# Coupled quartic anharmonic oscillators, Painlevé analysis, and integrability

M. Lakshmanan* and R. Sahadevan
*Department of Physics, Bharathidasan University, Tiruchirapalli 620 023, Tamilnadu, India*
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A detailed and systematic investigation of the Painlevé ($P$) properties of coupled quartic anharmonic oscillators is presented. Considering the two coupled oscillators we show that there exist four different parametric cases possessing $P$ properties, two are identified with strong $P$ property and the other two with weak $P$ property. For each of these four cases explicit second integrals of motion can also be constructed directly. We then consider the three-coupled-oscillator system and identify three cases, one with strong $P$ and the other two with weak $P$ nature. We have explicitly derived the second and third integrals of motion for a special case of the strong $P$ case, but for the remaining cases they have not yet been found. Finally, we extend the procedure to the $N$ coupled oscillators and succeed to show that there exist three cases possessing $P$ properties, which are the natural generalizations of the three-coupled-oscillator system.

## I. INTRODUCTION

It is well known that the evolution of typical dynamical systems[1–3] is described by nonlinear ordinary or partial differential equations and that their analysis is complicated due to the absence of systematic general methods of solving them. In general the phase trajectories of such systems are irregular, complicated, and sensitive to initial conditions. Particularly, completely integrable nonlinear dynamical systems are rather limited and therefore there is an intensive search for them in recent years. In this connection several years ago Painlevé and his contemporaries identified a class of second-order, nonlinear ordinary differential equations in which the movable singularities exhibited by the general solution are only poles in the complex time plane.[4,5] Such systems are called Painlevé type or simply $P$ type, possessing the $P$ property.

Recently the main objective of Painlevé has been exploited by Ablowitz, Ramani, and Segur (ARS),[6] who pointed out the intimate connection between the singularity structure and integrability of the system, particularly for soliton equations. They have also conjectured that every nonlinear ordinary differential equation obtained by an exact reduction of such soliton equations is of $P$ type. This has been verified for a large class of nonlinear partial differential equations by Lakshmanan and Kaliappan[7] using Lie point symmetries. The term *strong-P* is used in connection with ARS conjecture, wherein the solution in the neighborhood of an arbitrary singularity $t^*$ can be expressed as $\tau=(t-t^*)^{-p}$, where $p$ is an integer, determined solely from the leading order so that the movable algebraic or logarithmic branch points as well as essential singularities are excluded.

It is generally believed that the $P$ property of the solutions associated with equation of motion is an effective tool to test the integrability of nonlinear dynamical systems. In fact, the importance of this property was appreciated by Kowalevskaya[8] in the theory of motion of a rigid body rotating about a fixed point, 80 years ago. Bountis, Segur, and Vivaldi[9] have used this idea on the generalized Hénon-Heiles system, quartic anharmonic oscillators, and the Toda lattice system, and identified some of the integrable cases. The concept has been again applied to the Hénon-Heiles system by Chang, Tabor, and Weiss,[10] who investigated new integrable cases. Other recent related works include that of Menyuk, Chen, and Lee[11] on three-wave interactions in $2(2N+1)$ variables and that of Tabor and Weiss[12] on the integrability properties of the well-known Lorenz system, which is dissipative.

Recently Ramani, Dorizzi, and Grammaticos[13] have suggested that the existing ARS algorithm can be generalized by introducing the so-called *weak P* property. Weak $P$ property means that the solution in the neighborhood of the movable singularity $t^*$ can be expressed as an expansion in powers of $\bar{\tau}=(t-t^*)^{-1/n}$, where $n$ must be "natural" and that it depends purely on the leading-order behavior of the singularity and the nature of the potential.[14,15] They have also conjectured that the integrable systems, particularly with two degrees of freedom, have weak $P$ property, though, ordinarily, one could not recover poles either by raising the solution to the $n$th power or by a change of independent variables since the singularity is movable. In fact, they have presented several cases of polynomial potentials having this property. It is natural then to expect similar possibilities for systems having more than two degrees of freedom.

In this paper a systematic search of the $P$ properties is made for the equations of motion of two, three, and up to $N$ coupled quartic anharmonic oscillator systems[16,17] given by the Hamiltonians

$$H_2=\tfrac{1}{2}(p_x^2+p_y^2)+Ax^2+By^2+\alpha x^4+\beta y^4+\delta x^2y^2\ , \tag{1}$$

$$H_3=\tfrac{1}{2}(p_x^2+p_y^2+p_z^2)+Ax^2+By^2+Cz^2+\alpha x^4+\beta y^4+\gamma z^4+\delta x^2y^2+\epsilon y^2z^2+\omega x^2z^2 \tag{2}$$

and

 

$$H_N=\frac{1}{2}\sum_{i=1}^{N}p_{x_i}^2+\sum_{i=1}^{N}A_ix_i^2+\sum_{i=1}^{N}\alpha_ix_i^4+\frac{1}{2}\sum_{\substack{i,j=1\\(i\neq j)}}^{N}\beta_{ij}x_i^2x_j^2,\quad \beta_{ij}=\beta_{ji} \tag{3}$$

where $A$, $B$, $C$, $A_i$, $\alpha$, $\beta$, $\alpha_i$, $\beta_{ij}$, $\gamma$, $\delta$, $\epsilon$, and $\omega$ are parameters. It is well known that the above type of Hamiltonian systems are used widely as models in lattice dynamics,[18] condensed-matter theory,[19] field theory,[20] astrophysics,[21] etc. The equations of motion for the system (1) are

$$\ddot{x}=-2Ax-4\alpha x^3-2\delta xy^2, \tag{4a}$$

$$\ddot{y}=-2By-4\beta y^3-2\delta x^2y, \tag{4b}$$

while for the system (2) they are

$$\ddot{x}=-2Ax-4\alpha x^3-2\delta xy^2-2\omega xz^2, \tag{5a}$$

$$\ddot{y}=-2By-4\beta y^3-2\delta x^2y-2\epsilon yz^2, \tag{5b}$$

$$\ddot{z}=-2Cz-4\gamma z^3-2\omega x^2z-2\epsilon y^2z, \tag{5c}$$

and for the system (3), we obtain

$$\ddot{x}_i=-2A_ix_i-4\alpha_ix_i^3-\sum_{\substack{i,j=1\\(j\neq i)}}^{N}\beta_{ij}x_ix_j^2,\quad i=1,2,\ldots,N \tag{6}$$

where the double overdot denotes two differentiations with respect to time. We carry out a systematic singularity analysis of the systems (4)–(6) and identify those cases whose movable singularities are only poles (or transformable to poles), so that the solution has a sufficient number of arbitrary constants. We also identify integrals of motion wherever possible.

The plan of the paper is as follows. To be self-contained we briefly outline the salient features of $P$ analysis in Sec. II. In Sec. III we apply this procedure to the equation of motion (4) and show that there exist four different parametric cases possessing $P$ properties, two of which are associated with strong $P$ properties and the remaining two with weak $P$ properties. At the end of this section we explicitly derive the second integrals of motion, thereby substantiating the complete integrability. In Sec. IV a similar analysis is carried out for the three-coupled-oscillator system (5) where three $P$ cases are identified. At the end of this section an effort is made to derive the second and third integrals of motion but we succeed only for a special case of the strong-$P$ case. In Sec. V we extend this procedure to the equations of motion of $N$-coupled-oscillator system (6) and successfully isolate three sets of parametric values possessing $2N$ arbitrary constants which are generalizations of three-coupled-oscillator systems. In Sec. VI we give a brief discussion of our results and other features of the $P$ property.

## II. PAINLEVÉ ANALYSIS

Considering an $n$th order ordinary differential equation (ODE) having the form

$$\frac{d^nx}{dt^n}=F(t,x,\dot{x},\ldots,\dot{x}^{\,n-1}),\quad \dot{x}=\frac{dx}{dt} \tag{7}$$

where $F$ is real, analytic in $t$, and algebraic in its other arguments, and ignoring, for the present, essential singularities, the $P$ analysis essentially consists[6,7] of the following three steps.

(1) Determination of the leading-order behaviors of $x$ in a sufficiently small neighborhood of a movable singularity $t^*$ in the form $x\approx a_0(t-t^*)^q$, as $t\to t^*$, $a_0=\text{const}$. Then (i) if all the allowed $q$'s are negative integers, the solution may correspond to the strong-$P$ property; (ii) if any of the $q$'s is a rational fraction, the solution may be associated with the weak-$P$ property. In either case, the solution takes the form of a Laurent series,

$$x(t)=(t-t^*)^q\sum_{j=0}^{\infty}a_j(t-t^*)^j,\quad 0<|t-t^*|<R\ . \tag{8}$$

(2) Identification of the powers of (8) at which the arbitrary parameters can enter, called resonances. Apart from $t^*$, we have $(n-1)$ other arbitrary constants for (8) to be a general solution of (7). In order to find the resonances we substitute $x\approx a_0\tau^q+\Omega\tau^{q+r}$, $\tau=t-t^*$, in (7), retaining only the leading-order terms. The reduced equation will be of the form $Q(r)\tau^{q_1}\Omega=0$, $q_1\geq(q+r-n)$, where $Q(r)$ is polynomial of degree $n$. Then the resonance values are determined from the roots of $Q(r)=0$. For the general solution of (7), $Q(r)$ must have $(n-1)$ nonnegative distinct roots of real integers, apart from one value $r=-1$ representing the arbitrariness of $t^*$.

(3) The final step consists of verifying that in the Laurent series (8) at the resonance values sufficient number of arbitrary constants exist without the introduction of logarithmic branch points.

We may note that in typical cases, several leading-order behaviors can coexist and that it is essential to verify that the solution is free from movable branch points not only in the full (arbitrary) parameter branches of the leading order which we call *main branches* (MB), but also in the remaining lesser parameter branches [*subsidiary branches* (SB)] as well for the $P$ property to hold. In the following we apply the above method systematically to the coupled anharmonic oscillator systems (4)–(6).

## III. $P$ PROPERTIES OF TWO QUARTICALLY COUPLED OSCILLATORS

### A. Leading-order behaviors and resonances

Considering (4), we assume that the leading orders are

$$x\approx a_0\tau^p,\quad y\approx b_0\tau^q,\quad \tau=(t-t^*)\to 0\ . \tag{9}$$

To determine $p$, $q$, $a_0$, and $b_0$ we use (9) in (4) and obtain a pair of leading-order equations

$$a_0p(p-1)\tau^{p-2}=-4\alpha a_0^3\tau^{3p}-2\delta a_0b_0^2\tau^{p+2q}, \tag{10a}$$

$$b_0q(q-1)\tau^{q-2}=-4\beta b_0^3\tau^{3q}-2\delta a_0^2b_0\tau^{2p+q}\ . \tag{10b}$$

From Eqs. (10) we can identify the following two *distinct* sets of solutions:
case 1,

$$p=-1,\quad q=-1,\quad a_0^2=(\delta-2\beta)/\Delta_1,\quad b_0^2=(\delta-2\alpha)/\Delta_1,\quad \Delta_1=4\alpha\beta-\delta^2\ ; \tag{11}$$

case 2a,

$$p=-1,\quad q=\tfrac{1}{2}\{1+[1+(4\delta)/\alpha]^{1/2}\}\geq\tfrac{1}{2},\quad a_0^2=-\frac{1}{2\alpha},\quad b_0^2=\text{arbitrary}\ ; \tag{12a}$$

case 2b,

$$p=-1,\quad q=\tfrac{1}{2}\{1-[1+(4\delta)/\alpha]^{1/2}\}>-1,\quad a_0^2=-\frac{1}{2\alpha},\quad b_0^2=\text{arbitrary}\ . \tag{12b}$$

Due to the complete symmetry between $x$ and $y$ variables in $H_2$ or in (4), we do not treat the other possibility $q=-1$, $p>-1$ as distinct from (12).

