\ . \qquad (3.4)$$

Such a plane with normal $\mathbf{s}$ is shown in Fig. 3. In terms of this, the inverse transformation to Eq. (3.4) is

$$f(\mathbf{r})=-\frac{1}{8\pi^2}\int d^2\hat{e}_s\left[\frac{d^2}{ds^2}f(s=\hat{e}_s\cdot\mathbf{r},\ \hat{e}_s)\right]. \qquad (3.5)$$

The pair of equations, Eqs. (3.4) and (3.5), are a Radon transform pair in $L_2(R^3)$. If one exists among L_2 functions the inverse automatically exists (similar to Fourier transforms).

It remains to determine the projection of the potential onto planes from its line integrals. For a given direction of incidence, Eq. (3.4) gives

$$V(s,\hat{e}_s)=\int\delta(s-\hat{e}_s\cdot\mathbf{r})V(\mathbf{r})d^3r$$
$$=-2\int A(\mathbf{p}_{\|},z^+)\delta(s-\hat{e}_s\cdot\mathbf{p}_{\|})d^2p_{\|}\ . \qquad (3.6)$$

Here $\mathbf{s}$ and $\mathbf{p}_{\|}$ are vectors in the plane perpendicular to the direction of incidence. To complete the inversion it is necessary to determine $V(s,\hat{e}_s)$ for all possible directions of $\mathbf{s}$. This can be accomplished, for example, by taking directions of incidence to vary over all $0\le\phi\le\pi$ at $\theta=\pi/2$ in spherical coordinates. Once $V(s,\hat{e}_s)$ is known Eq. (3.5) determines $V(\mathbf{r})$.

IV. FAR-FIELD INVERSION

The fundamental identity simplifies the problem since it uses the wave only at the characteristic surface. Substituting the identity for the potential in the wave equation, Eq. (1.2), leads to

$$\Delta u-\frac{1}{c^2}u_{tt}-2\hat{e}_i\cdot\nabla\left[u\left[\mathbf{x},t=\frac{\hat{e}_i\cdot\mathbf{x}}{c}\right]\right.$$
$$\left.-u_0^+\left[\mathbf{x},t=\frac{\hat{e}_i\cdot\mathbf{x}}{c}\right]\right]u=0\ . \qquad (4.1)$$

If this nonlinear equation can be solved subject to both the initial and final conditions, then the wave field, and then the potential can be reconstructed. The initial condition for large negative times ($t\to-\infty$) is

$$u\sim u_0^+(\mathbf{x},t)=\delta(ct-\hat{e}_i\cdot\mathbf{x})\ , \qquad (4.2)$$

and the final condition ($t\to+\infty$) is

$$u\sim u_0^-(\mathbf{x},t)$$
$$=\delta(ct-\hat{e}_i\cdot\mathbf{x})+\frac{1}{|\mathbf{x}|}R(\hat{e}_i,\hat{e}_s,t-|\mathbf{x}|/c)\ . \qquad (4.3)$$

The existence of solutions to Eqs. (4.1)—(4.3) is not known. However, if the potential is sufficiently weak it is plausible that a solution may be obtained by starting with the final condition and integrating Eq. (4.1) backwards in time.

A coupled integral equation representation of the inversion procedure will also be obtained. The first equation is the time-reversed integral equation for the total wave given in Eq. (2.6) as

$$u(\mathbf{x},t)=u_0^-(\mathbf{x},t)$$
$$+\int d^3x'\int dt'G_0^-(\mathbf{x},t,\mathbf{x}',t')V(\mathbf{x}')u(\mathbf{x}',t')\ . \qquad (4.4)$$

Note that $u_0^-(\mathbf{x},t)$ is defined in Eqs. (2.7) and (2.8).

Defining $V_0(\mathbf{x})$ as

$$V_0(\mathbf{x})=-2\hat{e}_i\cdot\nabla_x\left[\lim_{t\to\frac{(\hat{e}_i\cdot\mathbf{x})}{c}}[u_0^-(\mathbf{x},t)]\right], \qquad (4.5)$$

and applying Eq. (2.11) to Eq. (4.4) yields the second equation,

$$V(\mathbf{x})=V_0(\mathbf{x})-2\int d^3x'\int dt'\left\{(\hat{e}_i\cdot\nabla_x)\left[G_0\left[\mathbf{x},\frac{\hat{e}_i\cdot\mathbf{x}}{c},\mathbf{x}',t'\right]\right]V(\mathbf{x}')u(\mathbf{x}',t')\right\} . \tag{4.6}$$

The inverse method proceeds as follows. Suppose that the impulse response $R(\hat{e}_i,\hat{e}_s,\tau)$ is known for one direction of incidence $\hat{e}_i$, all scattered directions $\hat{e}_s$ and all times $\tau>0$. For that direction of incidence u is obtained from Eq. (2.8) and V_0 is calculated from Eq. (4.5). Note that since $R(\hat{e}_i,\hat{e}_s,\tau)$ is required only for a fixed direction of incidence the data are three-variable functions.

