Brief Reports

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Improvement of the Eberly-Singh time-energy inequality by combination with the Mandelstam-Tamm approach

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It is demonstrated that Eberly and Singh's notion of partial stationarity of a quantum system can be reconciled with Mandelstam and Tamm's approach to deriving energy-time inequalities. The result of this merger is an improved stationarity time, which provides a more precise measure of the degree of stationarity of the system under study, and which permits a sharper statement of the energy-time uncertainty relation than that given by Eberly and Singh.

I. INTRODUCTION

It is generally appreciated that in most cases of interest the times appearing in the usual Mandelstam-Tamm (MT) inequality[1,2]

$$\tau_{A_n}^{(\mathrm{MT})}=\frac{\Delta A_n}{|d/dt\langle A_n\rangle|}\geq\frac{\hbar}{2\Delta H} \tag{1}$$

constitute very poor bounds in the sense that their product with the energy dispersion ΔH turns out to be much larger than $\hbar/2$ (Refs. 2 and 4). In (1), A_n may be the operator corresponding to any *observable* of the system under study, provided the time rate of change of its expectation value in the given state,

$$\frac{d}{dt}\langle A_n\rangle=\frac{i}{\hbar}\langle[H,A_n]\rangle+\left\langle\frac{\partial A_n}{\partial t}\right\rangle , \tag{2}$$

does not vanish. But apart from possessing the dimension of time, the set of times $\tau_{A_n}^{(\mathrm{MT})}$ thus produced does not seem to allow any obvious interpretation as a set of characteristic times of evolution of the system under study.[3,4]

Since the original paper by Mandelstam and Tamm,[1] there have been numerous attempts to improve this situation.[5] One of the most fruitful propositions was put forward by Eberly and Singh[6] (ES) through the concept of partial stationarity of a quantum system. This concept arises from the observation that for a stationary ensemble the density operator commutes with the Hamiltonian, and conversely,[7] whence $d\rho/dt=0$ from the equation of motion of the density operator (in the Schrödinger picture)

$$\frac{d\rho}{dt}=\frac{i}{\hbar}[\rho,H] . \tag{3}$$

On the other hand, the expectation value of (3) vanishes for all ensembles, stationary or nonstationary, i.e., $\langle d\rho/dt\rangle=0$, so that the dispersion of the time derivative of the density operator provides a natural measure of the degree of stationarity of the given ensemble. Its reciprocal value,

$$\tau_S^{(\mathrm{ES})}=\left[2\Delta\left(\frac{d\rho}{dt}\right)\right]^{-1} , \tag{4}$$

was introduced by Eberly and Singh[6] as the stationarity time of the given system in the given state, where, in addition to the original definition,[6] we have incorporated an inessential factor of 2 for later convenience. Eberly and Singh[6] proved that for any quantum system

$$\tau_S^{(\mathrm{ES})}\Delta H\geq\frac{\hbar}{2} , \tag{5}$$

with the equality holding for a pure state density operator. They remarked that while *not* being an element of the set (1) of MS times, $\tau_S^{(\mathrm{ES})}$ seemed to bound all those times from below.

However, it is the purpose of this note to demonstrate the interesting fact that if Eberly and Singh's[6] stationarity time is made a member of the family of MS times, or, in other words, if Eberly and Singh's[6] concept of partial stationarity is combined with Mandelstam and Tamm's[1] general approach to deriving energy-time inequalities for a given system, then an improved stationarity time emerges, which, in turn, bounds Eberly and Singh's[6] stationarity time (4) from below. Section II will give a proof of this assertion, and Sec. III will discuss the pertinence of this improved stationarity time.

II. IMPROVED STATIONARITY TIME

Eberly and Singh[6] overcame the unsatisfactory ambiguity of the many possible choices of an operator A_n in (1) by introducing the stationarity time (4), which through the operator $d\rho/dt$ refers to the system as a whole, rather than to an arbitrary observable A_n.

However, the different approaches manifesting themselves in Eqs. (1) and (4), respectively, can be reconciled by observing that—as was pointed out by Eberly and Singh[6] —the operator

$$s \equiv \frac{d\rho}{dt} \tag{6}$$

characterizes, in a certain sense, the stationarity properties of the given quantum system. Therefore, it seems natural to apply the Mandelstam-Tamm procedure to this "stationarity" operator, rather than to some observable A_n.