For finding the resonances, we substitute

$$x\approx a_0\tau^p+\Omega_1\tau^{p+r},\quad y\approx b_0\tau^q+\Omega_2\tau^{q+r} \tag{13}$$

into (4). From the dominant terms we obtain a system of linear algebraic equations,

$$\underline{M}_2(r)\underline{\Omega}=\underline{0},\quad \underline{\Omega}=(\Omega_1,\Omega_2) \tag{14}$$

where $\underline{M}_2(r)$ is a $2\times2$ matrix dependent on $r$. In order to have a nontrivial set of solutions $(\Omega_1,\Omega_2)$ we require that

$$\det\underline{M}_2(r)=0\ . \tag{15}$$

For case 1, the form of $\underline{M}_2(r)$ is

$$\underline{M}_2(r)=\begin{bmatrix}(r-1)(r-2)+8\alpha a_0^2-2 & 4\delta a_0b_0\\ 4\delta a_0b_0 & (r-1)(r-2)+8\beta b_0^2-2\end{bmatrix}, \tag{16a}$$

so that using (11), Eq. (15) becomes

$$(r^2-3r-4)(r^2-3r+\chi_0)=0,\quad \chi_0=4[1+2(\alpha a_0^2+\beta b_0^2)]\ . \tag{16b}$$

Thus for case 1, the resonances occur at

$$r=-1,\ 4,\ \tfrac{3}{2}\pm\tfrac{1}{2}(9-4\chi_0)^{1/2}\ . \tag{17}$$

The root $-1$ corresponds to the arbitrariness of $t^*$ in (9). Furthermore, for Painlevé, all the other resonances must be nonnegative integers. Equation (17) along with (11) then lead to the following two possibilities with associated parametric restrictions:

case 1 (i),

$$\chi_0=2,\quad \alpha a_0^2+\beta b_0^2=-\tfrac{1}{4},\quad r=-1,1,2,4,\quad \delta=2[\alpha+\beta\pm(3\alpha\beta-\alpha^2-\beta^2)^{1/2}]\ , \tag{18a}$$

case 1 (ii),

$$\chi_0=0,\quad \alpha a_0^2+\beta b_0^2=-\tfrac{1}{2},\quad r=-1,0,3,4,\quad \delta=\alpha+\beta\pm(6\alpha\beta-3\alpha^2-3\beta^2)^{+1/2}\ . \tag{18b}$$

For case 2, leading orders (12), the expression for $\underline{M}_2(r)$ degenerates to

$$\underline{M}_2(r)=\text{diag}[r^2-3r+8\alpha a_0^2,\ r(r+2q-1)]\ , \tag{19}$$

so that from (15), the resonance values become

$$r=-1,0,(1-2q),4\ . \tag{20}$$

Case 2(a). In (20), for $(1-2q)\geq0$, $q\leq0$. But this is in general contradictory to the leading-order singularity nature, $q\geq\frac{1}{2}$, Eq. (12a). The only consistent case $q=\frac{1}{2}$ requires both $a_0$, and $b_0$ to be arbitrary, which is not true as seen from (12a). So the associated $P$ branch can have a lesser number of arbitrary constants only.

Case 2(b). Using (12b) in (19), we infer two possibilities: $q=0$, and so $\delta=0$, the uncoupled case, and

$$q=-\tfrac{1}{2},\quad 3\alpha=4\delta,\quad r=-1,0,2,4\ . \tag{21}$$

Thus for the two-coupled-oscillator system (1) or (4), we identify three sets of full resonances, namely (18a), (18b), and (21).

### B. Identifying the arbitrary constants of integration

Introducing now the series expansions

$$\begin{aligned}x&=a_0\tau^p+\sum_{k>0}^{r_s=4}a_k\tau^{p+k}\ ,\\ y&=b_0\tau^q+\sum_{k>0}^{r_s=4}b_k\tau^{q+k},\quad \tau\to0\ ,\end{aligned} \tag{22}$$

in (4) and collecting and equating the coefficients of the powers of $(\tau^{p+k-2},\tau^{q+k-2})$ to zero, we obtain a system of linear algebraic equations for $a_k$ and $b_k$ which when solved will show the nature of $x(t)$ and $y(t)$. We will deal with each one of the cases 1 (i), 1 (ii), and 2b separately.

### 1. *Strong-P cases*

Case 1(i). Here the resonance values are $r=-1,1,2,4$ and the parameters $\alpha$, $\beta$, and $\delta$ are restricted as in (18a). From the coefficients of $(\tau^{-2},\tau^{-2})$ in (4), we obtain

$$\underline{\Gamma}_1\underline{X}_1=0,\quad \underline{X}_1=[a_1,b_1]^T, \tag{23}$$

$$\underline{\Gamma}_1=\begin{bmatrix}4\alpha a_0^2-1 & 2\delta a_0 b_0\\ 2\delta a_0 b_0 & 4\beta b_0^2-1\end{bmatrix},$$

where $a_0,b_0$ are given in (11). Then for $a_1$ (or $b_1$) to be arbitrary, we require det $\Gamma_1=0$, which means

$$\delta=2[(\alpha+\beta)\pm(\alpha^2+\beta^2-\alpha\beta)^{1/2}] .$$

Comparing this with the earlier one given in (18a), we conclude that $\alpha=\beta$, $\delta=2\alpha$, or $\delta=6\alpha$. However, from (10) and (11) we infer that for the value $\delta=2\alpha$, $\alpha=\beta$, either $a_0$ or $b_0$ becomes arbitrary, which is not indicated by the resonance values in (18a). We therefore infer that, at this stage, the only allowed set of parametric values for case 1(i) is

$$\alpha=\beta,\quad \delta=6\alpha . \tag{24}$$

Proceeding further with (24), and from the coefficients of $(\tau^{-1},\tau^{-1})$ we obtain

$$\tfrac{3}{2}a_2+\tfrac{3}{2}b_2=Aa_0,\quad \tfrac{3}{2}a_2+\tfrac{3}{2}b_2=Ba_0 , \tag{25}$$

so that the coefficient $a_2$ (or $b_2$) is arbitrary if

$$A=B . \tag{26}$$

In a similar way from the coefficients of $(\tau^0,\tau^0)$ and $(\tau^1,\tau^1)$ we uniquely determine the coefficients $a_3$ and $b_3$ in terms of the previous ones, while either one of the coefficients $a_4,b_4$ is arbitrary without any additional constraints on the parameters, and thus leading to a full four-parameter branch of solution for the choice (24) and (26).

Case 1(ii). The resonance values here are $r=-1,0,3,4$ with the parametric condition given in (18b). From the leading-order analysis we have

$$2\alpha a_0^2+\delta b_0^2=-1,\quad \delta a_0^2+2\beta b_0^2=-1 . \tag{27a}$$

For $a_0$ (or $b_0$) to be arbitrary, we require that

$$\alpha=\beta,\quad \delta=2\alpha , \tag{27b}$$

which also satisfies the form of $\delta$ in (18b) identically. Proceeding further we determine the coefficients $a_1$, $b_1$, $a_2$, and $b_2$. Comparing then the coefficients of $(\tau^0,\tau^0)$ in (4), we obtain a single equation

$$a_0a_3+b_0b_3=0 . \tag{28}$$

and so $a_3$ (or $b_3$) is arbitrary. Finally the coefficients of $(\tau^1,\tau^1)$ again lead to the single equation

$$b_0a_4-a_0b_4=\frac{\alpha}{2a_0b_0}(a_0^2b_2^2-b_0^2a_2^2) , \tag{29}$$

showing that $a_4$ (or $b_4$) is arbitrary. Thus for the choice (27b), without any restriction on $A,B$, we have a full four-parameter branch of solution.

### 2. *Weak-P cases*

Case 2b. From Eqs. (12b) and (21) we have $p=-1$, $q=-\frac{1}{2}$, $3\alpha=4\delta$, $r=-1,0,2,4$, $a_0^2=-1/2\alpha$, and $b_0$, arbitrary. Proceeding as before we can find $a_1$ and $b_1$ uniquely. Then from the coefficients of $(\tau^{-1},\tau^{-1/2})$ in (4), we obtain

$$(4\alpha a_0^2-1)a_2=-Aa_0+\delta(\delta-8\beta)a_0b_0^4 , \tag{30a}$$

$$2\delta a_0a_2+0\times b_2=-B-(\delta^2-48\beta\delta+288\beta^2)\frac{b_0^4}{12} . \tag{30b}$$

Using (12b) and (21) we infer that $b_2$ is arbitrary only if

$$A=4B,\quad \delta^2-18\beta\delta+72\beta^2=0 , \tag{31}$$

and that $a_2$ can be determined uniquely. We further verify that in $a_3$, $b_3$ there is no arbitrariness, while $a_4$ resulting from the coefficients $(\tau^1,\tau^{3/2})$ is arbitrary and $b_4$ is fixed without any further restrictions on the parameters. Thus $x(t)$, $y(t)$ in (4) have a full four-parameter branch of solution for the parametric values, case 2b(i),

$$3\alpha=4\delta,\quad \beta=\frac{\delta}{12},\quad A=4B ,$$

or (32a)

$$\alpha=16\beta,\quad \delta=12\beta,\quad A=4B$$

and case 2b(ii),

$$3\alpha=4\delta,\quad \beta=\frac{\delta}{6},\quad A=4B$$

or (32b)

$$\alpha=8\beta,\quad \delta=6\beta,\quad A=4B .$$

The above details of the $P$ properties, for the main branches (MB) of cases 1(i), 1(ii), 2b(i), and 2b(ii) are summarized in Table I. It remains only to check that for each of the above four parametric choices, the remaining branches [subsidiary branches (SB)] do not exhibit any movable branch points. We do verify explicitly that this is indeed the case for all the above four possibilities. The details are also included in Table I. Thus we conclude that all the four possibilities given in Table I indeed possess the Painlevé property.

