

Quantum theory of spontaneous and stimulated resonant transition radiation

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Resonant transition radiation generated by high-energy electron beams traversing a periodic medium has been considered by many researchers as a potential source of both spontaneous and stimulated emission at short wavelengths. To our knowledge, this problem has only been treated classically. This paper presents a quantum-mechanical theory that leads to a unified description of both spontaneous and stimulated emission and agrees closely with the classical results.

I. INTRODUCTION

Electrons traveling at high speed emit electromagnetic waves when they move from one medium to another with a different dielectric constant. This is known as transition radiation.¹ In a spatially periodic medium there is an interference of the waves emitted at different interfaces producing a resonant transition radiation when the following condition is satisfied (Fig. 1):²⁻⁴

$$\cos\theta = \frac{c'}{v} - \frac{n\lambda'}{l}, \tag{1}$$

where λ is the wavelength of radiation in free space, l is the period of the spatially varying dielectric constant $\epsilon(z)$ (the variations are usually assumed to be small), $\bar{\epsilon}$ is the "mean" relative dielectric constant, c is the velocity of light, $c' = c/\sqrt{\bar{\epsilon}}$, $\lambda' = \lambda/\sqrt{\bar{\epsilon}}$, v is the velocity of the electrons in the z direction perpendicular to the interfaces, θ is the angle between the direction of wave propagation and the z axis, and n is an integer.

Resonant transition radiation has been considered by a number of workers as a possible source of short-wavelength radiation. Usually the period l is much greater than λ so that ultrarelativistic electron beams⁴ with $v/c \approx 1$ are required to satisfy Eq. 1. The possibility of using nonrelativistic electron beams with short-period solid-state superlattices to generate x-ray radiation has also been considered recently.^{5,6}

To our knowledge the problem of resonant transition radiation has only been treated classically. In classical theory spontaneous and stimulated emission are two different problems. Spontaneous emission is obtained from Maxwell's equations, treating the electron beam as a fixed current source. To obtain stimulated emission, however, we must consider the effect of the electromagnetic wave on the electron beam. This can be done using either a collective approach⁴ (Boltzmann equation) or a single-particle approach.^{5,6}

In this paper we will present a quantum-mechanical treatment of resonant transition radiation. In the quantum-mechanical approach, the electrons and the electromagnetic wave are treated simultaneously and both spontaneous and stimulated emission come out of the same formalism. A simple fundamental relation is established between the spontaneous emission rate and the stimulated emission gain. In the quantum-mechanical theory we consider the interaction between the electron beam and the electromagnetic normal modes of the periodic medium. These normal modes consist of an infinite number of spatial harmonics with wave numbers $k\cos\theta + 2n\pi/l$ (n is an integer, $k = 2\pi/\lambda'$, θ is an angle between the direction of wave propagation and the direction perpendicular to the layers) whose amplitudes depend on the Fourier components of $\epsilon(z)$. The spatial harmonics have phase velocities smaller than that of the fundamental ($n = 0$) by the factor $(1 + n\lambda'/l \cos\theta)$. Consequently it is possible for the electron beam to Cerenkov radiate into the higher spatial harmonics even though $v < c$. In this

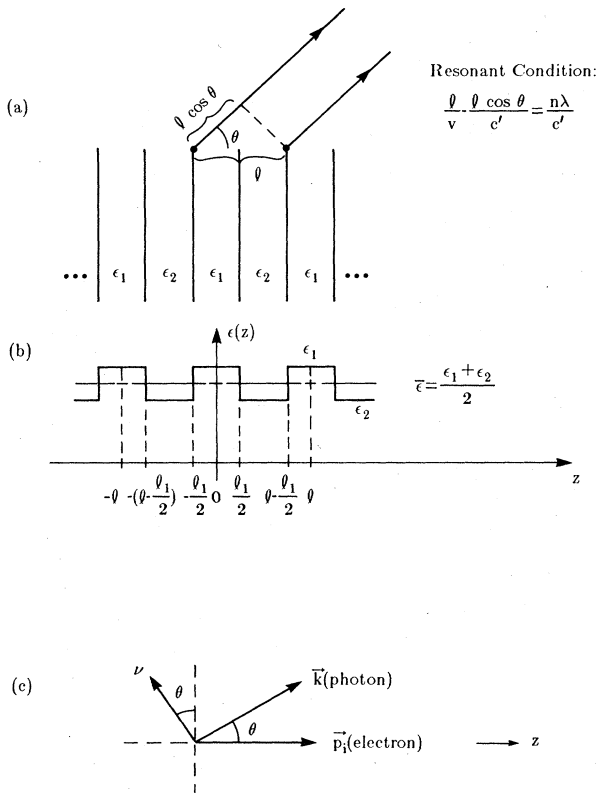


FIG. 1. Emission of electromagnetic waves by an electron traversing a periodic medium. (a) Configuration, (b) permittivity $\epsilon(z)$ vs z , (c) direction of electron current (\vec{p}_i), electromagnetic wave vector (\vec{k}) and wave polarization (ν).

paper we will view resonant transition radiation as a process of Čerenkov emission into the higher spatial harmonics; the results for both spontaneous and stimulated resonant transition radiation are obtained from those for ordinary Čerenkov radiation by multiplying by the squared amplitude of the n th spatial harmonic of an appropriate field component. It is interesting that the spontaneous emission obtained using this approach agrees exactly with the classical Ginzburg-Frank result obtained from a totally different approach.⁵ The advantage of this viewpoint is that it reduces resonant transition radiation to a special case of Čerenkov radiation.

