Path-integral approach to nonlinear self-excited oscillators

L. N. Epele, H. Fanchiotti, A. Spina, and H. Vucetich
Laboratorio de Física Teórica, Departamento de Física, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina
(Received 31 January 1984; revised manuscript received 4 June 1984)

Noise-driven self-excited oscillators in their quasisinusoidal regime are treated within a functional-integral approach. The zero modes are eliminated from the generating functional with a change of variables to collective coordinates. The van der Pol oscillator is analyzed as a prototype of such systems. The main generating functional is introduced, taking from the Lagrangian the nonfluctuating terms and linearizing the resulting integral in the neighborhood of the limit-cycle solution of the oscillator. By functional differentiation the response and correlation functions associated with the problem are calculated from the main generating functional. Some well-established properties of these noise-driven oscillators are easily recovered with this method.

I. INTRODUCTION

Self-excited oscillators were introduced in physics a century ago by Lord Rayleigh, who first analyzed an autonomous nonlinear oscillator possessing a limit-cycle solution.[1] Studies concerning the existence of periodic solutions to nonlinear equations were pioneered by Poincaré, Liapounov, and Bendixon. These equations regained new attention after van der Pol's work on triode generators,[2] where an equation closely related to Rayleigh's was derived. Autonomous nonlinear systems driven by noise have recently been used as models of many chemical, biological, and engineering systems.[3] Also their value as toy models in quantum-field theory has been stressed.[4]

When the external noise source is a Gaussian white noise, one is faced with a Langevin equation defining a continuous Markovian process. This stochastic process is completely described by the transition probability density $P(x,t;x_0,t_0)$ of finding the system at position x at time t if it was known to be at x_0 at an earlier time t_0. This transition probability density can be expressed as a path integral over all possible trajectories which bring the system from (x_0,t_0) to (x,t).[5] In the present paper the generating functional of correlation and response functions for the statistical stationary state of such systems is investigated in some detail. Procedures to evaluate this generating functional for general Markovian systems involve standard perturbation schemes. However, if one attempts to use a naive perturbation approach, the presence of limit cycles easily leads to problems, the origin of which is the presence of a zero mode that arises because of the breakdown of time-translation invariance. In fact, the free-oscillating system is invariant under time translations (actually this is the very definition of an autonomous system), but when a limit cycle is present the solution is no longer time-translation invariant. This is because the oscillating solution of the unperturbed oscillator is not unique: There are infinitely many solutions differing in the initial phase. Under the noise action the system can freely pass from one solution to another without energy expediture. This situation is analogous to that arising in phase transitions, i.e., the symmetry is spontaneously broken and the system selects one of the admissible solutions. Nevertheless there may exist a continuum of admissible states to which the system may pass without doing work. Any naive attempt to evaluate the Green's functions of the theory does not properly handle such a zero mode and leads to incorrect results in some order of the loop expansion.[6]

This zero mode must be extracted before developing the perturbation expansion. Our purpose here is to show that this goal can be achieved by generalizing a method previously developed for the treatment of the Poincaré model.[7] The technique, which we present in Sec. II, consists of performing an appropriate change of variables to a set of "collective coordinates," thus leading to a separation of the zero mode from the other dynamical degrees of freedom. Details of the computation are outlined in Sec. III, where it is shown that it is possible to carry out a systematic and consistent treatment of the generating functional. By using this functional, earlier results are recovered in a direct way, such as that described in Sec. IV. Our results are summarized and commented upon in Sec. V.

