Nonlinear coupling of electrostatic waves in magnetized plasmas

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The most general low-frequency electrostatic plasma response to the high-frequency upper-hybrid oscillations is obtained. Accounting for the relativistic electron mass variation and the ponderomotive force nonlinearities, a pair of coupled nonlinear equations are derived. The latter describe the coupling of the high- and low-frequency potential oscillations in magnetized plasmas. Localized envelope wave packets are found in limiting cases. The importance of the relativistic mass variation nonlinearity is examined.

In their classical paper, Kaufman and Stenflo[1] discovered super-Alfvénic upper-hybrid solitons. The latter consist of an envelope of negative group dispersive upper-hybrid wave packets together with compressional magnetic field perturbations. Porkolab and Goldman[2] presented a detailed investigation of upper-hybrid envelope soliton formation. Some new results for the positive group dispersive waves were found. Yu and Shukla[3] pointed out the existence of cusped solitons on the time scale of ion gyroperiod. More works[4–10] have later emerged providing an extensive study of nonlinear effects at the upper-hybrid layer.

In a recent paper, Dysthe, Mjølhus, Pécseli, and Stenflo[10] have derived a set of nonlinear equations which describe the coupling of the upper-hybrid waves with the electrostatic fluctuations near the lower-hybrid frequency. In this Brief Report, we generalize the results of Ref. 10 by including the relativistic effect[9] and also present the most general low-frequency electrostatic plasma response to the upper-hybrid oscillations. On imposing restrictions on the phase velocity and the angle of propagation of the low-frequency modulations, we can then obtain various kinds of driven low-frequency responses which are already known in the literature.[7] Finally, we investigate the effects of the relativistic electron mass variation and the ponderomotive nonlinearities on the soliton formation in magnetized plasmas.

Consider the propagation of a finite amplitude upper-hybrid wave in the form

$$\vec{E}=\tfrac{1}{2}(\hat{x}E_x+\hat{z}E_z)\exp(-i\omega t+i\vec{k}\cdot\vec{r})+\text{c.c.}\ , \tag{1}$$

where the external magnetic field $\vec{B}_0$ is directed along the z axis. The frequency ω and the wave number $\vec{k}$ are related by

$$\omega^2=\omega_H^2+\frac{3k_\perp^2 v_{te}^2}{\omega_H^2-4\Omega_e^2}-\frac{k_z^2}{k^2}\frac{\Omega_e^2}{\omega_H^2}\omega_{pe}^2\ , \tag{2}$$

where $\omega_H^2=\omega_{pe}^2+\Omega_e^2$, $\omega_{pe}^2=4\pi ne^2/m_e$, $\Omega_e=eB/m_ec$, $m_e=m_{e0}\gamma$, $\gamma=(1-v_e^2/c^2)^{-1/2}$; m_{e0} is the rest mass of the electron, v_{te} the electron thermal velocity, and v_e the quiver velocity of the electrons in the wave field. The factor γ incorporates the relativistic nonlinearities due to the electron mass variation.

The nonlinear interaction of the upper-hybrid wave with the slow plasma motion can give rise to an envelope of wave. The amplitudes of the latter vary on the time and the space scales of the low-frequency modulations. Within the WKB approximation ($\partial_t \ll \omega$), the development of the high-frequency wave envelope is governed by

$$2i\omega\partial_t E+6ik\alpha\partial_x E+\alpha\partial_x^2 E-\left(\omega_{p0}^2\frac{\delta n_e}{n_0}+2\Omega_{e0}^2\frac{\delta B_z}{B_0}\right)E+(\omega_{p0}^2+2\Omega_{e0}^2)\frac{|E|^2}{E_c^2}E=0\ , \tag{3}$$

where $E\equiv E_x \gg E_z$, $\partial_z \ll \Omega_{e0}\omega_{pe0}\partial_x/\omega_{H0}^2$, $\omega_{H0}^2=\omega_{p0}^2+\Omega_{e0}^2$, $\omega_{p0}^2=4\pi n_0e^2/m_{e0}$, $\Omega_{e0}=eB_0/m_{e0}c$, $\alpha=3v_{te}^2/(\omega_{p0}^2-3\Omega_{e0}^2)$, $E_c^2=\frac{4}{3}m_{e0}^2c^2\omega_{p0}^4/e^2(\omega_{p0}^2+\Omega_{e0}^2)$. Here, δn_e and δB_z are the electron density and magnetic field perturbations associated with the slow plasma motion.