Section II gives a brief summary of the classical results for spontaneous and stimulated resonant transition radiation. In Sec. III we first describe the quantum theory of ordinary Čerenkov radiation using the Schrödinger equation for nonrelativistic electrons; relativistic effects are then introduced using the Dirac equation. The results are in agreement with a quantum theory of stimulated Čerenkov radiation that has been presented recently⁷ using the Klein-Gordon equation for relativistic electrons. We discuss both the spontaneous emission rate and the stimulated emission gain and establish a simple fundamental relationship between them. In Sec. IV we first calculate the spatial harmonic amplitudes of an electromagnetic wave in a periodic medium assuming that the dielectric-constant variation is small ($\Delta\epsilon/\bar{\epsilon} \ll 1$); however, we do not make the usual WKB approximation so that the results are valid even if the period l is comparable to a wavelength λ .^{5,6} The results for ordinary Čerenkov radiation from Sec. III are then adapted to resonant transition radiation, using these spatial harmonic amplitudes.

II. CLASSICAL THEORY

A. Spontaneous emission

The energy U_0 radiated in a frequency range $d\omega$ in a solid angle $d\Omega$ by an electron traversing an interface between two media with dielectric constants ϵ_1 and ϵ_2 can be obtained directly from Maxwell's equations:⁸

$$\frac{d^2U_0}{d\omega d\Omega} = \frac{e^2}{4\pi\epsilon_0\bar{\epsilon}c'} \frac{\sin^2\theta}{4\pi^2} \left| \frac{\Delta\epsilon}{\bar{\epsilon}} \right|^2 \times \beta'^2 \left| \frac{1 - \beta' \cos\theta - \beta'^2}{(1 - \beta' \cos\theta)(1 - \beta'^2 \cos^2\theta)} \right|^2, \quad (2a)$$

where $\Delta\epsilon = |\epsilon_1 - \epsilon_2|/\epsilon_0$, $\bar{\epsilon} = (\epsilon_1 + \epsilon_2)/2\epsilon_0$, ϵ_0 is permittivity of vacuum, $\beta = v/c$, $\beta' = \beta\sqrt{\bar{\epsilon}}$. Using Eq. (1) we can rewrite Eq. (2a) as

$$\frac{d^2U_0}{d\omega d\Omega} = \frac{e^2}{4\pi\epsilon_0\bar{\epsilon}c'} \frac{\sin^2\theta}{4\pi^2} \left| \frac{\Delta\epsilon}{\bar{\epsilon}} \right|^2 \left| \frac{(nq - \beta')}{n^2q^2(2 - nq\beta')} \right|^2, \quad (2b)$$

where $q = \lambda'/l$.

In a periodic medium with multiple interfaces the total radiated energy U is given by

$$\frac{d^2U}{d\omega d\Omega} = \frac{d^2U_0}{d\omega d\Omega} \frac{4 \sin^2(\xi l_1/2l) \sin^2(\xi N/2)}{\sin^2(\xi/2)},$$

where l_1 is the length of one of the layers, $2N$ is the num-

ber of layers, and $\xi = (2\pi l/\lambda')(1/\beta' - \cos\theta)$. If N is sufficiently large, the radiation shows very narrow spectral peaks around values of θ for which ξ is an integer multiple of 2π . The location of the peaks is given by

$$\xi = 2\pi n$$

which is identical to Eq. (1). The narrow spectral peaks can be approximated by δ functions around a particular value of n ,

$$\begin{aligned} \frac{d^2U}{d\omega d\Omega} &= \frac{d^2U_0}{d\omega d\Omega} 8\pi N \sin^2 \left[\frac{n\pi l_1}{l} \right] \delta(\xi - 2n\pi) \\ &= \frac{d^2U_0}{d\omega d\Omega} 4Nq \sin^2 \left[\frac{n\pi l_1}{l} \right] \delta(\cos\theta + nq - 1/\beta'). \end{aligned}$$