## C. Integrals of motion

For each one of the four Painlevé cases given in Table I, we now explicitly construct a second integral of motion (the first being the Hamiltonian) thereby substantiating the previous discussions. For this purpose we apply the modified Whittaker[15] analysis to construct the integrals of motion for arbitrary energy (for generalization see also Hall[22]). Restricting ourselves to velocities up to fourth power, the second integral of motion may be written as

TABLE I. Painlevé cases of the two-coupled quartic anharmonic oscillator system. (Note: In the subsidiary branches $r<-1$ do not contribute to the Laurent series.)

| Cases (see text) | Parametric restrictions | Main or subsidiary branch | Leading order $p$ | $q$ | Resonances $r$ | No. of arbitrary constants | Second integral |
|---|---|---|---|---|---|---|---|
| 1(i) | $\alpha=\beta,\ \delta=6\alpha,$ | MB | $-1$ | $-1$ | $-1,\ 1,\ 2,\ 4$ | 4 | $p_xp_y+2Axy+4\alpha xy(x^2+y^2)$ |
| | $A=B$ | SB | $-1$ | 2 | $-5,\ -1,\ 0,\ 4$ | 3 | |
| 1(ii) | $\alpha=\beta,\ \delta=2\alpha$ | MB | $-1$ | $-1$ | $-1,\ 0,\ 3,\ 4$ | 4 | $(xp_y-yp_x)^2+\frac{2}{\alpha}(B-A)(p_x^2+\alpha x^4+\alpha x^2y^2+Ax^2)$ |
| | | SB | $-1$ | 2 | $-3,\ -1,\ 0,\ 4$ | 3 | |
| 2b(i) | $\alpha=16\beta,\ \delta=12\beta,$ | MB | $-1$ | $-\frac{1}{2}$ | $-1,\ 0,\ 2,\ 4$ | 4 | $-xp_y^2+yp_xp_y+2(B+4\beta x^2)xy^2+4\beta xy^4$ |
| | $A=4B$ | SB-1 | $-1$ | $-1$ | $-2,\ -1,\ 4,\ 5$ | 3 | |
| | | SB-2 | $-1$ | $\frac{3}{2}$ | $-2,\ -1,\ 0,\ 4$ | 3 | |
| 2b(ii) | $\alpha=8\beta,\ \delta=6\beta,$ | MB | $-1$ | $-\frac{1}{2}$ | $-1,\ 0,\ 2,\ 4$ | 4 | $p_y^4+4(B+\beta y^2+6\beta x^2)y^2p_y^2$ |
| | $A=4B$ | SB-1 | $-1$ | $\frac{3}{2}$ | $-5,\ -1,\ 4,\ 8$ | 3 | $-16\beta xy^3p_xp_y+4\beta y^4p_x^2$ |
| | | SB-2 | $-1$ | $\frac{3}{2}$ | $-2,\ -1,\ 0,\ 4$ | 3 | $+4B(B+2\beta y^2+4\beta x^2)y^4+4\beta^2(2x^2+y^2)^2y^4$ |

$$I=\xi_1p_x^4+\xi_2p_x^3p_y+\xi_3p_x^2p_y^2+\xi_4p_xp_y^3+\xi_5p_y^4+\xi_6p_x^2+\xi_7p_xp_y+\xi_8p_y^2+\xi_9\ , \tag{33}$$

where $\xi_i$'s are functions of $(x,y)$ alone. To obtain $\xi_i$'s we demand that the Poisson bracket $\{I,H\}_{\rm PB}$ vanishes. Equating now the coefficients of each power of the velocities $p_x^mp_y^n, m,n=1,2,\ldots,5$ separately to zero, we obtain a system of overdetermined partial differential equations:

$$\frac{\partial\xi_1}{\partial x}=0,\quad \frac{\partial\xi_1}{\partial y}+\frac{\partial\xi_2}{\partial x}=0,\quad \frac{\partial\xi_2}{\partial y}+\frac{\partial\xi_3}{\partial x}=0\ ,$$
$$\frac{\partial\xi_3}{\partial y}+\frac{\partial\xi_4}{\partial x}=0,\quad \frac{\partial\xi_4}{\partial y}+\frac{\partial\xi_5}{\partial x}=0,\quad \frac{\partial\xi_5}{\partial y}=0\ . \tag{34}$$

By successively solving (34) we find that

$$\xi_1=\epsilon_0y^4+\epsilon_1y^3+\epsilon_2y^2+\epsilon_3y+\epsilon_4\ , \tag{35a}$$
$$\xi_2=-(4\epsilon_0y^3+3\epsilon_1y^2+2\epsilon_2y+\epsilon_3)x+\eta_0y^3+\eta_1y^2+\eta_2y+\eta_3\ , \tag{35b}$$
$$\xi_3=(6\epsilon_0y^2+3\epsilon_1y+\epsilon_2)x^2-(3\eta_0y^2+2\eta_1y+\eta_2)x+\lambda_0y^2+\lambda_1y+\lambda_2\ , \tag{35c}$$
$$\xi_4=-(4\epsilon_0y+\epsilon_1)x^3+(3\eta_0y+\eta_1)x^2-(2\lambda_0y+\lambda_2x)x+\gamma_0y+\gamma_1\ , \tag{35d}$$

and

$$\xi_5=\epsilon_0x^4-\eta_0x^3+\lambda_0x^2-\gamma_0y+\gamma_2\ , \tag{35e}$$

where $\epsilon_i$'s, $\eta_i$'s, $\lambda_i$'s, and $\gamma_i$'s are integration constants. The next set of partial differential equations is

$$4\xi_1\ddot{x}+\xi_2\ddot{y}+\frac{\partial\xi_6}{\partial x}=0,\quad 3\xi_2\ddot{x}+2\xi_3\ddot{y}+\frac{\partial\xi_6}{\partial y}+\frac{\partial\xi_7}{\partial x}=0\ ,$$
$$2\xi_3\ddot{x}+3\xi_4\ddot{y}+\frac{\partial\xi_7}{\partial y}+\frac{\partial\xi_8}{\partial x}=0,\quad \xi_4\ddot{x}+4\xi_5\ddot{y}+\frac{\partial\xi_8}{\partial y}=0\ ,$$
$$2\xi_6\ddot{x}+\xi_7\ddot{y}+\frac{\partial\xi_9}{\partial x}=0\ ,$$

and

$$\xi_7\ddot{x}+2\xi_8\ddot{y}+\frac{\partial\xi_9}{\partial y}=0\ . \tag{36}$$

Using Eqs. (4) and (35) in Eqs. (36), we find that the consistency holds only for the following parametric values so that second integrals of motion of the form (33) exist:

$$\alpha=\beta,\quad \delta=6\alpha,\quad A=B,\quad I_1=p_xp_y+2Axy+4\alpha xy(x^2+y^2)\ , \tag{37}$$

$$\alpha=\beta,\quad \delta=2\alpha,\quad I_2=(xp_y-yp_x)^2+\frac{2}{\alpha}(B-A)\left[\frac{p_x^2}{2}+\alpha x^4+\alpha x^2y^2+Ax^2\right], \tag{38}$$

$$\alpha=16\beta,\quad \delta=12\beta,\quad A=4B,\quad I_3=-xp_y^2+yp_xp_y+2(B+4\beta x^2)xy^2+4\beta xy^4\ , \tag{39}$$

$$\alpha=8\beta,\quad \delta=6\beta,\quad A=4B\ ,$$
$$I_4=p_y^4+4(B+\beta y^2+6\beta x^2)y^2p_y^2-16\beta xy^3p_xp_y+4\beta y^4p_x^2+4B(B+2\beta y^2+4\beta x^2)y^4+4\beta^2(2x^2+y^2)^2y^4\ . \tag{40}$$

It is interesting to show that the first three cases above are not only integrable but also separable.

(i) In the case of (37), Eqs. (4) can be decoupled in terms of the linearly transformed variables $u=(x+y)$, $v=(x-y)$, to obtain $\ddot{u}+2Au+4\alpha u^3=0$, $\ddot{v}+2Av+4\alpha v^3=0$. These simply comprise the one-dimensional anharmonic oscillator equation of motion whose solution may be readily given in terms of the Jacobian elliptic functions $u=s_1\mathrm{cn}(\omega_1 t+\delta_1)$, $v=s_2\mathrm{cn}(\omega_2 t+\delta_2)$, where $\omega_i^2=4(A^2+\alpha s_i^2)$, $s_i$ and $\delta_i$, $i=1,2$ are integration constants. The associated energy integrals are

$$H_u=\tfrac{1}{2}(p_u^2+2Au^2+2\alpha u^4)$$

and

$$H_v=\tfrac{1}{2}(p_v^2+2Av^2+2\alpha v^4)$$

so that $H_2=\frac{1}{2}(H_u+H_v)$, while the second integral of motion is

$$I_1=\tfrac{1}{2}(H_u-H_v)=p_xp_y+2Axy+4\alpha xy(x^2+y^2)\ ,$$

as given by (37). The form of $I_1$ has also been noted in Ref. 9.

(ii) In the case of (38), if we transform the coordinates $x$ and $y$ to polar coordinates $x=\rho\cos\theta$, $y=\rho\sin\theta$, then for the parametric values $\alpha=\beta$, $\delta=2\beta$, and $A=B$, the Hamiltonian $H_2$ is independent of $\theta$. Then it is straightforward to check that the angular momentum $I_2=xp_y-yp_x=\text{const}$. Therefore, the Hamiltonian may be written as

$$H_2=\tfrac{1}{2}\left[\dot{\rho}^2+2A\rho^2+2\alpha\rho^4+\frac{I_2^2}{\rho^2}\right]$$

so that the equation of motion (4) in radial coordinates becomes

$$\ddot{\rho}+2A\rho+4\alpha\rho^3=\frac{I_2^2}{\rho^3}\ .$$

Lakshmanan and Kaliappan[23] integrated it as

$$\rho(t)=a_1[1-\beta_1^2\mathrm{sn}^2(\hat{\Gamma}t)]^{1/2}\ ,$$

where

$$\beta_1^2=\tfrac{1}{2}\left[3\left[1+\frac{A}{3\alpha a_1^2}\right]-\left[1+\frac{A}{\alpha a_1^2}+\frac{A^2}{\alpha^2a_1^4}+\frac{2I_2^2}{\alpha a_1^6}\right]^{1/2}\right],$$

and

$$\hat{\Gamma}^2=(A+3\alpha a_1^2)+\left[A^2+2\alpha A a_1+\alpha^2a_1^2+\frac{2\alpha}{a_1^2}I_2^2\right]^{1/2}\ .$$

Here, the square of the modulus of the Jacobian elliptic function is given by $k^2=2\alpha a_1^2\beta_1^2/\hat{\Gamma}^2$, $0\le k^2\le\beta_1^2\le 1$. For $A\neq B$, the system is not separable in spherical polar coordinates. However the resulting Hamilton-Jacobi equation for the action $S_0$,

$$\frac{1}{2}\left[\frac{1}{\xi^2-c^2}\left[\frac{\partial S_0}{\partial\xi}\right]^2+\frac{1}{c^2-\eta^2}\left[\frac{\partial S_0}{\partial\eta}\right]^2\right]+\frac{f(\xi)-f(\eta)}{\xi^2-\eta^2}=E\ ,$$

where $\xi$ and $\eta$ are confocal coordinates, is separable.[24] The form of $I_2$ has also been mentioned by Hietarinta.[25]

(iii) Similarly, in the case of (39), if we transform the Cartisean coordinates to parabolic cylinder coordinates $x=\epsilon\eta$, $y=\frac{1}{2}(\epsilon^2-\eta^2)$ for the parametric values $\alpha=16\beta$, $\delta=12\beta$, $A=4B$, then the resulting Hamilton-Jacobi equation is separable. This result has also been noted by Ankiewicz and Pask[26] recently.

## IV. P PROPERTIES OF THREE QUARTICALLY COUPLED OSCILLATORS

### A. Leading-order behaviors

Assuming now that the leading-order terms of the solutions of (5) are

$$x\approx a_0\tau^p,\quad y\approx b_0\tau^q,\quad z\approx c_0\tau^s,\quad \tau\to 0\ , \tag{41}$$

we find from (5) that

$$a_0p(p-1)\tau^{p-2}=-4\alpha a_0^3\tau^{3p}-2\delta a_0b_0^2\tau^{p+2q}-2\omega a_0c_0^2\tau^{p+2s}\ , \tag{42a}$$

$$b_0q(q-1)\tau^{q-2}=-4\beta b_0^3\tau^{3q}-2\delta a_0^2b_0\tau^{2p+q}-2\epsilon b_0c_0^2\tau^{q+2s}\ , \tag{42b}$$

$$c_0s(s-1)\tau^{s-2}=-4\gamma c_0^3\tau^{3s}-2\omega a_0^2c_0\tau^{2p+s}-2\epsilon b_0^2c_0\tau^{2q+s}\ . \tag{42c}$$

We have essentially three distinct sets of solutions [noting the complete symmetry between the coordinates $x,y,z$ in (2) or (5)].