Berezhiani[9] has investigated the magnetohydrodynamic response of the plasma to the upper-hybrid waves. In this report, we present the most general low-frequency electrostatic fluctuatons [thus setting $\delta B_z=0$ in Eq. (3)] driven by the upper-hybrid waves. For $\partial_t \ll \Omega_{e0}$, the electron velocities involved in the plasma slow motion are given by

$$\vec{v}_{e\perp}=\frac{c}{B_0}\hat{z}\times\vec{\nabla}_\perp\phi-\frac{cT_e}{eB_0}\hat{z}\times\vec{\nabla}_\perp\frac{\delta n_e}{n_0}-\frac{cT_e}{eB_0\Omega_{e0}}\partial_t\vec{\nabla}_\perp\frac{\delta n_e}{n_0}+\frac{1}{\Omega_{e0}^2}\partial_t\nabla_\perp\left(\frac{e\phi}{m_{e0}}-\phi_{p\perp}\right)\ , \tag{4}$$

$$\partial_t v_{ez}=\frac{e}{m_{e0}}\partial_z\phi-\partial_z\phi_{pz}-v_{te}^2\partial_z\frac{\delta n_e}{n_0}\ , \tag{5}$$

where ϕ is the ambipolar potential, and the perpendicular (to $\vec{B}_0$) and parallel components of the ponderomotive potentials[2] are, respectively, given by

$$\phi_{p\perp}=\frac{e^2\omega^2|E|^2}{2m_{e0}^2\omega_{p0}^4}\ ,$$

$$\phi_{pz}=\frac{\omega_{p0}^2}{\omega^2}\phi_{p\perp}\ .$$

Substituting Eqs. (4) and (5) into the electron continuity equation, we obtain

$$(\partial_t^2 - v_{te}^2\partial_z^2)\frac{\delta n_e}{n_0} = -\frac{e}{m_{e0}}(\partial_z^2 + \Omega_{e0}^{-2}\partial_t^2\nabla_\perp^2)\phi + \left[\partial_z^2 + \frac{\omega^2}{\omega_{p0}^2}\frac{\partial_t^2}{\Omega_{e0}^2}\nabla_\perp^2\right]\frac{e^2|E|^2}{2m_{e0}^2\omega_{p0}^2} , \quad (6)$$

where $\rho_e^2\nabla_\perp^2 << 1$ and ρ_e is the electron Larmor radius.

Combining the ion continuity and the momentum equatons, one can readily derive a relationship between the ion number density perturbation δn_i and the ambipolar potential ϕ. One finds

$$[\partial_t^2(\partial_t^2 + \Omega_i^2 - v_{ti}^2\nabla^2) - \Omega_i^2 v_{ti}^2\partial_z^2]\frac{\delta n_i}{n_0} = \frac{e}{m_i}(\partial_t^2\nabla^2 + \Omega_i^2\partial_z^2)\phi , \quad (7)$$

where v_{ti} and Ω_i are the thermal velocity and the gyrofrequency of the ions, and $\nabla^2 = \nabla_\perp^2 + \partial_z^2$. The ponderomotive force acting on the ions is smaller, and is, therefore, neglected. However, the ions are coupled to the electrons by the ambipolar potential.