Integrating over the solid angle we get the energy U radiated into the frequency interval $d\omega$ for a particular value of n ,

$$\begin{aligned} \frac{dU}{d\omega} &= \frac{e^2}{4\pi\epsilon_0\bar{\epsilon}c'} \sin^2\theta \left| \frac{\Delta\epsilon}{\bar{\epsilon}} \right|^2 \frac{2Nq}{\pi} \sin^2 \left[\frac{n\pi l_1}{l} \right] \\ &\times \left| \frac{nq - \beta'}{n^2q^2(2 - nq\beta')} \right|^2, \quad (3) \end{aligned}$$

where

$$\omega = k_n v, \quad (4a)$$

$$k_n = k \cos\theta + 2\pi n/l, \quad (4b)$$

$$k = 2\pi/\lambda'. \quad (4c)$$

Here we have used Eq. (2b) for $d^2U_0/d\omega d\Omega$ and assumed that it varies slowly compared to the δ function. Equations (4) are really another way of writing the resonance condition expressed by Eq. (1). Equation (3) gives the total energy radiated by one electron; multiplying by J/e (J = current density) we get the radiated power I per unit area,

$$\frac{dI}{d\omega} = T \frac{eJ\omega \sin^2\theta}{4\pi\epsilon_0\bar{\epsilon}c'} \beta' |\rho_n|^2, \quad (5a)$$

where

$$\rho_n = \frac{1}{n\pi} \frac{\Delta\epsilon}{\bar{\epsilon}} \frac{(nq - \beta')}{nq(2 - nq\beta')} \sin \left[\frac{n\pi l_1}{l} \right], \quad (5b)$$

and

$$T = Nl/v \quad (5c)$$

is the time of interaction.

B. Stimulated emission

Stimulated emission can be treated classically using either a collective approach⁴ (Boltzmann equation) or a single-particle approach^{5,6} (Lorentz equation); the latter is probably more explicit and straightforward. In this approach we calculate the change in energy of an individual electron due to its interaction with an electromagnetic wave. The energy lost by the electrons contributes to the

gain of the wave. The change in the electron energy U is given by

$$\frac{dU}{dt} = ec\boldsymbol{\beta} \cdot \mathbf{E}, \quad (6a)$$

where \mathbf{E} is the electric field at the location of the electron $\mathbf{r}(t)$ and

$$\mathbf{E} = \sum_n \mathbf{E}_n \exp(i\mathbf{k}_n \cdot \mathbf{r} - \omega t + \phi_n).$$

The index n runs over all the spatial harmonics. A large exchange of energy between an electron and the electromagnetic wave becomes possible if it moves synchronously with the wave so as to see almost a constant electric field. Assuming $\mathbf{r}(t) \approx \boldsymbol{\beta}' ct$, we can see that this resonant condition is achieved if

$$\mathbf{k}_n \cdot \boldsymbol{\beta}' c \approx \omega$$

which is the same condition as Eqs. (4). Assuming that the resonant condition is satisfied, we can calculate the energy exchange between the electron and the wave from Eq. (6a); depending on the phase ϕ_n of the wave, the electron can either gain or lose energy. If we average this energy over all possible phases ϕ_n , the result is zero. To get a nonzero result we need to take into account the perturbation of the electron trajectory $\mathbf{r}(t)$ due to the electromagnetic wave. We then get from Eq. (6a),

$$\Delta U = \left\langle ec \int_0^T (\mathbf{E} \cdot \Delta \boldsymbol{\beta} + \boldsymbol{\beta} \cdot \Delta \mathbf{E}) dt \right\rangle, \quad (6b)$$

where \mathbf{E} and $\boldsymbol{\beta}$ are the unperturbed values, $\Delta \boldsymbol{\beta}$ and $\Delta \mathbf{E} [= \mathbf{E}(\mathbf{r} + \Delta \mathbf{r})]$ are the deviations due to the effect of the electromagnetic wave, T is the time of interaction, and the angular brackets $\langle \rangle$ represent the averaging over ϕ_n . $\Delta \boldsymbol{\beta}$ and $\Delta \mathbf{E}$ are obtained from the Lorentz equation. Assuming a plane wave, we can show that the wave is amplified by the electron beam only if its electric field \mathbf{E} is polarized in the plane of incidence (x - z plane); for this polarization only the z components of $\boldsymbol{\beta}$ and \mathbf{E} contribute to the change of energy. We have from the Lorentz equation,

$$\Delta \beta_z = \frac{e}{mc\gamma^3} \int_0^T E_z dt. \quad (6c)$$

Also,

$$\Delta z = c \int_0^T \Delta \beta_z dt, \quad (6d)$$

$$\Delta E_z = \frac{\partial E_z}{\partial z} \Delta z. \quad (6e)$$

Using Eqs. (6c)–(6e) in Eq. (6b) we can calculate the energy exchange ΔU between the electron and the electromagnetic wave. In order to obtain the amplification of an electromagnetic wave due to an electron beam carrying an electric current density J , we have to multiply ΔU by J/e and divide it by the energy flux of the incident wave per unit area ($= \frac{1}{2} E_0^2 \epsilon_0 c' \cos \theta$). The small signal gain Γ at a frequency ω is then obtained as^{5,6}