Equations (4.4)—(4.6) may be solved either self-consistently or by iteration. The series obtained from jointly iterating u and V in Eq. (4.6) becomes

$$V(\mathbf{x})=V_0(\mathbf{x})-2\int d^3x'\int dt'\left\{(\hat{e}_i\cdot\nabla_x)\left[G_0^-\left[\mathbf{x},\frac{\hat{e}_i\cdot\mathbf{x}}{c},\mathbf{x}',t'\right]\right]V_0(x')u_0^-(\mathbf{x}',t')\right\}+\cdots . \tag{4.7}$$

For sufficiently weak potentials, this series is expected to converge. In the weak scattering limit, Eq. (4.6) reduces to

$$V(\mathbf{x})=\frac{1}{2\pi c}\int d^2\hat{e}_s\,|\hat{e}_1-\hat{e}_s|^2 \times\frac{d^2\tilde{R}(\hat{e}_i,\hat{e}_s,(\hat{e}_i-\hat{e}_s)\cdot\mathbf{x}/c)}{dt'^2} . \tag{4.8}$$

Also note that Eq. (4.8) can also be obtained by inverting the first term in the Born-Neumann expansion[26] of Eq. (2.5).

This inversion procedure is designed to reconstruct $u(\mathbf{x},t)$ and $V(\mathbf{x})$ for a fixed direction of incidence. If the field $u(\mathbf{x},t)$ is needed for other directions of incidence, it can be obtained most simply by solving the direct scattering problem using $V(\mathbf{x})$.

V. CONCLUSIONS

The time-domain pure inverse problem for the plasma wave equation has been investigated. The heuristics are much clearer in the time domain since the pulse evolves in time. Additionally such unphysical idealization as monochromatic plane waves are not needed. The causality and hyperbolic singularity propagation properties give an identity,[6,12,16] here Eq. (2.12). Based upon this result *rather than triangularity,* inversions methods using near-field or far-field scattering data were proposed.

The time-domain gives considerable insight into the stability of the inverse problem. Rather generally, the derivations presented depend (either implicitly or explicitly) on the second derivative of the measured data. See, for example, Eqs. (3.4), (4.11), and (5.5). Thus, the inversion methods for the plasma wave equation (and Schrödinger's equation) presented here are expected to be sensitive to small errors in the measured data.

Although this paper discusses only the plasma wave equation, the methods developed bear a striking resemblance to well-established acoustic methods. The nonlinear differential equation is observed to be similar to wave migration.[27] The near-field inversion of Sec. III is similar to Devaney's acoustic diffraction tomography.[28]

The far-field inverse-scattering methods discussed in this paper have the general property of depending upon a three-variable subspace of the impulse-response function. This situation is to be contrasted with the inverse methods of Newton[14] which require a five-variable data function.

Additional work is required to determine the existence of solutions and their realization via numerical algorithms. In any case, a physically clear discussion of the problem has provided several results with a smaller data set than customary. The compact support of V guarantees that a free wave occurs outside the region containing this support. The distorted wave perturbation calculations of Morawetz and Kriegsmann in one space dimension[29] give considerable hope for the formalism presented here. Finally, the time-domain work presented here provides a natural interpretation of Newton's work[14,30] in the frequency domain. Causality together with the restrictions on the potential here combine to imply (more than) the analyticity used there.[15]

APPENDIX

The full Green's function is defined by

$$\left[\Delta-\frac{1}{c^2}\partial_{tt}-V(\mathbf{x})\right]G(\mathbf{x},\mathbf{x}',t-t')=\delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t') . \tag{A1}$$

Low's equation in operator notation is

$$G^{\pm}=G^{0\pm}+G^{0\pm}VG^{\pm} . \tag{A2}$$

The representation theorem yields

$$u(\mathbf{x},t)=\int_{\partial S'}dA'\int_{-\infty}^{\infty}c\,dt'\left[u(\mathbf{x}',t')\frac{\partial G^-(\mathbf{x},t,\mathbf{x}',t')}{\partial n}-G^-(\mathbf{x},t,\mathbf{x}',t')\frac{\partial u(\mathbf{x}',t')}{\partial n}\right] . \tag{A3}$$

Here the integration is over the surface of the sphere S defined just below Eq. (2.7). Substitution of (A2) into (A3) yields the time-reversed wave equation (2.6) and $u_0^-(\mathbf{x},t)$ as given in Eq. (2.7).

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