II. PATH-INTEGRAL REPRESENTATION OF THE GENERATING FUNCTIONAL

Self-sustained oscillators are described by equations of the form

$$\ddot{x}+g(x,\dot{x})+x=0\ , \tag{2.1}$$

where the nonlinear function $g(x,\dot{x})$ has the property that in the absence of external sources stable vibrations occur. Although the treatment of this section is general and can be applied to any self-oscillating system obeying an equation of the type (2.1), in order to fix ideas we will concentrate on quasisinusoidal oscillators that when driven by white noise are governed by the stochastic equation

$$\ddot{x}+\epsilon f(x,\dot{x})+x=\xi(t) \tag{2.2}$$

with

$$\langle \xi(t) \rangle = 0 ,$$
$$\langle \xi(t)\xi(t') \rangle = \sigma^2 \delta(t-t') , \tag{2.3}$$

where σ^2 is a measure of the noise strength and

$$|\epsilon| \ll 1 . \tag{2.4}$$

In particular, the van der Pol oscillator which we will use as an example is obtained for

$$f(x,\dot{x}) = (x^2-1)\dot{x} . \tag{2.5}$$

It will undergo limit-cycle oscillations for $\epsilon > 0$. The case $\epsilon < 0$, when the only stable stationary solution is the quiescent one ($x = \dot{x} = 0$), has already been studied with success, but when the same technique used for the quiescent regime is applied to the oscillating case, divergences appear in the perturbation expansion terms.[6] We will next identify the origin of this divergence showing that it is a common feature of self-oscillating systems and also show a way of eliminating them.

The standard approach to this sort of stochastic problems is to set

$$\dot{x} = -y \tag{2.6}$$

and to compute the transition probability function $P(x,y,t;x_0,y_0,t_0)$ whose path-integral representation can be found to be[5,6]

$$P(x,y,t;x_0,y_0,t_0) = \int Dx\, Dy\, Dp_x Dp_y \exp\left[\int_{t_0}^{t} -\frac{\sigma^2}{2}p_y^2 + ip_x(\dot{x}+y) + ip_y(\dot{y}-\epsilon f(x,-y)-x) \right] , \tag{2.7}$$

where the functional integral is taken between the limits $x(t_0)=x_0$, $y(t_0)=y_0$, and $x(t)=x$, $y(t)=y$.

The generating functional for the response and correlation functions in the stationary state is defined as

$$Z(u_x,u_y,v_x,v_y) = \lim_{\substack{t\to\infty \\ t_0\to-\infty}} \int dx_0 dy_0 P_s(x_0,y_0,t;x_0,y_0,t_0) , \tag{2.8}$$

where P_s is defined similar to P with the addition of nonrandom sources for coordinates and momenta so that the Lagrangian in Eq. (2.7) will now read

$$-\frac{\sigma^2}{2}p_y^2 + ip_x(\dot{x}+y) + ip_y(\dot{y}-\epsilon f(x,-y)-x) + iu_x x + iu_y y + iv_x p_x + iv_y p_y . \tag{2.9}$$

As we already mentioned, a perturbation expansion based on this approach fails since the individual terms in the series diverge. It was already noted that these divergences originate in the presence of a zero mode, which was not made explicit by the previous analysis.[6,7] The source of this mode was traced back to the existence of an arbitrary phase in quasisinusoidal self-sustained oscillations. We intend now to make a systematic analysis of this problem, reconsidering the quasisinusoidal case for simplicity, but at the same time pointing out the generality of the problem.

For small values of the nonlinearity parameter ϵ, limit cycles (LC) will be circlelike

$$x_{\rm LC}(t) = r\cos(t+\phi) + O(\epsilon) . \tag{2.10}$$

The particular equation under consideration determines the amplitude r (for the van der Pol case we will have $r=2$). The phase ϕ remains the only arbitrary integration constant. The other constant disappears after the transient regime dies out. This feature is common to every self-oscillating system. The simple geometrical meaning that can be assigned to this constant explains by itself why this must be so: Different points on the limit cycle can be labeled by a single variable, but since the cycle is a closed continuous curve this variable must appear as an argument of a periodic function in order to ensure that after traversing the whole cycle one returns to the starting point, closing the curve. The remaining constant fixes the departure point for traversing the cycle. In the quiescent regime this constant is meaningless since the only stable steady-state solution reduces to a point in the phase plane and ϕ disappears in the limit $t \to \infty$. In the oscillating regime this integration constant defines a whole family of solutions of the differential equation, each of them differing in its initial "phase." The problem is now analogous to that presented in quantum theory of collective motions,[8] the phase ϕ being the collective variable. In order to isolate this internal coordinate responsible for the zero mode we give to the phase the status of a time-dependent dynamical variable and allow for fluctuations about the cycle. Going back to the quasisinusoidal case we fulfill these two requirements by defining new coordinates $r(t)$ and $\phi(t)$ through