$$\Gamma = (\pi F) \frac{T^3}{\gamma^3 m} |\rho_n|^2 \frac{eJ\omega \sin^2 \theta}{4\pi \epsilon_0 \bar{c} c' \cos \theta}, \quad (7a)$$

where

$$F = \frac{d}{d\phi} \left[\frac{\sin^2 \phi}{\phi^2} \right], \quad (7b)$$

$$\phi = (1 - \beta' \cos \theta - \beta' q) \omega T / 2 = (\omega - k_n v) T / 2, \quad (7c)$$

$$\gamma = (1 - \beta'^2)^{-1/2}. \quad (7d)$$

This is a one-dimensional analysis assuming a guided mode. As discussed in Ref. 9, when the oblique angle of the EM wave is taken into account, the gain is multiplied by the factor $(1 - \beta^2 \cos^2 \theta) \gamma^2 / \cos^2 \theta$, so that

$$\Gamma = (\pi F) \frac{T^3}{\gamma m} |\rho_n|^2 \frac{eJ\omega \tan^2 \theta}{4\pi \epsilon_0 \bar{c} c' \cos \theta} (1 - \beta^2 \cos^2 \theta). \quad (8a)$$

Using Eq. (6a) for $dI/d\omega$ we can write Eq. (8a) as

$$\Gamma = \pi F \frac{T^2}{\gamma m} \frac{dI}{d\omega} \frac{(1 - \beta^2 \cos^2 \theta) \cos \theta}{\beta'}. \quad (8b)$$

III. QUANTUM THEORY OF ČERENKOV RADIATION

In this section we will describe spontaneous and stimulated Čerenkov radiation using a quantum-mechanical formalism. As we have mentioned in the Introduction, the results for Čerenkov radiation are readily adapted to resonant transition radiation simply by multiplying with the squared amplitude of the spatial harmonic of an appropriate field component; this is done in Sec. IV.

A. Nonrelativistic electron beam

We first consider a free nonrelativistic electron interacting with a radiation field; relativistic effects are incorporated in Sec. III B with a simple modification,

$$H_0 = \frac{p^2}{2m} + \sum_{\mathbf{k}, \nu} \hbar \omega_{\mathbf{k}} a_{\mathbf{k}, \nu}^* a_{\mathbf{k}, \nu}, \quad (9a)$$

$$H_{\text{int}} = \sum_{\mathbf{k}, \nu} K_{\mathbf{k}, \nu} a_{\mathbf{k}, \nu}^* + K_{\mathbf{k}, \nu}^* a_{\mathbf{k}, \nu}, \quad (9b)$$

$$K_{\mathbf{k}, \nu} = - \left[\frac{\hbar}{2\bar{\epsilon}\epsilon_0 V \omega_{\mathbf{k}}} \right]^{1/2} \frac{e}{2m} (p_{\nu} e^{-i\mathbf{k} \cdot \mathbf{R}} + e^{-i\mathbf{k} \cdot \mathbf{R}} p_{\nu}), \quad (9c)$$

where \mathbf{k}, ν represent the wave vector and polarization of the photon mode, $\omega_{\mathbf{k}} = c' k$, V is the volume of normalization, p is the momentum operator for the electron, m is the electron mass, \mathbf{R} is the electron position operator, and a, a^* are the annihilation and creation operators for photon modes. The initial state $|I\rangle$ has the photon modes in harmonic oscillator states $|n_{\mathbf{k}, \nu}\rangle$ and the electron in the momentum state $|\mathbf{p}_i\rangle [\equiv (V)^{-1/2} \exp(i\mathbf{p}_i \cdot \mathbf{R} - E_i t) / \hbar, E_i = p_i^2 / 2m]$,

$$|I\rangle = |\mathbf{p}_i\rangle \prod_{\mathbf{k}, \nu} |n_{\mathbf{k}, \nu}\rangle. \quad (10a)$$

We consider transitions to a final state $|F\rangle$ with the electron in state $|\mathbf{p}_f\rangle$ and one more (or one less) photon in

mode \mathbf{k}, ν corresponding to photon emission (or absorption),

$$|F\rangle = |\mathbf{p}_f\rangle \prod_{\mathbf{k}, \nu} |n_{\mathbf{k}, \nu} + \frac{1}{2} \pm \frac{1}{2}\rangle. \quad (10b)$$