Case 1

$$\begin{gathered}p=q=s=-1\ ,\\ a_0^2=\frac{(\epsilon^2-4\beta\gamma+2\gamma\delta-\epsilon\omega-\epsilon\delta+2\beta\omega)}{\tilde{\Delta}_1}\ ,\\ b_0^2=\frac{(\omega^2-4\alpha\gamma+2\alpha\epsilon-\epsilon\omega-\omega\delta+2\gamma\delta)}{\tilde{\Delta}_1}\ ,\\ c_0^2=\frac{(\delta^2-4\alpha\beta+2\alpha\epsilon-\omega\delta-\epsilon\delta+2\beta\omega)}{\tilde{\Delta}_1}\ ,\\ \tilde{\Delta}_1=[8\alpha\beta\gamma+2\delta\epsilon\omega-2(\alpha\epsilon^2+\beta\omega^2+\gamma\delta^2)]\ ,\end{gathered} \tag{43}$$

and the following:

case 2(a)

$$\begin{gathered}p=-1,\quad q=\tfrac{1}{2}+\tfrac{1}{2}[1-8(\delta a_0^2+\epsilon c_0^2)]^{1/2}\ge\tfrac{1}{2},\quad s=-1\ ,\\ a_0^2=\frac{\omega-2\gamma}{\Delta_2},\quad b_0^2=\text{arbitrary},\quad c_0^2=\frac{\omega-2\alpha}{\Delta_2},\quad \Delta_2=4\alpha\gamma-\omega^2\ ,\end{gathered} \tag{44a}$$

case 2b

$$p=-1,\quad q=\tfrac{1}{2}-\tfrac{1}{2}[1-8(\delta a_0^2+\epsilon c_0^2)]^{1/2}>-1,\quad s=-1\,,$$
$$a_0^2=\frac{\omega-2\gamma}{\Delta_2},\quad b_0^2=\text{arbitrary},\quad c_0^2=\frac{\omega-2\alpha}{\Delta_2}\,, \tag{44b}$$

case 3a

$$p=-1,\quad q=\tfrac{1}{2}+\tfrac{1}{2}[1+4(\delta/\alpha)]^{1/2}\geq\tfrac{1}{2},\quad s=\tfrac{1}{2}+\tfrac{1}{2}[1+4(\omega/\alpha)]^{1/2}\geq\tfrac{1}{2}\,,$$
$$a_0^2=-\frac{1}{2\alpha},\quad b_0^2=\text{arbitrary},\quad c_0^2=\text{arbitrary}\,, \tag{44c}$$

case 3b

$$p=-1,\quad q=\tfrac{1}{2}+\tfrac{1}{2}[1+4(\delta/\alpha)]^{1/2}\geq\tfrac{1}{2},\quad s=\tfrac{1}{2}-\tfrac{1}{2}[1+4(\omega/\alpha)]^{1/2}>-1\,,$$
$$a_0^2=-\frac{1}{2\alpha},\quad b_0^2=\text{arbitrary},\quad c_0^2=\text{arbitrary}\,, \tag{44d}$$

case 3c

$$p=-1,\quad q=\tfrac{1}{2}-\tfrac{1}{2}[1+4(\delta/\alpha)]^{1/2}>-1,\quad s=\tfrac{1}{2}-\tfrac{1}{2}[1+4(\omega/\alpha)]^{1/2}>-1\,,$$
$$a_0^2=-\frac{1}{2\alpha},\quad b_0^2=\text{arbitrary},\quad c_0^2=\text{arbitrary}\,. \tag{44e}$$

### B. Resonances

In order to find the resonances, we substitute

$$x\approx a_0\tau^p+\Omega_1\tau^{p+r},\quad y\approx b_0\tau^q+\Omega_2\tau^{q+r}\,,$$
$$z\approx c_0\tau^s+\Omega_3\tau^{s+r},\quad \tau\to 0\,, \tag{45}$$

into the leading-order terms of (5) and obtain a system of linear algebraic equations in $\Omega_1$, $\Omega_2$, and $\Omega_3$ and for a nontrivial set of solutions $(\Omega_1,\Omega_2,\Omega_3)$ we require that

$$\det\underline{M}_3(r)=\begin{vmatrix}(r+p)(r+p-1)+8\delta a_0^2-2 & 4\delta a_0b_0 & 4\omega a_0c_0\\ 4\delta a_0b_0 & (r+q)(r+q-1)+8\beta b_0^2-2 & 4\epsilon b_0c_0\\ 4\omega a_0c_0 & 4\epsilon b_0c_0 & (r+s)(r+s-1)+8\gamma c_0^2-2\end{vmatrix}=0\,. \tag{46}$$

For case 1, $p=q=s=-1$ and so (46) becomes

$$\begin{vmatrix}r'+8\alpha a_0^2 & 4\delta a_0b_0 & 4\omega a_0c_0\\ 4\delta a_0b_0 & r'+8\beta b_0 & 4\epsilon b_0c_0\\ 4\omega a_0c_0 & 4\epsilon b_0c_0 & r'+8\gamma c_0^2\end{vmatrix}=0,\quad r'=r^2-3r\,. \tag{47}$$

It is easy to check that $r'=4$ is a root of (47), and so

$$(r'-4)(r'+\chi_1)(r'+\chi_2)=0\,, \tag{48}$$

where

$$\chi_1+\chi_2=4[1+2(\alpha a_0^2+\beta b_0^2+\gamma c_0^2)]\,, \tag{49a}$$
$$\chi_1\chi_2=16\{\tfrac{1}{4}(\chi_1+\chi_2)+4[(\alpha\beta-\delta^2/4)a_0^2b_0^2+(\beta\gamma-\epsilon^2/4)b_0^2c_0^2+(\alpha\gamma-\omega^2/4)a_0^2c_0^2]\}\,. \tag{49b}$$

From (49), we infer that the resonances occur at

$$r=-1,4,\tfrac{3}{2}\pm\tfrac{1}{2}(9-4\chi_1)^{1/2},\tfrac{3}{2}\pm\tfrac{1}{2}(9-4\chi_2)^{1/2}\,. \tag{50}$$

As before, the restriction that the resonance values (except $r=-1$) be nonnegative integers leads to the following possibilities:

case 1(i)

$$\chi_1=2,\ \ \chi_2=2,\ \ P=0,\ \ Q=-\tfrac{3}{16},\ \ r=-1,1,1,2,2,4\ , \tag{51}$$

case 1(ii)

$$\chi_1=0,\ \ \chi_2=0,\ \ P=-\tfrac{1}{2},\ \ Q=0,\ \ r=-1,0,0,3,3,4\ , \tag{52}$$

case 1(iii)

$$\chi_1=2,\ \ \chi_2=0,\ \ \text{or}\ \ \chi_1=0,\ \ \chi_2=2,\ \ P=-\tfrac{1}{4},\ \ Q=-\tfrac{1}{8},\ \ r=-1,0,1,2,3,4\ . \tag{53}$$

In Eqs. (51)–(53), $P=\alpha a_0^2+\beta b_0^2+\gamma c_0^2$, and

$$Q=(\alpha\beta-\delta^2/4)a_0^2b_0^2+(\beta\gamma-\epsilon^2/4)b_0^2c_0^2+(\alpha\gamma-\omega^2/4)a_0^2b_0^2\ ,$$

where $a_0^2$, $b_0^2$, and $c_0^2$ are as given in Eq. (43).

For case 2, $p=-1$, $q>-1$, and $s=-1$, we obtain the resonance condition (46) after omitting the coefficients of lesser dominant terms as

$$\begin{vmatrix} r'+8\alpha a_0^2 & 0 & 4\omega a_0c_0 \\ 4\delta a_0b_0 & r(r+2q-1) & 4\epsilon b_0c_0 \\ 4\omega a_0c_0 & 0 & r'+8\gamma c_0^2 \end{vmatrix}=0,\ \ r'=r^2-3r\ , \tag{54a}$$

which on expanding gives

$$r(r+2q-1)(r'-4)(r'+\psi_0)=0,\ \ \psi_0=4[1+2(\alpha a_0^2+\gamma c_0^2)]\ . \tag{54b}$$

Thus the associated resonances are

$$r=-1,0,4,(1-2q),\tfrac{3}{2}\pm\tfrac{1}{2}(9-4\psi_0)^{1/2}\ . \tag{55}$$

It is interesting to note that the resonance values of the case 1 given by (17) of two coupled oscillators merges with the case 2 resonances given by (55) of the three-coupled oscillators with the additional resonances at 0 and $(1-2q)$, and with appropriate parametric rearrangement.

Case 2a. From Eq. (55), we find that the nonnegative integer resonance for $r=(1-2q)$ is possible only if $q\le 0$, which leads to a contradiction that $q\ge\frac{1}{2}$ given in (44a) unless $q=\frac{1}{2}$. However, for the latter the associated solution does not have sufficient number of arbitrary constants.

Case 2b. Here we have the following four possibilities:

$$q=0,\ \ \psi_0=0,\ \ \delta a_0^2+\epsilon c_0^2=0,\ \ \alpha a_0^2+\gamma c_0^2=-\tfrac{1}{2},\ \ r=-1,0,0,1,3,4\ , \tag{56a}$$

$$q=0,\ \ \psi_0=2,\ \ \delta a_0^2+\epsilon c_0^2=0,\ \ \alpha a_0^2+\gamma c_0^2=-\tfrac{1}{4},\ \ r=-1,0,1,1,2,4\ , \tag{56b}$$

$$q=-\tfrac{1}{2},\ \ \psi_0=0,\ \ \delta a_0^2+\epsilon c_0^2=-\tfrac{3}{8},\ \ \alpha a_0^2+\gamma c_0^2=-\tfrac{1}{2},\ \ r=-1,0,0,2,3,4\ , \tag{56c}$$

$$q=-\tfrac{1}{2},\ \ \psi_0=2,\ \ \delta a_0^2+\epsilon c_0^2=-\tfrac{3}{8},\ \ \alpha a_0^2+\gamma c_0^2=-\tfrac{1}{4},\ \ r=-1,0,1,2,2,4\ . \tag{56d}$$

Case 3. Here $p=-1$, $q>-1$, and $s>-1$, and so the resonant determinant (46) becomes

$$\begin{vmatrix} r^2-3r-4 & 0 & 0 \\ 4\delta a_0b_0 & r(r+2q-1) & 0 \\ 4\omega a_0c_0 & 0 & r(r+2s-1) \end{vmatrix}=0\ . \tag{57}$$

Thus we have

$$r=-1,0,0,(1-2q),(1-2s),4\ . \tag{58}$$

It is easy to check that for nonnegative integer resonances $r=(1-2q)$ and $(1-2s)$ in Eq. (58), we should have $q\le 0$ and $s\le 0$ which contradict the fact that $q\ge\frac{1}{2}$ and $s\ge\frac{1}{2}$ for the cases 3a and 3b, Eqs. (44c) and (44d), and thus leading to lesser parameter solutions only.

Case 3c. The requirement of nonnegative integer resonances now leads to the following four choices:

$$q=0,\ \ s=0,\ \ \delta=0,\ \ \omega=0,\ \ r=-1,0,0,1,1,4\ , \tag{59a}$$

$$q=0,\ \ s=-\tfrac{1}{2},\ \ \delta=0,\ \ \omega=\tfrac{3}{4}\alpha,\ \ r=-1,0,0,1,2,4\ , \tag{59b}$$

$$q=-\tfrac{1}{2},\ \ s=0,\ \ \delta=\frac{3}{4}\alpha,\ \ \omega=0,\ \ r=-1,0,0,1,2,4\ , \tag{59c}$$

$$q=-\tfrac{1}{2},\ \ s=-\tfrac{1}{2},\ \ \delta=\tfrac{3}{4}\alpha,\ \ \omega=\tfrac{3}{4}\alpha,\ \ r=-1,0,0,2,3,4\ . \tag{59d}$$

Thus we have isolated eleven distinct sets of integer resonances corresponding to Eqs. (51)—(53). In the next section we proceed to analyze the arbitrariness of the coefficients which enter at these resonance values. It is important to note that in the case of three degrees of freedom, the coincidence of two resonance values in the above is possible without the introduction of logarithmic terms in the solution, unlike the case of two degrees of freedom.