$$x(t) = r(t)\cos[t+\phi(t)] . \tag{2.11}$$

This duplicates the number of degrees of freedom. In order to restore them to 1 a fixing condition must be imposed. We chose for this purpose the condition that is obtained by taking the time derivative of the limit-cycle solution and then replacing t and ϕ by $r(t)$ and $\phi(t)$

$$\dot{x}(t) = -r(t)\sin[t+\phi(t)] . \tag{2.12}$$

Equations (2.11) and (2.12) implicitly define a constraint equation between $r(t)$ and $\phi(t)$

$$\dot{r}(t)\cos[t+\phi(t)] + r(t)\dot{\phi}(t)\sin[t+\phi(t)] = 0 . \tag{2.13}$$

Equations (2.11) and (2.12) are similar to those used in the Bogoliubov-Krilov method for treating quasisinusoidal oscillations,[9] but this similitude is circumstantial and may be misleading. It is not a mere change from rectangular (x,y) to polar (r,ϕ) coordinates. We could have chosen another way to account for fluctuations about the cycle, for instance,

$$x(t)=r\cos[t+\phi(t)]+\eta(t)\ , \tag{2.14}$$

with constant r, and $\eta(t)$ measuring the fluctuations about the cycle. In this case the similitude disappears. If the oscillator limit cycle is not circlelike, polar coordinates may not even be well suited for describing it in the phase plane as it is the case for strong relaxing oscillators.

Shifting now from the original coordinates to the new set of collective coordinates, the transition probability function becomes

$$P(r,\phi,r;r_0,\phi_0,t_0)=\int Dr\,D\phi\,Dp_r Dp_\phi \exp\left[i\int_{t_0}^{t}d\tau[p_r(\tau)\dot r(\tau)+p_\phi(\tau)\dot\phi(\tau)-h(p_r(\tau),p_\phi(\tau),r(\tau),\phi(\tau),\tau)]\right]\ , \tag{2.15}$$

where the limits of the path integral are now $r(t_0)=r_0$, $\phi(t_0)=\phi_0$ and $r(t)=r$, $\phi(t)=\phi$, and with

$$h(p_r,p_\phi,r,\phi,t)=i\frac{\sigma^2}{2}p_r^2\sin^2\psi+i\frac{\sigma^2}{2}\frac{p_\phi^2}{r^2}\cos^2\psi+i\sigma^2 p_r\frac{p_\phi}{r}\cos\psi\sin\psi$$
$$+p_r\left[\epsilon f(r\cos\psi,-r\sin\psi)\sin\psi+\frac{\sigma^2}{2}\frac{\cos^2\psi}{r}\right]+\frac{p_\phi}{r}\left[\epsilon f(r\cos\psi,-r\sin\psi)\cos\psi-\sigma^2\frac{\cos\psi\sin\psi}{r}\right] \tag{2.16}$$

with

$$\psi(t)=t+\phi(t) \tag{2.17}$$

and

$$Z(j,J,k_r,k_\phi)=\sum_{n=-\infty}^{\infty}\lim_{\substack{t\to\infty\\ t_0\to-\infty}}\int r_0dr_0d\phi_0P_s(r_0,\phi_0+2n\pi,t;r_0,\phi_0,t_0) \tag{2.18}$$

with

$$h_s(p_r,p_\phi,r,\phi,t)=h(p_r,p_\phi,r,\phi,t)+jr+J\phi+k_rp_r+k_\phi\frac{p_\phi}{r}\ . \tag{2.19}$$

Since ϕ is a cyclic variable, the condition of returning to the starting point contained in the definition of the generating functional can be achieved, ending not only at ϕ_0 but also at $\phi_0+2n\pi$, where $n=0,\pm1,\pm2,\ldots$.