$$|M| = (n_{\mathbf{k}, \nu} + \frac{1}{2} \pm \frac{1}{2})^{1/2} \left[\frac{1}{2\epsilon_0 V \hbar \omega_{\mathbf{k}}} \right]^{1/2} \frac{e}{2mV} (\mathbf{p}_i + \mathbf{p}_f)_{\nu} L_x L_y L_z \cdot T \\ \times \frac{\sin(\Omega_x L_x/2)}{(\Omega_x L_x/2)} \frac{\sin(\Omega_y L_y/2)}{(\Omega_y L_y/2)} \frac{\sin(\Omega_z L_z/2)}{(\Omega_z L_z/2)} \frac{\sin(\Omega T/2)}{(\Omega T/2)}, \quad (11)$$

where

$$\Omega_{x,y,z} = (\mathbf{p}_i - \mathbf{p}_f \mp \hbar \mathbf{k})_{x,y,z} / \hbar,$$

$$\Omega = (E_i - E_f \mp \hbar \omega_{\mathbf{k}}) / \hbar.$$

The total transition probability is given by $|M|^2$ summed over the possible final electron states $d\mathbf{p}_f$. If L_x, L_y, L_z are sufficiently large, we can treat the $\sin^2 x/x^2$ factors as δ functions in this integration, so that $\Omega_{x,y,z} = 0$. We can also assume $L_x L_y L_z = V$ so that the wave functions are normalized to the interaction volume

$$\pm \Delta n_{\mathbf{k}} = (n_{\mathbf{k}} + \frac{1}{2} \pm \frac{1}{2}) \frac{e^2 p_i^2 \sin^2 \theta}{2\epsilon_0 V m^2 \hbar \omega_{\mathbf{k}}} T^2 \left[\frac{\sin(\Omega T/2)}{\Omega T/2} \right]^2, \quad (12a)$$

$$\mathbf{p}_f = \mathbf{p}_i \mp \hbar \mathbf{k}. \quad (12b)$$

The electron couples only to the photons which are polarized in the plane containing \mathbf{p}_i and \mathbf{k} [Fig. (1c)]; it is evident from Eq. (9c) that the photon polarization must have a component along the electron momentum for nonzero coupling. Using the momentum conservation condition [Eq. (12b)] and the energy momentum relation ($E = p^2/2m$), we can write

$$\Omega = \pm(\omega_{\mathbf{k}} - kv \cos\theta \pm \hbar k^2/2m) \quad (12c)$$

$$= \pm kv \left[\frac{1}{\beta'} - \cos\theta \mp \frac{\hbar k}{2p_i} \right], \quad (12d)$$

where $v = p_i/m$ is the electron velocity.

The total spontaneous emission U , by an electron, is obtained by integrating Eq. (12a) (taking the upper sign with $n_{\mathbf{k}} = 0$) over all photon modes \mathbf{k} . If the time of interaction T is long enough, we can treat the $\sin^2 x/x^2$ as a δ function in this integration to get

$$\frac{dU}{d\omega} = T \frac{e^2 \omega v}{4\pi\epsilon_0 \bar{c}^2} \sin^2 \theta, \quad (13a)$$

where θ is fixed by the condition $\Omega = 0$.

To obtain the gain per pass we need the difference between stimulated emission and absorption which arises from the small difference in Ω for the two cases [Eq. (12c)]

$$\frac{\Delta n_{\mathbf{k}}}{n_{\mathbf{k}}} = \frac{e^2 \omega v^2 \sin^2 \theta}{4\epsilon_0 m c^2 V} T^3 F, \quad (13b)$$

where

The upper sign corresponds to emission and the lower sign to absorption. Assuming that the length of the interaction region is L_x, L_y, L_z in the x, y, z directions and that the time of interaction is T , we can write the first-order matrix element M for this transition

$$F = \frac{d}{d\phi} \left[\frac{\sin^2 \phi}{\phi^2} \right] = \frac{16}{\pi^3} (\text{maximum}),$$

$$\phi = \Omega T/2.$$

Equations (13a) and (13b) give the spontaneously radiated energy in a frequency interval ω to $\omega + d\omega$ and the gain per pass at a frequency ω due to a single electron traveling at a velocity v . For a beam of electrons with a current density J ($\equiv env$, n = electron density), we should multiply Eq. (13a) by J/e to get the intensity I (W/m^2) and Eq. (13b) by the total number of electrons JV/ev in the volume V to get the total gain Γ ,

$$\frac{dI}{d\omega} = T \frac{eJ\omega \sin^2 \theta}{4\pi\epsilon_0 \bar{c}^2} \beta', \quad (14a)$$

$$\Gamma = \pi F \frac{T^3}{m} \frac{eJ\omega \sin^2 \theta}{r\pi\epsilon_0 \bar{c}^2} \beta'. \quad (14b)$$