Eliminating ϕ from Eqs. (6) and (7) and using the quasineutrality condition ($\delta n_e = \delta n_i$), we derive our nonlinear low-frequency equation:

$$\left\{[\partial_t^2(\partial_t^2 + \Omega_i^2 - v_{ti}^2\nabla^2) - \Omega_i^2 v_{ti}^2\partial_z^2](\partial_z^2 + \Omega_e^{-2}\partial_t^2\nabla_\perp^2) + \frac{m_{e0}}{m_i}(\partial_t^2 - v_{te}^2\partial_z^2)(\partial_t^2\nabla^2 + \Omega_i^2\partial_z^2)\right\}\frac{\delta n_e}{n_0} = \frac{m_{e0}}{m_i}(\partial_t^2\nabla^2 + \Omega_i^2\partial_z^2)\left[\partial_z^2 + \frac{\omega^2}{\omega_{p0}^2}\frac{\partial_t^2}{\Omega_{e0}^2}\nabla_\perp^2\right]\frac{e^2|E|^2}{2m_{e0}^2\omega_{p0}^2} . \quad (8)$$

Equations (3) and (8) constitute a pair of coupled nonlinear equations which describe the nonlinear interaction of the upper-hybrid waves with low-frequency potential fluctuations in magnetized plasmas. This set could also be considered as the Zakharov equations[11] for magnetized plasmas. Clearly, we have generalized the previous work[10] by including the relativistic nonlinearity and arbitrary frequencies ($\partial_t << \Omega_{e0}$) of the slow modulations.

In what follows, we consider some limiting cases. First, we consider the driven lower-hybrid modulations. Here, $\partial_t^2 >> \Omega_i^2$, $v_{te}^2\partial_z^2$, and $\nabla_\perp^2 >> \partial_z^2$. For $\partial_z^2 << (\omega^2/\omega_{p0}^2) \times \Omega_{e0}^{-2}\partial_t^2\nabla_\perp^2$, Eq. (8) reduces to[7]

$$(\partial_t^2 + \omega_{LH}^2 - v_{ti}^2\partial_x^2)\frac{\delta n_e}{n_0} = \frac{e^2\omega^2}{2m_{e0}m_i\omega_{p0}^2}\partial_x^2|E|^2 . \quad (9)$$

In this case, the slow ions follow a straight-line orbit and the slow electrons are highly magnetized. Here, $\omega_{LH}^2 = \Omega_e\Omega_i$. Second, for $\partial_t^2 << v_{te}^2\partial_z^2$, $\Omega_i^2 << \partial_t^2$, $\partial_z^2 << \nabla_\perp^2$, $\partial_z^2 >> \Omega_{e0}^{-1}\partial_t^2\nabla_\perp^2$, and $(\omega/\omega_{p0})^2\Omega_{e0}^{-2}\partial_t^2\nabla_\perp^2$, Eq. (7) yields[7]

$$(\partial_t^2 - c_s^2\partial_x^2)\frac{\delta n_e}{n_0} = \frac{e^2}{2m_{e0}m_i\omega_{p0}^2}\partial_x^2|E|^2 . \quad (10)$$

Here, the slow electrons establish the Boltzmann-type equilibrium along $B_0\hat{z}$. We have defined $c_s^2 = (T_e + \gamma_i T_i)/m_i$. Note that Eq. (10) describes the driven fast ion-acoustic wave. Third, we let $\partial_t \sim \Omega_i$, but impose other frequency and wave number restrictions similar to the fast ion-acoustic waves described above. In this case, Eq. (7) gives the driven ion-cyclotron waves[3]

$$(\partial_t^2 - c_s^2\partial_x^2 + \Omega_i^2)\frac{\delta n_e}{n_0} = \frac{e^2}{2m_{e0}m_i\omega_{p0}^2}\partial_x^2|E|^2 . \quad (11)$$

Finally, in the adiabatic limit ($\partial_t \to 0$), Eq. (7) simplifies to[2,7]

$$\frac{\delta n_e}{n_0} \equiv N = -\frac{|E|^2}{8\pi n_0(T_e + T_i)} . \quad (12)$$

We now look for the stationary solutions of our coupled system [Eqs. (3) and (9)–(12)]. Introducing $\xi = x - v_g t$, one finds the stationary density perturbations for the lower-hybrid ($v_g \approx v_{ti}$) and ion-cyclotron ($v_g \approx c_s$) modulations, respectively,

$$N = \frac{\omega^2}{\omega_{p0}^2}\frac{c_s^2}{\Omega_e\Omega_i}\partial_\xi^2\frac{|E|^2}{8\pi n_0 T} , \quad (13)$$

$$N = \rho_s^2\partial_\xi^2\frac{|E|^2}{8\pi n_0 T} , \quad (14)$$

where $\rho_s = c_s/\Omega_i$ and $T = T_e + T_i$. On the other hand, for the fast ion-acoustic modulations, one finds

$$N = \frac{|E|^2}{M^2 - 1}\frac{1}{8\pi n_0 T} , \quad (15)$$

where $M = v_g/c_s$ is the Mach number.