III. MAIN GENERATING FUNCTIONAL FOR SELF-SUSTAINED OSCILLATORS IN THE QUASISINUSOIDAL REGIME

The Lagrangian in Eqs. (2.15)–(2.19) can be separated into three groups,

$$L=L_0+\epsilon L_\epsilon+\frac{\sigma^2}{2}L_{\sigma^2} \tag{3.1}$$

with

$$L_0=p_r\dot r+p_\phi\dot\phi+jr+J\phi+k_rp_r+k_\phi\frac{p_\phi}{r}\ ,$$
$$L_\epsilon=-p_rf(r\cos\psi,-r\sin\psi)\sin\psi$$
$$-\frac{p_\phi}{r}f(r\cos\psi,-r\sin\psi)\cos\psi\ , \tag{3.2}$$
$$L_{\sigma^2}=i\left[p_r\sin\psi+\frac{p_\phi}{r}\cos\psi\right]^2-p_r\frac{\cos^2\psi}{r}$$
$$+2\frac{p_\phi}{r}\frac{\cos\psi\sin\psi}{r}\ ,$$

where L_0 corresponds to the harmonic oscillator part, $L_0+\epsilon L_\epsilon$ to the self-sustained oscillator, and L_{σ^2} to the corrections introduced by the external noise source.

It would be desirable to compute a nonperturbed generating functional for the self-sustained oscillator $(L_0+\epsilon L_\epsilon)$ and to obtain perturbatively the corrections introduced by the noise terms. This however is not possible because the presence of sine and cosine terms precludes a closed computation of such a functional. Since we are treating here the particular case of quasisinusoidal oscillations (small ϵ), we try another arrangement of the terms in Eq. (3.1) and for this purpose we write

$$L=L_0+L_1^c\cos\psi+L_1^s\sin\psi$$
$$+L_2^c\cos2\psi+L_2^s\sin2\psi+\cdots\ , \tag{3.3}$$

where we have grouped the terms according to their angular contributions. This grouping is suggested by analytical mechanics perturbation theory, where techniques have been devised for eliminating in a systematic way the oscillating terms through the use of successive canonical transformations.[10] Our first task will be to rewrite the generating functional as

$$Z(j,J,k_r,k_\phi)=\sum_{n=-\infty}^{\infty}\int Dr\,D\phi\,Dp_r Dp_\phi \exp\left\{\int_{-\infty}^{\infty}-\frac{\sigma^2}{4}p_r^2-\frac{\sigma^2}{4}\frac{p_\phi^2}{r^2}+ip_\phi\dot\phi + ip_r\left[\dot r-\frac{\epsilon}{2}\left[r-\frac{r^3}{4}\right]-\frac{\sigma^2}{4r}\right]+ijr+iJ\phi+ik_rp_r+ik_\phi\frac{p_\phi}{r}\right\}e^V, \tag{3.4}$$

where V contains the oscillating part of expansion (3.3). We shall call Eq. (3.4) with $V=0$ the "main generating functional Z_0."

Once the main generating functional has been found, higher orders in V can be computed, for instance,

$$Z(j,J,k_r,k_\phi)=Z_0(j,J,k_r,k_\phi)\langle \exp V\rangle_0 \simeq Z_0\exp[\langle V\rangle_0-\tfrac{1}{2}(\langle V^2\rangle_0-\langle V\rangle_0^2)+\cdots], \tag{3.5}$$

where the mean values are computed with respect to Z_0. Examples of computation with this sort of nonpolynomial Lagrangians will be given in Sec. IV. In this way higher-order corrections to Z can be found systematically. Although the convergence of the cumulant expansion (3.5) is not guaranteed, no troubles such as divergences of secular terms will appear in the expansion.