### C. Evaluation of arbitrary constants of integration

In order to compute the constants of integration we introduce the following power series representations into the equations of motion (5):

$$x=a_0\tau^p+\sum_{k=1}^{4}a_k\tau^{p+k},\quad y=b_0\tau^q+\sum_{k=1}^{4}b_k\tau^{q+k},$$
$$z=c_0\tau^s+\sum_{k=1}^{4}c_k\tau^{s+k},\quad \tau\to 0\,. \tag{60}$$

We will search for the arbitrary constants of integration of the cases 1, 2b, and 3c separately.

#### *1. Strong-P cases*

For the case 1(i), Eq. (51), the resonances occur at $r=-1,1,1,2,2,4$. Therefore for the strong-$P$ property to hold we will have to show that five of the coefficients in (60) are arbitrary in addition to $t^*$. Now the coefficients of $(\tau^{-2},\tau^{-2},\tau^{-2})$ in (5) are given by

$$(4\alpha a_0^2-1)a_1+2\delta a_0b_0b_1+2\omega a_0c_0c_1=0\,, \tag{61a}$$

$$2\delta a_0b_0a_1+(4\beta b_0^2-1)b_1+2\epsilon b_0c_0c_1=0\,, \tag{61b}$$

$$2\omega a_0c_0a_1+2\epsilon b_0c_0b_1+(4\gamma c_0^2-1)c_1=0\,, \tag{61c}$$

where $a_0$, $b_0$, and $c_0$ are as given in (43). As the resonance has a double value at $r=1$, it indicates that two arbitrary constants can enter here. From (61) this requires that $\alpha a_0^2=\beta b_0^2=\gamma c_0^2$. However, this on comparison with the value of $P$ in (51) shows that $a_0=b_0=c_0=0$, and so contradicting the form of $Q$ in (51). We thus infer that none of the coefficients $a_1$, $b_1$, and $c_1$ is arbitrary, and so this case is of no further interest.

For case 1(ii), Eq. (52), the resonances are $r=-1,0,0,3,3,4$. From the leading-order analysis we obtain

$$2\alpha a_0^2+\delta b_0^2+\omega c_0^2=-1,\quad \delta a_0^2+2\beta b_0^2+\epsilon c_0^2=-1\,,$$
$$\omega a_0^2+\epsilon b_0^2+2\gamma c_0^2=-1\,. \tag{62}$$

Then the coefficients $a_0,b_0$ (or $b_0,c_0$ or $c_0,a_0$) are arbitrary only if

$$\alpha=\beta=\gamma,\quad \delta=\epsilon=\omega=2\alpha\,. \tag{63}$$

Proceeding further we determine $a_k$, $b_k$, and $c_k$, $k=1,2$ uniquely. Then the coefficients of $(\tau^0,\tau^0,\tau^0)$ lead to the single equation $a_0a_3+b_0b_3+c_0c_3=0$, so that two of the quantities $a_3$, $b_3$, and $c_3$ are arbitrary. Collecting now the coefficients of $(\tau^1,\tau^1,\tau^1)$ in (5), and rearranging, we obtain the two equations

$$b_0a_4-a_0b_4=\frac{1}{2a_0b_0}(a_0b_2^2-b_0a_2^2)\,, \tag{64a}$$

$$a_0b_0a_4-(c_0^2+a_0^2)b_4+b_0c_0c_4$$
$$=-\frac{B}{4\alpha}b_2-\tfrac{1}{2}(3b_0b_2^2+b_0a_2^2+2a_0a_2b_2$$
$$+b_0c_2^2+2c_0c_2b_2)\,, \tag{64b}$$

thereby showing that the coefficient $a_4$ or $b_4$ is arbitrary irrespective of the values of $A,B,C$ in (5). Thus for the choice (63), a full six-parameter $P$ branch of solution exists for (5).

For case 1(iii), Eq. (53), $r=-1,0,1,2,3,4$, proceeding in a similar way we check that the coefficient equations (62) and (61) hold here also. But the fact that arbitrariness is needed both for $r=0$ and 1 leads to a contradiction on the coefficients thereby disallowing this case.

#### *2. Weak-P cases*

For case 2b, we isolated four distinct possibilities in Eqs. (56). Again using (60) in (5) with the appropriate leading orders, (44), we find that three of the possibilities corresponding to (56a), (56b), and (56d) do not satisfy the Painlevé criteria. For example, in the case of resonances given by (56a), $r=-1,0,0,1,3,4$. Since $r=0$ appears twice, two of the coefficients $a_0$, $b_0$, and $c_0$ should be arbitrary. The leading-order analysis, (44b), identifies that $b_0$ is arbitrary with the parametric equations $2\alpha a_0^2+\omega c_0^2=-1$, $\omega a_0^2+2\gamma c_0^2=-1$. But the resonance analysis, (56a) shows that $\delta a_0^2+\epsilon c_0^2=0$, $\alpha a_0^2+\gamma c_0^2=-\frac{1}{2}$. Consistency requires that there exists no parametric value for which the coefficient $a_0$ or $c_0$ is arbitrary and hence the solution can have only lesser number of arbitrary coefficients. Similar conclusion can be arrived at for the cases of the resonances given by (56b) and (56d).

However, for the choice (iii) given in Eq. (56c), we have $r=-1,0,0,2,3,4$. From the leading-order analysis we know that $t^*$ and $b_0$ are arbitrary and that $2\alpha a_0^2+\omega c_0^2=-1$, $\alpha a_0^2+2\gamma c_0^2=-1$, while the resonance analysis, (56c), shows that $\delta a_0^2+\epsilon c_0^2=-\frac{3}{8}$, $\alpha a_0^2+\gamma c_0^2=-\frac{1}{2}$. Now unlike the previous cases, it is possible to choose a set of parametric values.

$$\alpha=\gamma,\quad \omega=2\alpha,\quad \delta=\epsilon,\quad 3\alpha=4\delta\,, \tag{65}$$

such that the coefficient $a_0$ (or $c_0$) is arbitrary. Proceeding further, with the restriction (65), we determine $a_1$, $b_1$, and $c_1$ uniquely and that

$$(4\alpha a_0^2-1)a_2+4\alpha a_0c_0c_2=-Aa_0+\delta(\delta-8\beta)a_0b_0^4\,, \tag{66a}$$

$$2\delta a_0a_2+0\times b_2+2\epsilon c_0c_2$$
$$=-B-\tfrac{1}{12}(\delta^2-48\beta\delta+288\beta^2)b_0^4\,, \tag{66b}$$

$$4\alpha a_0c_0a_2+(4\alpha c_0^2-1)c_2=-Cc_0+\delta(\delta-8\beta)c_0b_0^4\,. \tag{66c}$$

Solving (66), we find that for consistency it is required that

$$A=4B=C\,,$$

and (67)

TABLE II. Painlevé cases of three-coupled quartic anharmonic oscillator system. (Note: In case 1, for the additional restriction $A=B=C$, $I_1=(xp_y-yp_x)$, $I_2=(xp_y-yp_x)^2+(yp_z-zp_y)^2+(zp_x-xp_z)^2$ are the second and third integrals of motion.)

| Cases (see text) | Parametric restriction | Main or subsidiary branch | Leading orders $p$ | $q$ | $s$ | Resonances $r$ | No. of arbitrary constants |
|---|---|---|---|---|---|---|---|
| 1 | $\alpha=\beta=\gamma$, | MB | $-1$ | $-1$ | $-1$ | $-1,0,0,3,3,4$ | 6 |
| | $\delta=\epsilon=\omega=2\alpha$ | SB-1 | $-1$ | 2 | $-1$ | $-4,-1,0,0,3,4$ | 5 |
| | | SB-2 | $-1$ | 2 | 2 | $-3,-3,-1,0,0,4$ | 4 |
| 2b(iii)-1 | $\alpha=16\beta=\gamma$, | MB | $-1$ | $-\frac{1}{2}$ | $-1$ | $-1,0,0,2,3,4$ | 6 |
| | $\delta=\epsilon=12\beta$, | SB-1 | $-1$ | $-1$ | $-1$ | $-2,-1,0,3,4,5$ | 5 |
| | $\omega=32\beta$, $A=4B=C$ | SB-2 | $-1$ | $\frac{3}{2}$ | $-1$ | $-2,-1,0,0,3,4$ | 5 |
| | | SB-3 | $-1$ | $\frac{3}{2}$ | 2 | $-3,-2,-1,0,0,4$ | 4 |
| 2b(iii)-2 | $\alpha=8\beta=\gamma$, | MB | $-1$ | $-\frac{1}{2}$ | $-1$ | $-1,0,0,2,3,4$ | 6 |
| | $\delta=\epsilon=6\beta$, | SB-1 | $-1$ | $-1$ | $-1$ | $-5,-1,0,3,4,8$ | 5 |
| | $\omega=16\beta$, $A=4B=C$ | SB-2 | $-1$ | $\frac{3}{2}$ | $-1$ | $-2,-1,0,0,3,4$ | 5 |
| | | SB-3 | $-1$ | $\frac{3}{2}$ | 2 | $-3,-2,-1,0,0,4$ | 4 |

$$\delta^2-18\beta\delta+72\beta^2=0\ ,$$

from which the coefficient $b_2$ becomes arbitrary. Continuing further we find that the coefficients $a_3$ (or $c_3$) and $a_4$ (or $c_4$) are arbitrary without any additional parametric constraints. Thus from (65) and (67) we conclude that for the following two sets of parametric values, case 2b(iii)-1,

$$\alpha=16\beta=\gamma,\ \ \delta=\epsilon=12\beta,\ \ \omega=32\beta,\ \ A=4B=C \tag{68a}$$

and, case 2b(iii)-2,

$$\alpha=8\beta=\gamma,\ \ \delta=\epsilon=6\beta,\ \ \omega=16\beta,\ \ A=4B=C\ , \tag{68b}$$

Eqs. (5) possess a full $P$ branch of solution with sufficient number (six) of arbitrary constants.

For the case 3c we have four sets of resonances (59a)–(59d). Substituting (60) in (5), we compute $a_k$, $b_k$, and $c_k$, $k=1,2,3,4$, by comparing the coefficients of $(\tau^{k-3},\tau^{k-5/2},\tau^{k-5/2})$. The detailed calculations show that none of the four choices lead to sufficient number (six) of arbitrary constants. For example, considering the resonance values (59a), $r=-1,0,0,1,1,4$, and using the fact that $b_0$, $c_0$ are arbitrary, Eq. (44e), from the coefficients of $(\tau^{-2},\tau^{-3/2},\tau^{-3/2})$ in (5), we obtain a set of equations for $a_1$, $b_1$, and $c_1$ and solving them we find $a_1=0$, $b_1=8b_0(2\beta b_0^2+\epsilon c_0^2)$, $c_1=8c_0(2\gamma c_0^2+\epsilon b_0^2)$. This clearly shows that two of the coefficients $a_1,b_1,c_1$ are not arbitrary. Thus the corresponding solution will have lesser number of arbitrary constants. Similar conclusion is also arrived at for the remaining possibilities.

The salient features of the above analysis are schematically sketched in Table II. Finally we have to explicitly show that for the above three parametric choices, the solution in the remaining subsidiary branches does not introduce any movable singularity. We have checked in all the branches the solution remains single valued (within the strong- and weak-$P$ criteria). The details are also given in Table II. Thus for the three-coupled anharmonic oscillators, (2), the three choices exhibit Painlevé property.