To this end, we require the equation of motion of s, which we obtain by taking the time derivative of Eq. (3),

$$\frac{ds}{dt} = \frac{i}{\hbar}[s,H] + \frac{i}{\hbar}\left[\rho, \frac{\partial H}{\partial t}\right] , \tag{7}$$

where we have made allowance for a possible time dependence of the Hamiltonian. Taking the expectation value of (7), we find by invoking Schwarz's inequality

$$\left|\left\langle \frac{ds}{dt}\right\rangle\right| = \frac{1}{\hbar}|\langle [s,H]\rangle| \leq \frac{2}{\hbar}\Delta s\,\Delta H , \tag{8}$$

since

$$\left\langle\left[\rho, \frac{\partial H}{\partial t}\right]\right\rangle = \mathrm{Tr}\left(\rho\left[\rho, \frac{\partial H}{\partial t}\right]\right) = 0 .$$

Defining the stationarity time (labeled with the initials of the present authors) of the system by

$$\tau_S^{(\mathrm{LK})} = \frac{\Delta s}{|\langle ds/dt\rangle|} , \tag{9}$$

we are led to

$$\tau_S^{(\mathrm{LK})}\Delta H \geq \frac{\hbar}{2} . \tag{10}$$

It remains to be demonstrated that, in general, (10) provides a sharper statement of the energy-time uncertainty relation than does (5). To this end, we note that the numerator of (9), which is just $(2\tau_S^{(\mathrm{ES})})^{-1}$, can be written as

$$\Delta s = \langle s^2\rangle^{1/2} = \hbar^{-1}\{\mathrm{Tr}(\rho^3H^2 - \rho^2 H\rho H)\}^{1/2} , \tag{11}$$

while the denominator of (9) can be represented as

$$\left|\left\langle \frac{ds}{dt}\right\rangle\right| = 2\hbar^{-2}|\mathrm{Tr}(\rho^2H^2 - \rho H\rho H)|. \tag{12}$$

Since a pure state statistical operator is idempotent, (4), (9), (11), and (12) immediately imply that

$$\tau_S^{(\mathrm{LK})} = \tau_S^{(\mathrm{ES})} , \tag{13}$$

for pure states.

For mixed states, we recall [cf. Eq. (7) of Ref. 6] that in the basis of eigenstates of ρ, (11) takes the form

$$\Delta s = \hbar^{-1}\left(\sum_{k,m} p_k(p_k - p_m)^2 |H_{km}|^2\right)^{1/2} . \tag{14}$$

Similarly, we find for the right-hand side of (12),

$$\left|\left\langle \frac{ds}{dt}\right\rangle\right| = 2\hbar^{-2}\left|\sum_{k,m} p_k(p_k - p_m)|H_{km}|^2\right| . \tag{15}$$

For our purposes, it is convenient to write (15) and (14) in the alternative form

$$\left|\left\langle \frac{ds}{dt}\right\rangle\right| = \hbar^{-2}\sum_{k,m}(p_k - p_m)^2|H_{km}|^2 , \tag{16}$$

and

$$\Delta s = \hbar^{-1}\left(\frac{1}{2}\sum_{k,m}(p_k + p_m)(p_k - p_m)^2|H_{km}|^2\right)^{1/2}$$

$$\leq \hbar^{-1}\left(\frac{1}{2}\sum_{k,m}(p_k - p_m)^2|H_{km}|^2\right)^{1/2} , \tag{17}$$

respectively. The inequality sign in (17) stems from the sum of any two probabilities of a mixed state being less or equal to unity,

$$p_k + p_m \leq 1 , \tag{18}$$

the equality holding for mixed states composed of exactly two pure states.

Inserting (16) and the right-hand side of (17) into (9), we find

$$\tau_S^{(\mathrm{LK})} \leq \hbar\left(2\sum_{k,m}(p_k - p_m)^2|H_{km}|^2\right)^{-1/2}$$

$$\leq \hbar\left(2\sum_{k,m}(p_k + p_m)(p_k - p_m)^2|H_{km}|^2\right)^{-1/2} = \tau_S^{(\mathrm{ES})} ,$$

where (18) has been used. Thus, we have shown that

$$\tau_S^{(\mathrm{LK})} \leq \tau_S^{(\mathrm{ES})} , \tag{19}$$

for mixed states, where the equality holds for mixed states composed of exactly two pure states.

III. DISCUSSION

The improved stationarity time (9) enjoys a number of desirable features. Firstly, it reconciles Eberly and Singh's[6] proposition for an energy-time inequality with that of Mandelstam and Tamm,[1] thereby removing the alleged "roundabout way" of the introduction of the former,[8] which has prevented some authors from appreciating its appropriateness.[9] At the same time, we have fully retained the favorable characteristic of Eberly and Singh's[6] stationarity time, namely, of referring to the quantum system as a whole, rather than to one of its observables.

Secondly, for all mixed states composed of more than two pure states, the improved stationarity time provides a sharper statement of the time-energy uncertainty relation than does Eberly and Singh's version.[6] This is because for a particular mixed state of this kind, the improved stationarity time $\tau_S^{(\mathrm{LK})}$ is always less than $\tau_S^{(\mathrm{ES})}$, thus furnishing a more precise measure of the degree of stationarity of the given quantum system in that state. This is of relevance to certain investigations in quantum chemistry, where the degree of stationarity of only approximately known stationary states is tested by the stationarity time $\tau_S^{(\mathrm{ES})}$ of the system in that state.[10]

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