We shall carry out the computation of the main generating functional in several stages. First we shall compute the integral over ϕ and p_ϕ since they can be carried out exactly using the discretized expression for the path integral. After a straightforward calculation we find

$$Z(j,J,k_r,k_\phi)=\delta\left[\int_{-\infty}^{\infty}J\right]\sum_{n=-\infty}^{\infty}\int Dr\,Dp_r\exp\left\{\int_{-\infty}^{\infty}-\frac{\sigma^2}{4}p_r^2+ik_\phi\frac{n+K}{r}-\frac{\sigma^2}{4}\frac{(n+K)^2}{r^2} + ip_r\left[\dot r-\frac{\epsilon}{2}\left[r-\frac{r^3}{4}\right]-\frac{\sigma^2}{4r}\right]+ijr+ik_rp_r\right\}, \tag{3.6}$$

where

$$K(t)=\int_{-\infty}^{t}J(\tau)d\tau\,. \tag{3.7}$$

The generating functional Z_0 will be zero unless the sources J satisfy the condition $\int J=0$. Since we will need only point sources the sum of their intensities must vanish.

The remaining integral is not Gaussian. For small σ^2 the main contribution to the integral comes from trajectories close to the equilibrium solution r_0 of

$$\dot r=\frac{\epsilon}{2}\left[r-\frac{r^3}{4}\right]+\frac{\sigma^2}{4r}, \tag{3.8}$$

which is found to be

$$r_0^2=4+\frac{\sigma^2}{2\epsilon}+O(\sigma^4)\,. \tag{3.9}$$

Furthermore in the limit cycle only the term $n=0$ survives in Eq. (3.6). We shall follow the usual technique and develop the integrand around the equilibrium solution r_0. Setting

$$\rho=r-r_0 \tag{3.10}$$

and retaining only linear terms in the expansion, we obtain

$$Z_0'(j,J,k_r,k_\phi)=\delta\left[\int_{-\infty}^{\infty}J\right]\exp\left[\frac{i}{r_0}\int_{-\infty}^{\infty}k_\phi K\right]\exp\left[-\frac{\sigma^2}{4r_0^2}\int_{-\infty}^{\infty}K^2\right]\exp\left[ir_0\int_{-\infty}^{\infty}j\right] \times\int D\rho\,Dp_r\exp\left[\int_{-\infty}^{\infty}-\frac{\sigma^2}{4}p_r^2+ip_r(\dot\rho+b\rho)+ip_rk_r+\rho\left[ij-\frac{i}{r_0^2}k_\phi K+\frac{\sigma^2}{2r_0^3}K^2\right]\right], \tag{3.11}$$

where

$$b=\frac{\epsilon}{2}\left[\frac{3r_0^2}{4}-1\right]+\frac{\sigma^2}{4r_0^2} . \tag{3.12}$$

The functional integral in Eq. (3.1) is Gaussian in character and therefore its dependence on the sources can easily be worked out. Thus we write it as

$$\int D\rho\, Dp_r \exp\left[\int_{-\infty}^{\infty} -\frac{\sigma^2}{4}p_r^2+ip_r[L(\rho)+k_r] +\rho\left[ij-\frac{i}{r_0^2}k_\phi K+\frac{\sigma^2}{4r_0^3}K^2\right]\right] . \tag{3.13}$$

The linear operator $L(\rho)$ defined by,

$$L(\rho)=\dot{\rho}+b\rho$$

with

$$\lim_{t\to\infty}\rho(t)=0 \text{ when } \sigma^2=0 , \tag{3.14}$$

has the following Green's function

$$G(t-t')=\exp[-b(t-t')]\Theta(t-t') , \tag{3.15}$$

where Θ is the Heaviside's unit step function. Now the procedure to follow is standard and consists in making a shift of variables so as to cancel the linear terms in the exponential function. Setting

$$\rho=\tilde{\rho}+\rho_0 ,$$
$$p_r=\tilde{p}_r+p_{r0} , \tag{3.16}$$

the values of ρ_0 and p_{r0} are obtained by solving the system of equations which result from equating to zero the functional derivatives of the terms in the exponent with respect to ρ and p_r,