### D. Integrals of motion

We now investigate the form of the associated second and third integrals of motion for the Painlevé cases in Table II. For this purpose we again consider a general integral of motion which contains velocities up to fourth power and proceed as in the previous section. However, for the sake of clarity, here we present the results explicitly for the case with quadratic velocities alone:

$$\begin{aligned}I=\ &\theta_1p_x^2+\theta_2p_y^2+\theta_3p_z^2+\theta_4p_xp_y+\theta_5p_yp_z\\&+\theta_6p_xp_z+\theta_7p_x+\theta_8p_y+\theta_9p_z+\theta_{10}\ ,\end{aligned} \tag{69}$$

where $\theta_i$'s are functions of $(x,y,z)$ alone. When the coefficients of $p_x^lp_y^mp_z^n$, in the Poisson bracket $\{I,H\}_{\rm PB}=0$ are set equal to zero, the following partial differential equations result:

$$\frac{\partial\theta_1}{\partial x}=0,\ \ \frac{\partial\theta_1}{\partial y}+\frac{\partial\theta_4}{\partial x}=0,\ \ \frac{\partial\theta_2}{\partial x}+\frac{\partial\theta_4}{\partial y}=0,\ \ \frac{\partial\theta_1}{\partial z}+\frac{\partial\theta_6}{\partial x}=0,\ \ \frac{\partial\theta_3}{\partial x}+\frac{\partial\theta_6}{\partial z}=0,\ \ \frac{\partial\theta_2}{\partial y}=0,\ \ \frac{\partial\theta_2}{\partial z}+\frac{\partial\theta_5}{\partial y}=0\ ,$$

$$\frac{\partial\theta_3}{\partial y}+\frac{\partial\theta_5}{\partial z}=0,\ \ \frac{\partial\theta_3}{\partial z}=0,\ \ \frac{\partial\theta_4}{\partial z}+\frac{\partial\theta_5}{\partial x}+\frac{\partial\theta_6}{\partial y}=0,\ \ \frac{\partial\theta_7}{\partial x}=0,\ \ \frac{\partial\theta_8}{\partial y}=0,\ \ \frac{\partial\theta_9}{\partial z}=0,\ \ \frac{\partial\theta_7}{\partial y}+\frac{\partial\theta_8}{\partial x}=0\ ,$$

$$\frac{\partial\theta_8}{\partial z}+\frac{\partial\theta_9}{\partial y}=0,\ \ \frac{\partial\theta_7}{\partial z}+\frac{\partial\theta_9}{\partial x}=0,\ \ 2\theta_1\ddot{x}+\theta_4\ddot{y}+\theta_6\ddot{z}+\frac{\partial\theta_{10}}{\partial x}=0,\ \ \theta_4\ddot{x}+2\theta_2\ddot{y}+\theta_5\ddot{z}+\frac{\partial\theta_{10}}{\partial y}=0\ ,$$
$$\theta_6\ddot{x}+\theta_5\ddot{y}+2\theta_3\ddot{z}+\frac{\partial\theta_{10}}{\partial z}=0,\ \ \theta_7\ddot{x}+\theta_8\ddot{y}+\theta_9\ddot{z}=0\ . \tag{70}$$

By solving (70) successively we find that nontrivial solutions exist for the parametric case,

$$\alpha=\beta=\gamma,\ \ \delta=\epsilon=\omega=2\alpha,\ \ A=B=C\ , \tag{71a}$$

and the corresponding second and third integrals of motion become

$$I_1=(xp_y-yp_x)\ , \tag{71b}$$

$$I_2=(xp_y-yp_x)^2+(yp_z-zp_y)^2+(zp_x-xp_z)^2\ . \tag{71c}$$

Obviously this case corresponds to a special choice of the Eq. (63), and is separable in spherical polar coordinates. Apart from this, we have not yet succeeded to isolate any other integrals of motion in calculations involving up to fourth power in velocities. Therefore it is not clear to us at present as to what general form one has to assume to proceed with the above type of analysis to obtain the second and third integrals of motion for the remaining $P$ cases.

## V. $P$ PROPERTIES OF $N$ QUARTICALLY COUPLED OSCILLATORS

In this section we extend our $P$ analysis to the case of arbitrary $N$ quartically coupled oscillators obeying the equation of motion (6).

### A. Leading-order behaviors

Assuming now

$$x_i\approx a_{i0}\tau^{p_i},\ \ \tau\to 0,\ \ i=1,2,\ldots,N \tag{72}$$

in (6) we isolate $N$ *distinct* sets of leading-order behaviors. Case 1,

$$p_1=p_2=\cdots=p_N=-1\ ,$$
$$2\alpha_i a_{i0}^2+\sum_{\substack{j=1\\(j\neq i)}}^{N}\beta_{ij}a_{j0}^2=-1,\ \ i=1,2,\ldots,N\ . \tag{73}$$

Case 2,

$$p_1=p_3=\cdots=p_N=-1\ ,$$
$$p_2=\tfrac{1}{2}+\tfrac{1}{2}\left[1-8\sum_{\substack{j=1\\(j\neq 2)}}^{N}\beta_{2j}a_{j0}^2\right]^{1/2}\geq\tfrac{1}{2}\ , \tag{74a}$$

$$p_1=p_3=\cdots=p_N=-1\ ,$$
$$p_2=\tfrac{1}{2}-\tfrac{1}{2}\left[1-8\sum_{\substack{j=1\\(j\neq 2)}}^{N}\beta_{2j}a_{j0}^2\right]^{1/2}>-1\ , \tag{74b}$$

where the leading coefficients are given by

$$a_{20}=\text{arbitrary}$$
and
$$2\alpha_i a_{i0}^2+\sum_{\substack{j=1\\(j\neq 2)}}^{N}\beta_{ij}a_{j0}^2=-1,\ \ i=1,3,4,\ldots,N\ . \tag{74c}$$

Case 3,

$$p_1=p_4=\cdots=p_N=-1\ ,$$
$$p_2=\tfrac{1}{2}+\tfrac{1}{2}\left[1-8\sum_{\substack{j=1\\(j\neq 2,3)}}^{N}\beta_{2j}a_{j0}^2\right]^{1/2}\geq\tfrac{1}{2}\ ,$$
$$p_3=\tfrac{1}{2}+\tfrac{1}{2}\left[1-8\sum_{\substack{j=1\\(j\neq 2,3)}}^{N}\beta_{3j}a_{j0}^2\right]^{1/2}\geq\tfrac{1}{2}\ , \tag{75a}$$

$$p_1=p_4=\cdots=p_N=-1\ ,$$
$$p_2=\tfrac{1}{2}+\tfrac{1}{2}\left[1-8\sum_{\substack{j=1\\(j\neq 2,3)}}^{N}\beta_{2j}a_{j0}^2\right]^{1/2}\geq\tfrac{1}{2}\ ,$$
$$p_3=\tfrac{1}{2}-\tfrac{1}{2}\left[1-8\sum_{\substack{j=1\\(j\neq 2,3)}}^{N}\beta_{3j}a_{j0}^2\right]^{1/2}>-1\ , \tag{75b}$$

$$p_1=p_4=\cdots=p_N=-1\ ,$$
$$p_2=\tfrac{1}{2}-\tfrac{1}{2}\left[1-8\sum_{\substack{j=1\\(j\neq 2,3)}}^{N}\beta_{2j}a_{j0}^2\right]^{1/2}>-1\ ,$$
$$p_3=\tfrac{1}{2}-\tfrac{1}{2}\left[1-8\sum_{\substack{j=1\\(j\neq 2,3)}}^{N}\beta_{3j}a_{j0}^2\right]^{1/2}>-1\ . \tag{75c}$$

Here

$$a_{20},a_{30}=\text{arbitrary}$$
and
$$2\alpha_i a_{i0}^2+\sum_{\substack{j=1\\(j\neq 2,3)}}^{N}\beta_{ij}a_{j0}^2=-1,\ \ i=1,4,\ldots,N\ . \tag{75d}$$

We can continue this process by considering $i-1$ $p$'s at a time to be greater than $-1$ and the remaining $p$'s to be equal to $-1$, $i=1,2,\ldots,N$ and write down the corresponding values of $p$'s and $a_0$'s. Thus for the case $N$ we have

$$p_1=-1,\ p_2=\tfrac{1}{2}+\tfrac{1}{2}(1-8\beta_{21}a_{10}^2)^{1/2}\geq\tfrac{1}{2}\ ,\quad p_3=\tfrac{1}{2}+\tfrac{1}{2}(1-8\beta_{31}a_{10}^2)^{1/2}\geq\tfrac{1}{2},\ \ldots\ ,\quad p_N=\tfrac{1}{2}+\tfrac{1}{2}(1-8\beta_{N1}a_{10}^2)^{1/2}\geq\tfrac{1}{2}\ , \tag{76a}$$

$$p_1=-1,\ p_2=\tfrac{1}{2}+\tfrac{1}{2}(1-8\beta_{21}a_{10}^2)^{1/2}\geq\tfrac{1}{2}\ ,\quad p_3=\tfrac{1}{2}+\tfrac{1}{2}(1-8\beta_{31}a_{10}^2)^{1/2}\geq\tfrac{1}{2},\ \ldots\ ,\quad p_N=\tfrac{1}{2}-\tfrac{1}{2}(1-8\beta_{N1}a_{10}^2)^{1/2}>-1\ , \tag{76b}$$

and this continues up to

$$p_1=-1,\ p_2=\tfrac{1}{2}-\tfrac{1}{2}(1-8\beta_{21}a_{10}^2)^{1/2}>-1\ ,\quad p_3=\tfrac{1}{2}-\tfrac{1}{2}(1-8\beta_{21}a_{10}^2)^{1/2}>-1,\ \ldots\ ,\quad p_N=\tfrac{1}{2}-\tfrac{1}{2}(1-8\beta_{N1}a_{10}^2)^{1/2}>-1\ , \tag{76c}$$

with

$$a_{10}^2=-\frac{1}{2\alpha};\quad a_{20},\ a_{30},\ \ldots,\ a_{N0}=\text{arbitrary}\ . \tag{76d}$$

### B. Determination of resonances

Here again we substitute

$$x_i\approx a_{i0}\tau^{p_i}+\Omega_i\tau^{p_i+r},\quad \tau\to 0,\quad i=1,2,\ldots,N\ , \tag{77}$$

into the leading-order terms of (6) and obtain a system of $N$ linear algebraic equations for the $\Omega_i$'s. To have a nontrivial set of solutions $(\Omega_1,\Omega_2,\ldots,\Omega_N)$ we require that

$$\det\underline{M}_N(r)=0,\quad \underline{M}_N(r)=\underline{A}_N+\underline{D}_N(r)\ , \tag{78a}$$

$$\underline{A}_N=\begin{bmatrix} 8\alpha_1 a_{10}^2 & 4\beta_{12}a_{10}a_{20} & \cdots & 4\beta_{1N}a_{10}a_{N0} \\ 4\beta_{21}a_{20}a_{10} & 8\alpha_2 a_{20}^2 & \cdots & 4\beta_{2N}a_{20}a_{N0} \\ \vdots & \ddots & \ddots & \ddots \\ 4\beta_{N1}a_{N0}a_{10} & 4\beta_{N2}a_{N0}a_{20} & \cdots & 8\alpha_N a_{N0}^2 \end{bmatrix}, \tag{78b}$$

$$\underline{D}_N(r)=\mathrm{diag}[(r+p_1)(r+p_1-1)-2,\ (r+p_2)(r+p_2-1)-2,\ \ldots,\ (r+p_N)(r+p_N-1)-2]\ . \tag{78c}$$