B. Relativistic electron beam

To consider relativistic effects we need to use the Dirac equation rather than the Schrödinger equation for the electrons so that the initial- and final-state electronic wave functions are four-component spinors of the form¹⁰

$$\{\psi\} = \left[\frac{E + mc^2}{2EV} \right]^{1/2} e^{i(\mathbf{p}\cdot\mathbf{R} - Et)/\hbar} \{u\}, \quad (15a)$$

where

$$\{u\} = \begin{Bmatrix} 1 \\ 0 \\ p_z c / (E + mc^2) \\ (p_x + ip_y) c / (E + mc^2) \end{Bmatrix} \text{ for up spin} \quad (15b)$$

and

$$\{u\} = \begin{Bmatrix} 0 \\ 1 \\ (p_x - ip_y) c / (E + mc^2) \\ -p_z c / (E + mc^2) \end{Bmatrix} \text{ for down spin.}$$

The interaction Hamiltonian is given by $-e\mathbf{c}\boldsymbol{\alpha}\cdot\mathbf{A}$ [instead of the nonrelativistic $-e(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p})/2m$] where $\alpha_{x,y,z}$

are 4×4 matrices given by

$$\alpha_v = \begin{pmatrix} 0 & \sigma_v \\ \sigma_v & 0 \end{pmatrix},$$

σ_i being the Pauli matrices. Consequently, in calculating the matrix element M [Eq. (11)], instead of $e(\mathbf{p}_i + \mathbf{p}_f)\gamma/2mV$, we get

$$\frac{ec}{2V} \left[\frac{(E_i + mc^2)(E_f + mc^2)}{E_i E_f} \right]^{1/2} u^\dagger(\mathbf{p}_f) \alpha_v u(\mathbf{p}_i), \quad (16a)$$

where $u(\mathbf{p})$ is the four-component spinor defined in Eq. (15b) and $u^\dagger(\mathbf{p})$ is its complex conjugate transpose. In most cases of practical interest the energy and momentum of the photon is very small compared to that of the electron, so that $E_f \simeq E_i$ and $\mathbf{p}_f \simeq \mathbf{p}_i$. Assuming that the electron is initially in the up-spin state with momentum along the z direction, we get

$$u^\dagger(\mathbf{p}_f) \alpha_x u(\mathbf{p}_i) \simeq u^\dagger(\mathbf{p}_f) \alpha_y u(\mathbf{p}_i) \simeq 0, \quad (16b)$$

$$u^\dagger(\mathbf{p}_f) \alpha_z u(\mathbf{p}_i) \simeq 2p_i c / (E + mc^2).$$

Here we have assumed $\mathbf{p}_f \simeq \mathbf{p}_i$; in this approximation the electron spin is unchanged by the transition. Using Eqs. (16a) and (16b), we see that in the relativistic case ep_i/mV is replaced by $ep_i c^2/EV$; that is, m is replaced by γm where

$$E = \gamma mc^2, \quad (17)$$

$$\gamma = (1 - \beta^2)^{-1/2}.$$

Since the relativistic velocity is defined as $p/\gamma m$ rather than p/m [Eq. (13a)], there is no change in v . Consequently Eq. (14a) for spontaneous emission remains unchanged, as

$$\frac{dI}{d\omega} = T \frac{eJ\omega \sin^2\theta}{4\pi\bar{\epsilon}\epsilon_0 c'} \beta'. \quad (18a)$$

It is tempting to conclude that the stimulated emission gain Γ in the relativistic case is obtained from Eq. (13a) [and (14b)] by replacing m with γm in the denominator. However, the m comes from the last term in the resonance factor Ω [Eq. (12c)]. This term needs to be reevaluated taking into account the relativistic energy-momentum relationship $E^2 = p^2 c^2 + m^2 c^4$. Using a Taylor's series expansion for E , assuming that the change in the electron momentum (which is equal to the photon momentum $\hbar\mathbf{k}$) is small compared to electron momentum, we get

$$\begin{aligned} \Omega &= (E_i - E_f \mp \hbar\omega_k) / \hbar \\ &= \pm \left[\omega_k - kv \cos\theta \pm \frac{\hbar k^2}{2\gamma m} (1 - \beta^2 \cos^2\theta) \right], \end{aligned} \quad (18b)$$

where $v = p/\gamma m$. Consequently Eq. (14b) for the stimulated emission gain Γ is modified by a factor $(1 - \beta^2 \cos^2\theta)/\gamma$:

$$\Gamma = \pi F \frac{T^3}{\gamma m} \frac{eJ\omega \sin^2\theta}{4\pi\epsilon_0 \bar{\epsilon} c'} \beta' (1 - \beta^2 \cos^2\theta). \quad (18c)$$

The spontaneous emission given by Eq. (18a) agrees exactly with the classical result for spontaneous Čerenkov radiation.³ It is also in agreement with the quantum-mechanical result derived in Ref. 11 using the Klein-Gordon equation for electrons. The stimulated emission gain given by Eq. (18c) agrees with a quantum-mechanical theory of stimulated Čerenkov radiation that has been presented recently⁷ using the Klein-Gordon equation for relativistic electrons. It also agrees with the classical result in Ref. 9. The classical theory in Ref. 12, however, gives a gain $\sim 1/\gamma^3$ rather than $1/\gamma$; the reason for this lies in the difference in geometry as explained in Ref. 9.