Inserting Eqs. (12)–(15) into Eq. (3), one gets in the moving frame

$$2i\omega_0\partial_t E + \alpha_\xi^2 E + EQ_j|E|^2 = 0 , \quad (16)$$

where the electric field E is normalized by $(8\pi n_0 T)^{1/2}$. The coefficient Q_j of the nonlinear term of the static and fast ion-acoustic modulations are, respectively, given by

$$Q_a = \omega_{p0}^2(1 + X) , \quad (17)$$

$$Q_{\text{FIA}} = \omega_{p0}^2[(1 - M^2)^{-1} + X] , \quad (18)$$

where

$$X = (v_{te}^2/2c^2)(1 + \Omega_{e0}^2/\omega_{p0}^2)(1 + 2\Omega_{e0}^2/\omega_{p0}^2)$$

is the contribution of relativistic nonlinearity.

On the other hand, for the lower-hybrid and the ion-cyclotron modulations, one finds

$$Q_{\text{LHICM}} = \omega_{p0}^2(X - \mu\rho_s^2\partial_\xi^2) , \quad (19)$$

where $\mu = 1(\omega^2 m_{e0}/m_i\omega_{p0}^2)$ for the ion-cyclotron (lower-hybrid) modulations.

Stationary solutions of Eq. (16) are well known in the literature.[2,3,5,7] For completeness, we briefly summarize our results. For the adiabatic modulations, the relativistic nonlinearity is added to the ponderomotive force nonlinearity. The combined effects then lead to the standing bright (dark) solitons[7] for positive (negative) group dispersive upper-hybrid waves. Inclusion of the ion inertia gives rise to the moving perturbations. For $M^2 < 1$, the properties of the solitons are similar to those of the standing solitons. However, for $M^2 > 1$, Eq. (18) becomes

$$Q_{\mathrm{FIA}} = \omega_{p0}^2 (X - M^{-2}) \quad . \tag{20}$$

Note that for $\alpha(X - M^{-2}) > 0$ (< 0) bright (dark) solitons are encountered.

If one combines Eqs. (16) and (19), one can then obtain the nonlinear Schrödinger equation with nonlocal nonlinearity

$$2i\omega_0 \partial_t E + \alpha \partial_\xi^2 E + \omega_{p0}^2 E (X - \mu \rho_s^2 \partial_\xi^2) |E|^2 = 0 \quad . \tag{21}$$

Equation (21) has been extensively studied by Litvak and Sergeev.[5] According to these authors, spiky solitons may appear in a magnetized plasma. It also seems likely that the combined effects of the relativistic and the nonlocal nonlinearities may lead to collapse of the upper-hybrid wave packets even in one space dimension.

In summary, we have generalized the work of Ref. 10 in two respects. Firstly, we have incorporated the nonlinearities associated with the relativistic electron mass variation.[9] Secondly, the low-frequency electrostatic response to the upper-hybrid wave is considered to be fairly general. It is shown that the previously derived low-frequency responses are the special cases of our general formulation. Thus, our results are well suited for numerical investigation of strong turbulence in magnetoplasmas.

Finally, we should like to point out that our investigation can be useful to the understanding of nonlinear upper-hybrid wave phenomena. The latter might appear during the radio-frequency heating of fusion plasmas as well as the beat-wave-particle accelerators.[12] In particular, Katsouleas and Dawson[12] have proposed that a large amplitude upper-hybrid electrostatic wave propagating perpendicular to $B_0\hat{z}$ can accelerate the electrons to very high energies. However, if the present nonlinear mechanism is operative it can have important consequences to the idea of electron acceleration by the beating of laser beams.

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