Considering case 1, (73) implies that $\underline{D}_N(r)=(r^2-3r)\underline{I}_N$, so that $\det\underline{M}_N(r)$ is the characteristic polynomial of $[A_N]$ in the variable $r^2-3r$. The coefficients of this polynomial can be expressed on the one hand in terms of its roots $\chi_0,\chi_1,\chi_2,\ldots,\chi_{N-1}$, and on the other hand in terms of the quantities $\mathrm{tr}A_N^i$, $i=1,2,\ldots,N$. Using (73), we can prove that there is a root, $\chi_0$ say, equal to 4. As a result, for the case 1, we find that (78) becomes

$$\det\underline{M}_N=(r+1)(r-4)(r^2-3r+\chi_1)(r^2-3r+\chi_2)\cdots(r^2-3r+\chi_{N-1})=0\ , \tag{79a}$$

so that the resonances occur at

$$r=-1,4,\tfrac{3}{2}\pm\tfrac{1}{2}(9-4\chi_1)^{1/2},\tfrac{3}{2}\pm\tfrac{1}{2}(9-4\chi_2)^{1/2},\ldots,\tfrac{3}{2}\pm(9-4\chi_{N-1})^{1/2}\ . \tag{79b}$$

Here the quantities $\chi_l$, $l=1,2,\ldots,N-1$, satisfy the following conditions:

$$\sum_{l=1}^{N-1}\chi_l=4\left[1+2\sum_{i=1}^{N}\alpha_i a_{i0}^2\right], \tag{80a}$$

$$\sum_{\substack{m,n=1\\(m\neq n)}}^{N-1}\tfrac{1}{2}(\chi_m\chi_n)=4^2\left\{\tfrac{1}{4}\sum_{l=1}^{N-1}\chi_l+4\left[\sum_{\substack{i,j=1\\(i\neq j)}}^{N}\frac{1}{2}\left[\alpha_i\alpha_j-\frac{\beta_{ij}^2}{4}\right]a_{i0}^2a_{j0}^2\right]\right\}, \tag{80b}$$

$$\sum_{\substack{m,n,p=1\\(m\neq n\neq p)}}^{N-1}\tfrac{1}{3}(\chi_m\chi_n\chi_p)=4^3\left[\tfrac{1}{16}\sum_{\substack{m,n=1\\(m\neq n)}}^{N-1}\tfrac{1}{2}(\chi_m\chi_n)+16\left[\sum_{\substack{i,j,k=1\\(i\neq j\neq k)}}^{N}\tfrac{1}{3}[\alpha_i\alpha_j\alpha_k-\tfrac{1}{4}(\alpha_i\beta_{jk}^2+\alpha_j\beta_{ik}^2+\alpha_k\beta_{ij}^2-\beta_{ij}\beta_{jk}\beta_{ki})]a_{i0}^2a_{j0}^2a_{k0}^2\right]\right], \tag{80c}$$

and so on. Continuing further we can finally write down the value of the product $(\chi_1\chi_2\chi_3\cdots\chi_{N-1})$ from the expansion of the determinant in (78). We can easily verify that our results for $N=2,3$ given in Secs. III and IV follow straightforwardly.

Again, the allowed values of resonances such that all the $r$'s (except $r=-1$) in (79) are nonnegative integers correspond to the following possibilities:

$$l \text{ of the } \chi_i=2,\quad (N-1-l) \text{ of the } \chi_i=0,\quad i,l=1,2,\ldots,N-1\,, \tag{81}$$

$$r=-1;\ 0,0,\ldots,(N-l-1) \text{ times};\ 1,1,\ldots,l \text{ times};\ 2,2,\ldots,l \text{ times};\ 3,3,\ldots,(N-l-1) \text{ times};\ 4\,. \tag{82}$$

For case 2, $p_i=-1$, $i=1,3,\ldots,N$, and $p_2>-1$ and hence the elements of the second column of the matrix $[A_N]$, Eq. (78b), become zero, except the diagonal one. As a consequence, we find that

$$r=-1,0,(1-2p_2),\tfrac{3}{2}\pm\tfrac{1}{2}(9-4\chi_1)^{1/2},\tfrac{3}{2}\pm\tfrac{1}{2}(9-4\chi_3)^{1/2},\ldots,\tfrac{3}{2}\pm\tfrac{1}{2}(9-4\chi_{N-1})^{1/2},4\,, \tag{83}$$

which correspond to the resonance values of case 1 of the $(N-1)$ degrees of freedom system with additional roots at 0 and $(1-2p_2)$. Here the quantities $\chi_l, l=1,3,\ldots,N-1$, are obtained from (80) by omitting the coefficient $a_{20}$ and the parameter $\beta_{2j}$ on the right-hand sides of (80). As the resonances are to be nonnegative integers, from Eq. (83), we necessarily require that $p_2\le\frac{1}{2}$ which contradicts the fact that $p_2\ge\frac{1}{2}$, [cf. (74a)] except for $p_2=\frac{1}{2}$ and case 2a necessarily corresponds to lesser parameter branches. Further, for case 2b, the restriction on nonnegative integer resonances leads to the following. (i) $p_2=0$, $l$ of the $\chi_i=2$, $(N-2-l)$ of the $\chi_i=0,i,l=1,3,\ldots,N-1$ in (83) and so

$$r=-1,\ 0,0,\ldots,N-l-1 \text{ times};\ 1,1,\ldots,(l+1) \text{ times};\ \ 2,2,\ldots,l \text{ times};\ \ 3,3,\ldots,(N-l-2) \text{ times};\ 4\,. \tag{84a}$$

(ii) $p_2=-\frac{1}{2}$, $l$ of the $\chi_i=2$, $(N-2-l)$ of the $\chi_i=0$, $i,l=1,3,\ldots,N-1$ in (83) and so

$$r=-1,0,0,\ldots,(N-l-1) \text{ times};\ 1,1,\ldots,l \text{ times};\ 2,2,\ldots,(l+1) \text{ times};\ 3,3,\ldots,(N-l-2) \text{ times};\ 4\,. \tag{84b}$$

In a similar fashion we derive the resonances for the remaining cases. Finally, the resonance values of case $N$ become

$$r=-1,0,0,\ldots,(N-1)\text{times},\ (1-2p_2),(1-2p_3),\ldots,(1-2p_N),4\,. \tag{85}$$

From (85), it is easy to check that nonnegative integer resonances are possible only if $p_i\le 0$, $i=2,3,\ldots,N$ which in general contradicts the leading-order behaviors, Eqs. (76a) and (76b), that $p_i\ge\frac{1}{2}$ except for the last possibility given by Eq. (76c). In the latter case we may then obtain $2N-2$ sets of integer resonances corresponding to one of the $p_i$'s, $i=2,3,\ldots,N$ equal to zero and the rest equal to $-\frac{1}{2}$ and so on.

### C. Identification of arbitrary constants

We now explicitly evaluate the coefficients of $a_{i\mu}$, $\mu=1,2,3,4$ by using the series representation

$$x_i=a_{i0}\tau^{p_i}+\sum_{\mu=1}^{4}a_{i\mu}\tau^{p_i+\mu},\quad \tau\to 0 \tag{86}$$

in the equation of motion (6). The following analysis is used to compute the coefficients $a_{i\mu}$, $\mu=1,2,3,4$. By equating the coefficients of $(\tau^{p_1+\mu-2},\tau^{p_2+\mu-2},\ldots,\tau^{p_N+\mu-2})$ to zero in (6), at each $\mu$, we obtain an equation of the form

$$\underline{M}_{N,\mu}\underline{a}_{i\mu}=\underline{S}_\mu(a_{i(\mu-1)})\,, \tag{87}$$

where $\underline{M}_{N,\mu}$ is an $N\times N$ matrix, $\underline{a}_{i\mu}$ and $\underline{S}_\mu$ are $N\times 1$ column matrices. The matrix $\underline{S}_\mu$ depends only on $a_{i,(\mu-1)}$ coefficients. The application of Cramer's rule for determinants yields a unique solution to (87),

$$a_{i\mu}=\frac{\det\overline{\underline{M}}_{N,\mu}}{\det\underline{M}_{N,\mu}},\quad \det\underline{M}_{N,\mu}\ne 0 \tag{88}$$

where the $N\times N$ matrix $\overline{\underline{M}}_{N,\mu}$ is obtained by replacing the $i$th column in $\underline{M}_{N,\mu}$ by the column matrix $\underline{S}_\mu$. At the resonance $\mu$, the $\det\underline{M}_\mu$ may vanish and so Eq. (87) may not have a unique solution. Suppose further that the det $\overline{\underline{M}}_{N,\mu}$ also vanishes at the same resonance; then one of the coefficients of $\underline{a}_{i\mu}$ may be arbitrary. Furthermore, if $m$ rows of $\underline{M}_{N,\mu}$ are identical and so also the corresponding $m$ elements of $\underline{S}_\mu$, then $m-1$ of the $\underline{a}_{i\mu}$'s will be arbitrary. Making use of the above technique we identified that only three cases, one from case 1, and the other two from case 2b, possess the required number of $2N$ arbitrary constants to be of Painlevé type, while the remaining cases do not satisfy the necessary conditions of Painlevé property. The details are briefly outlined below.

#### 1. *Strong-P case*

For the case 1 [cf. Eqs. (73)], $p_i=-1$, $i=1,2,\ldots,N$,

$$2\alpha_i a_{i0}^2+\sum_{\substack{j=1\\(j\ne i)}}^{N}\beta_{ij}a_{j0}^2=-1\,,i\ne j,\ i,j=1,2,\ldots,N\,.$$

For the choice all the $\chi_i=0$, $i=1,2,\ldots,N$ in (81), the resonance values (82) become $r=-1,0,0,\ldots,(N-1)$ times, $3,3,\ldots,(N-1)$ times, 4, and from (80) we have $\sum_{i=1}^{N}\alpha_i a_{i0}^2=-\frac{1}{2},\ldots,$ so that consistency requires that

$$\alpha_j=\alpha_1,\quad \beta_{ij}=2\alpha_1\quad (i\ne j),\ i,j,=1,2,\ldots,N \tag{89}$$

from which the $(N-1)$'s of the coefficients $a_{i0}$ are arbitrary. Using (89) in (6) we further infer that

$$(4\alpha_1 a_{i0}^2-1)a_{i1}+4\alpha_1\sum_{\substack{j=1\\(j\ne i)}}^{N}a_{i0}a_{j0}a_{j1}=0\,,\qquad i=1,2,\ldots,N\,. \tag{90}$$

From (90) we check that the determinant $\overline{M}_{N,1}$ vanishes, while $\det \underline{M}_{N,1} \neq 0$, cf. (88), and hence $a_{i1}=0$, $i=1,2,\ldots,N$. Moreover,

$$(4\alpha_1 a_{i0}^2-1)a_{i2}+4\alpha_2 \sum_{\substack{j=1\\(j\neq i)}}^{N} a_{i0}a_{j0}a_{j2}=-A_i a_{i0}\ , \qquad i=1,2,\ldots,N \quad (91)$$

from which $a_{i2}$, $i=1,2,\ldots,N$ can be found uniquely. Then considering the coefficients of $(\tau^0,\tau^0,\ldots,\tau^0)$ in (6), we find that they reduce to a single equation

$$\sum_{j=1}^{N} a_{j0}a_{j3}=0\ , \quad (92)$$

so that the $(N-1)$'s of the coefficients $a_{i3}$, $i=1,2,\ldots,N$ are arbitrary, while the coefficients of $(\tau^1,\tau^1,\ldots,\tau^1)$ after rearrangement [as in the case of three degrees of freedom, Eqs. (64)] lead to $(N-1)$ equations for the $N$ coefficients $a_{i4}$, thereby showing that one of them is arbitrary without any further parametric restrictions. Thus for the choice (89), (6) possesses a full $2N$ parameter $P$ branch of solution. For the other resonance possibilities given in Eq. (82), a similar analysis can be carried out to check that no other $2N$ parameter $P$ branch exists here.