Equations (18a) and (18c) apply to ordinary Čerenkov radiation; it is shown in Sec. IV that the results for resonant transition radiation are identical except for a multiplying factor depending on the relative amplitude of the appropriate spatial harmonic. From Eqs. (18a) and (18c) we can derive a simple relationship between the spontaneous emission intensity and the stimulated emission gain which will hold true even for resonant transition radiation

$$\Gamma = \frac{dI}{d\omega} F \pi \frac{T^2}{\gamma m} (1 - \beta^2 \cos^2\theta). \quad (19)$$

IV. QUANTUM THEORY OF RESONANT TRANSITION RADIATION

In our discussion of ordinary Čerenkov radiation we assumed that the electromagnetic waves were plane waves of the form $\exp[i(kz - \omega t)]$; the necessary condition for Čerenkov emission is then found to be $v > \omega/k$ [Eq. (12c)]. In a periodic medium with period l , the fields are composed of spatial harmonics of the form $e^{-i\omega t} \sum_n c_n e^{ik_n z}$ where $k_n = k \cos\theta + n \cdot 2\pi/l$. Even if the condition for Čerenkov emission is not satisfied by the lowest spatial harmonic ($n=0$), it may be satisfied by the higher-order harmonics ($n > 0$). Resonant transition radiation can be viewed as a process involving "Čerenkov" emission into the higher-order spatial harmonics. We have seen in Sec. III that the component of the electric field along the direction of electron motion (E_z) determines the matrix element for emission and absorption. So, in order to obtain the results for resonant transition radiation, we need to multiply the results for ordinary Čerenkov radiation by the squared magnitude of the n th spatial harmonic of E_z . We will now determine the spatial harmonic amplitudes starting directly from Maxwell's equations in inhomogeneous media following the approach of our previous work.^{5,6} As in Refs. 5 and 6 we do not make the WKB approximation so that the results are valid even if $l \gtrsim \lambda$. However, we make the usual assumption that $\Delta\epsilon/\bar{\epsilon} \ll 1$.

Consider an electromagnetic wave propagating at an angle θ to the z direction which is perpendicular to the plane of the layers [Fig. (1c)]. The coupling of the wave to the electrons is proportional to the component of the electron momentum along the direction of the electric field [Eq. (10)]; consequently the electrons couple only to the waves polarized in the plane containing \mathbf{p}_i and \mathbf{k} . For this polarization the field components are E_x, E_z , and H_y ; the component that determines the coupling strength is

E_z since the electron current is along z .

The problem is to find out the amplitudes of the spatial harmonics of E_z . Usually the WKB approximation is used which is not valid if the period l is comparable to the electromagnetic-wave wavelength. So we use an alternative approach starting directly from Maxwell's equation in inhomogeneous media (assuming $\Delta\epsilon/\bar{\epsilon} \ll 1$). In general the vector potential \mathbf{A} in a medium with an inhomogeneous dielectric constant $\epsilon(\mathbf{r})$ is described by the following equation (Lorentz gauge):¹³

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} - \frac{2}{k} (\nabla \cdot \mathbf{A}) \nabla k = 0, \quad (20)$$

where $k^2(\mathbf{r}) = \omega^2 \epsilon(\mathbf{r}) / c^2$. In the present problem ϵ is a periodic function of z (Fig. 1); also,

$$A_x = A_y = 0, \quad (21)$$

so that $\mathbf{A} = A_z \hat{e}_z$. The electric field is polarized in the plane of incidence (i.e., $E = 0$), and its components can be expressed in terms of A_z :¹³

$$E_z = i\omega \left[A_z + \frac{\partial}{\partial z} \left(\frac{1}{k^2} \frac{\partial A_z}{\partial z} \right) \right], \quad (22a)$$

$$E_x = \frac{i\omega}{k^2} \frac{\partial^2 A_z}{\partial x \partial z}. \quad (22b)$$

Assuming $A_z = A(z) e^{ikx \sin \theta}$, we get the following equation for A from Eq. (20):

$$A'' - \frac{1}{\bar{\epsilon}} (\Delta\epsilon)' A' + k^2 A \left[\cos^2 \theta + \frac{\Delta\epsilon}{\bar{\epsilon}} \right] = 0, \quad (23)$$

where the prime denotes derivative with respect to z , and $\Delta\epsilon(z)$ is a periodic function with period l , which can be written as a Fourier series:

$$\frac{\Delta\epsilon}{\bar{\epsilon}} = \frac{1}{2} \sum_{m=1}^{\infty} (a_m e^{2i\pi m z/l} + \text{c.c.}). \quad (24)$$