### 2. *Weak-P case*

Considering now the case-2 [Eq. (74b)] singularity solutions associated with the resonance values (84), we can verify that they do not possess the required number of arbitrary constants as in the three-coupled-oscillator system except for the choice $p_i=-1$, $i=1,3,\ldots,N$, $p_2=-\frac{1}{2}$, and that all $\chi_i=0$, $i=1,3,\ldots,N-1$ in (84b) with $r=-1,0,0,\ldots,(N-1)$ times, $2,3,3,\ldots,(N-2)$ times, 4. Thus from (74) and (80) we obtain the conditions

$$2\alpha_i a_{i0}^2+\sum_{\substack{j=1\\(j\neq 2)}}^{N}\beta_{ij}a_{j0}^2=-1\ ,$$

$$\sum_{\substack{j=1\\(j\neq 2)}}^{N}\beta_{2j}a_{j0}^2=-\tfrac{3}{8}$$

and

$$\sum_{i=1}^{N}\alpha_i a_{i0}^2=-\tfrac{1}{2},\ldots,\quad i\neq j,\ i=1,3,\ldots,N\ .$$

Consistency then requires that

$$\alpha_i=\alpha_1,\ \ \beta_{ij}=2\alpha_1,\ \ 3\alpha_1=4\beta_{2j}\ , \qquad i\neq j,\ i,j=1,3,\ldots,N \quad (93)$$

so that the $(N-2)$ coefficients of $a_{i0}$, $i=1,3,\ldots,N$ are arbitrary in addition to $a_{20}$.

Making use of the parametric constraints (93) now in (6), we determine $a_{i1}$, $i=1,2,\ldots,N$ uniquely. Then we obtain

$$(4\alpha_1 a_{i0}^2-1)a_{i2}+4\alpha_1\sum_{\substack{j=1\\(j\neq 2)}}^{N}a_{i0}a_{j0}a_{j2} =-A_i a_{i0}+\beta_{2j}(\beta_{2j}-8\alpha_2)a_{i0}a_{20}^4\ , \quad (94a)$$

$$2\beta_{2j}\sum_{\substack{j=1\\(j\neq 2)}}^{N}a_{j0}a_{j2}+0a_{22} =-A_2-\tfrac{1}{12}(\beta_{2j}^2-48\alpha_2\beta_{2j}+288\alpha_2^2)a_{20}^4 \qquad i\neq j,\ i,j=1,3,\ldots,N\ . \quad (94b)$$

Solving (94a), we find that

$$a_{i2}=\frac{a_{i0}}{3}[A_i-\beta_{2j}(\beta_{2j}-8\alpha_2)a_{20}^4]\ ,\quad i=1,3,\ldots,N\ .$$

Using this in (94b), we infer that the coefficient $a_{22}$ is arbitrary if

$$A_i=4A_2,\ \ \beta_{2j}^2-18\alpha_2\beta_{2j}+72\alpha_2^2=0\ , \quad (95a)$$

or

$$A_i=4A_2,\ \ \beta_{2j}=12\alpha_2,\ 6\alpha_2,\ \ i,j=1,3,\ldots,N\ . \quad (95b)$$

Proceeding then with (95b), we can show that the $(N-2)$ coefficients of $a_{i3}$, $i=1,3,\ldots,N$ are arbitrary, and that one of the coefficients of $a_{i4}$, $i=1,3,\ldots,N$ is arbitrary without any new parametric restrictions. The latter two facts follow straightforwardly from our earlier assertion in Sec. V that the resonance values of case 2 of $N$th degrees of freedom merge with case 1 resonance values of $(N-1)$ degrees of freedom and that the general solution possess the required number $2(N-1)$ arbitrary constants. Thus collecting all the above facts, we conclude that for the following two parametric values, case 2b(i),

$$\alpha_i=16\alpha_2,\ \ \beta_{2j}=12\alpha_2,\ \ \beta_{ij}=32\alpha_2,\ \ A_i=4A_2\ , \quad (96a)$$

and case 2b(ii),

$$\alpha_i=8\alpha_2,\ \ \beta_{2j}=6\alpha_2,\ \ \beta_{ij}=16\alpha_2\ ,\ \ A_i=4A_2\ , \qquad i\neq j,\ i,j=1,3,\ldots,N \quad (96b)$$

Eq. (6) possesses a full $2N$ parameter $P$ branch of solutions.

Similar investigations are carried out for the remaining cases 3—$N$ and we find that the weak-$P$ property does not hold good for the above cases. As the procedure is analogous, we do not present the details here. Table III gives the essential features of our Painlevé cases for the $N$-coupled-oscillator system. The final task of checking that the remaining subsidiary branches (SB) for the above three choices, namely (89), (96a), and (96b), are also single valued within the Painlevé criteria proceeds in an analogous manner and we verify that these choices do indeed correspond to the Painlevé cases.

## D. Integrals of motion

If we transform the Cartesian coordinates $x_i$ to the polar coordinates of $N$ dimension then the Hamiltonain $H_N$ becomes radially symmetric for the parametric values $\alpha_i=\alpha_1$, $\beta_{ij}=2\alpha_1$, $A_i=A_N$, $i=1,2,\ldots,N$, and the equations of motion become separable and hence the system possess $N$ integrals of motion, thereby the integrability is established. It remains to be seen how to obtain the integrals of motion for the other cases.

TABLE III. Painlevé cases of the $N$-coupled quartic anharmonic oscillator system. (Note: In case 1, for the additional restriction, $A_i = A_2$, $i = 2,3,\ldots,N$, the system is radially symmetric.)

| Cases (see text) | Parametric restrictions | Main or subsidiary branch | Leading orders $p_1$ | $p_2$ | $p_3$ | $\cdots$ | $p_N$ | Resonances $r$ | No. of arbitrary constants |
|---|---|---|---|---|---|---|---|---|---|
| 1 | $\alpha_i=\alpha_1$, $\beta_{ij}=2\alpha_1$, | MB | $-1$ | $-1$ | $-1$ | $\cdots$ | $-1$ | $-1;\ 0,0,\ldots,(N-1)$ times, $3,3,\ldots,(N-1)$ times; 4 | $2N$ |
| | $i\neq j$, $i,j=1,2,\ldots,N$ | SB-1 | $-1$ | 2 | $-1$ | $\cdots$ | $-1$ | $-3;\ -1;\ 0,0,\ldots,(N-1)$ times; $3,3,\ldots,(N-2)$ times; 4 | $2N-1$ |
| | | SB-2 | $-1$ | 2 | 2 | $\cdots$ | $-1$ | $-3,-3;\ -1;\ 0,0,\ldots,(N-1)$ times; $3,3,\ldots,(N-3)$ times; 4 | $2N-2$ |
| | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| | | SB-$(N-1)$ | $-1$ | 2 | 2 | $\cdots$ | 2 | $-3,-3,\ldots,(N-1)$ times; $-1;\ 0,0,\ldots,(N-1)$ times; 4 | $N+1$ |
| 2b(i) | $\alpha_i=16\alpha_2$, $\beta_{2j}=12\alpha_2$, | MB | $-1$ | $-\frac{1}{2}$ | $-1$ | $\cdots$ | $-1$ | $-1;\ 0,0,\ldots,(N-1)$ times; 2; $3,3,\ldots,(N-2)$ times; 4 | $2N$ |
| | $\beta_{ij}=32\alpha_2$, $A_i=4A_2$, | SB-1 | $-1$ | $-1$ | $-1$ | $\cdots$ | $-1$ | $-2,-1;\ 0,0,\ldots,(N-2)$ times; $3,3,\ldots,(N-2)$ times; 4; 5 | $(2N-1)$ |
| | $i\neq j$, $i,j=1,3,\ldots,N$ | SB-2 | $-1$ | $-\frac{3}{2}$ | $-1$ | $\cdots$ | $-1$ | $-2,-1;\ 0,0,\ldots,(N-1)$ times; $3,3,\ldots,(N-2)$ times; 4 | $(2N-1)$ |
| | | SB-3 | $-1$ | $\frac{3}{2}$ | 2 | $-1,\ldots$ | $-1$ | $-3,-2,-1;\ 0,0,\ldots,(N-1)$ times; $3,3,\ldots,(N-3)$ times; 4 | $(2N-2)$ |
| | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| | | SB-$N$ | $-1$ | $\frac{3}{2}$ | 2 | $2,\ldots$ | 2 | $-3,-3,\ldots,(N-2)$ times; $-2;\ -1;\ 0,0,\ldots,(N-1)$ times; 4; | $(N+1)$ |
| 2b(ii) | $\alpha_i=8\alpha_2$, $\beta_{2j}=12\alpha_2$, | MB | $-1$ | $-\frac{1}{2}$ | $-1$ | $\cdots$ | $-1$ | $-1;\ 0,0,\ldots,(N-1)$ times; 2; $3,3,\ldots,(N-2)$ times; 4 | $2N$ |
| | $\beta_{ij}=16\alpha_2$, $A_i=4A_2$, | SB-1 | $-1$ | $-1$ | $-1$ | $\cdots$ | $-1$ | $-5,-1;\ 0,0,\ldots,(N-2)$ times; $3,3,\ldots,(N-2)$ times; 4; 8 | $2N-1$ |
| | $i\neq j$, $i,j=1,3,\ldots,N$ | SB-2 | $-1$ | $\frac{3}{2}$ | 2 | $\cdots$ | $-1$ | $-2,-1;\ 0,0,\ldots,(N-1)$ times; $3,3,\ldots,(N-2)$ times; 4 | $2N-1$ |
| | | SB-2 | $-1$ | $\frac{3}{2}$ | 2 | $-1,\ldots$ | $-1$ | $-3,-2,-1;\ 0,0,\ldots,(N-1)$ times; $3,3,\ldots,(N-3)$ times; 4 | $2N-2$ |
| | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| | | SB-$N$ | $-1$ | $\frac{3}{2}$ | 2 | $\cdots$ | 2 | $-3,-3,\ldots,(N-2)$ times; $-2;\ -1;\ 0,0,\ldots,(N-1)$ times; 4 | $N+1$ |

## VI. DISCUSSION

In this work we have applied two methods, namely the Painlevé analysis and the direct computation of integrals of motion, to investigate the integrability of two, three, and $N$ quartically coupled anharmonic oscillators. Considering the two coupled oscillators, the Painlevé analysis shows that this is integrable for four specific sets of parameters, among which three of them are separable under appropriate transformations. By direct computation, the associated second integrals of motion have also been presented for all the above four cases. For the three-coupled-oscillator system, we have shown that the general solution of (6) possesses the sufficient number of six arbitrary constants for three sets of parametric values. We have derived the second and third integrals of motion to one of the cases, namely a special case of the strong $P$, thereby substantiating the integrability. For the remaining cases we have not succeeded in obtaining them. Also in the two degrees of freedom case, we have shown that for the set of parametric values (24) and (26), the equation of motion (4) of $H_2$ is separable under linear transformations, but such a possibility does not occur in higher degrees of freedom.

We have successfully extended the Painlevé analysis for both strong- and weak-$P$ properties, to the arbitrary $N$-coupled-oscillator system and obtained a set of three integrable cases, which are the generalizations of the three-coupled-oscillators system, one corresponding to a strong-$P$ property and the remaining two to weak-$P$ property. It now remains to investigate the singularity structure, fractal structure associated with natural boundaries, etc., of the non-Painlevé cases.

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*Present address (until March 1985): Department of Physics, Kyoto University, Kyoto 606, Japan.

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