The Floquet theorem for the linear differential equations with periodic coefficients allows us to write a solution to Eq. (23) in the form (for a wave traveling in the positive z direction)

$$A = A_0 e^{ikz \cos \theta} \sum_{n=-\infty}^{\infty} c_n e^{2i\pi n z/l} \quad (c_0 = 1). \quad (25)$$

The fields thus contain an infinite set of spatial harmonics with $k_n = k \cos \theta + (2\pi n/l)$. The wave traveling in the negative z direction contains the spatial harmonics $-k \cos \theta + (2\pi n/l)$. In an infinite medium the two solutions are independent as long as $k \cos \theta$ does not lie in the vicinity of a multiple of π/l . Of course in a finite medium the two solutions are coupled together giving rise to the problem of retroreflections; this difficulty may be avoided in practice by choosing^{5,6} the angle of emission to match the Brewster angle.

In general, using Eq. (25), we can obtain an infinite system of linear-coupled algebraic equations for c_n . However, if $\Delta\epsilon/\bar{\epsilon} \ll 1$ so that $a_n \ll 1$ (which is of particular interest in the theory of x-ray generation), we may decouple these equations assuming that the amplitudes c_n are small ($c_n \ll c_0 = 1$). Substituting (24) and (25) into (23), collect-

ing together terms with the same spatial frequency $k_0 \cos \theta + nq$, retaining only terms linear in a_n and c_n , we can determine each amplitude c_n separately^{5,6} with

$$c_n = \frac{a_n}{2} \frac{1 + nq \cos \theta}{(2 \cos \theta + nq) nq}. \quad (26)$$

Now, using (26), (25), and (22), we can obtain spatial harmonics of the z component of the electric field E_z ,^{5,6}

$$E_z = -E_0 \sin \theta e^{ik(x \sin \theta + z \cos \theta)} \left(1 + \sum c_{nz} e^{iqnz} \right),$$

where

$$c_{nz} = \frac{a_n}{2} \frac{1 - nq \cos \theta - n^2 q^2}{nq(2 \cos \theta + nq)} \quad (27)$$

and E_0 is the amplitude of the principal spatial harmonic. Using the resonant condition (1), c_{nz} can be rewritten in the form

$$c_{nz} = \frac{a_n}{2} \frac{(\beta' - nq)}{nq(2 - nq\beta')}. \quad (28)$$

The results [Eqs. (18)] in Sec. III have to be multiplied by $|c_{nz}|^2$ in order to get the results for resonant transition radiation

$$\frac{dI}{d\omega} = T \frac{eJ\omega \sin^2 \theta}{4\pi\epsilon_0 c} \beta |c_{nz}|^2. \quad (29)$$

Assuming $\epsilon(z)$ has a rectangular form, we can readily calculate its Fourier coefficients a_n as

$$a_n = \frac{2}{n\pi} \frac{\Delta\epsilon}{\bar{\epsilon}} \sin \left[\frac{n\pi l_1}{l} \right]. \quad (30)$$

Using Eq. (30) in (28) we see that $c_{nz} = \rho_n$ [Eq. (5b)] so that the quantum-mechanical result [Eq. (29)] is identical to the classical result [Eq. (5a)] for spontaneous emission. The stimulated emission gain Γ is related to $dI/d\omega$ by the same relation as for Cerenkov radiation [Eq. (19)] since both quantities are multiplied by the same factor $|\rho_n|^2$ and

$$\Gamma = \frac{dI}{d\omega} F \pi \frac{T^2}{\gamma m} (1 - \beta^2 \cos^2 \theta). \quad (31)$$

This result agrees with the classical result [Eq. (8b)] except for a factor of $\beta'/\cos \theta$ which is almost 1 in the relativistic limit; the origin of this slight discrepancy is not known. The resonant factor Ω for ordinary Cerenkov radiation is readily modified for transition radiation: $k \cos \theta$ is replaced by $(k \cos \theta + n2\pi/l)$. Assuming $\hbar k \ll p_i$, we have from Eq. (18b)

$$\begin{aligned} \Omega &\simeq \mp (\omega - k_n v) \\ &= \mp kv \left[\frac{1}{\beta'} - \cos \theta - \frac{n\lambda'}{l} \right]. \end{aligned} \quad (32)$$

Setting $\Omega = 0$ yields the resonant condition [Eqs. (1) or (4)] for transition radiation.

V. CONCLUSIONS

In this paper we have described a quantum-mechanical theory for spontaneous and stimulated resonant transition

radiation. Resonant transition radiation is viewed as a process of Čerenkov radiation into the slow spatial harmonics present in an electromagnetic field in a periodic medium. The results for transition radiation are thus obtained readily from those for ordinary Čerenkov radiation by multiplying by the squared amplitude of the spatial harmonic of an appropriate field component. The spatial harmonic amplitude is calculated directly from Maxwell's equations in an inhomogeneous medium without recourse

to the WKB approximation. The quantum-mechanical results are in close agreement with the classical results.

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