$$iL(\rho_0)+ik_r-\tfrac{1}{2}\sigma^2 p_{r0}=0 ,$$
$$iL^\dagger(p_{r0})+\left[ij+\frac{\sigma^2}{2r_0^3}K^2-\frac{i}{r_0^2}k_\phi K\right]=0 , \tag{3.17}$$

where $L^\dagger$ is the adjoint of L. Equations (3.17) are easily solved and finally we obtain for ρ_0 and p_{r0} the following expressions in terms of the sources:

$$p_{r0}=iG^\dagger\left[ij+\frac{\sigma^2}{2r_0^3}K^2-\frac{i}{r_0^2}k_\phi K\right] ,$$
$$\rho_0=-G(k_r)+\tfrac{1}{2}\sigma^2 GG^\dagger\left[ij+\frac{\sigma^2}{2r_0^3}K^2-\frac{i}{r_0^2}k_\phi K\right] , \tag{3.18}$$

where $G^\dagger$ is the adjoint of G and

$$GG^\dagger=\int_{-\infty}^{\infty} G(t,\tau)G(\tau,t')d\tau=\Delta(t,t') =\frac{\exp(-b\,|t-t'|)}{2b} . \tag{3.19}$$

The resulting expression is then

$$Z_0'(j,J,k_r,k_\phi)=\delta\left[\int_{-\infty}^{\infty} J\right]\exp\left[-\frac{\sigma^2}{4r_0^2}\int_{-\infty}^{\infty} K^2\right]\exp\left[\frac{i}{r_0}\int_{-\infty}^{\infty} k_\phi K\right]\exp\left[ir_0\int_{-\infty}^{\infty} j\right] \times\exp\left[-\int_{-\infty}^{\infty} Q(t_1)G(t_1,t_2)k_r(t_2)dt_1dt_2\right]\exp\left[\frac{\sigma^2}{4}\int_{-\infty}^{\infty} Q(t_1)\Delta(t_1,t_2)Q(t_2)dt_1dt_2\right] , \tag{3.20}$$

where

$$Q(t)=ij(t)+\frac{\sigma^2}{2r_0^3}K^2(t)-\frac{i}{r_0^2}k_\phi(t)K(t) . \tag{3.21}$$

This is one of our principal results. This main generating functional can be used to obtain rather general properties of quasisinusoidal self-sustained oscillators in the vicinities of their limit-cycle solution. We worked out the van der Pol case, but for any other quasisinusoidal oscillator described by Eq. (2.2) we should obtain the same result, just replacing the constants b and r_0 by their appropriate values. In Sec. IV we shall use Eq. (3.21) to compute mean values and probability functions of the relevant parameters of the problem.

IV. RESPONSE, CORRELATION, AND PROBABILITY DENSITY FUNCTIONS

Let us now show how several statistical properties of the system can be computed with the help of the main generating functional.

We will first calculate the response function of the system. In terms of the quantities defined in Sec. III we can write

$$R(t-t')=i\left\langle r\cos\psi\left[p_r'\sin\psi'+\frac{p_\phi'}{r'}\cos\psi'\right]\right\rangle . \tag{4.1}$$

This is a common example of computation with this kind of nonpolynomial Lagrangians. Mean values of expressions containing trigonometric functions are to be evaluated. In order to obtain a meaningful result the whole series defining this function must be summed. This infinite series evaluation, which is characteristic of nonpolynomial Lagrangians,[11] can be done with the help of a pointlike source J. This simple example will be indicative of the way the cumulant expansion (3.5), where similar mean values are involved, must be undertaken. One can expand Eq. (4.1) and simplify it using the restriction $\int J=0$. The resulting expression is

$$R(t-t')=-\tfrac{1}{4}\langle rp_r' e^{i\phi}e^{-i\phi'}\rangle e^{i(t-t')}+\tfrac{1}{4}\langle rp_r' e^{-i\phi}e^{i\phi'}\rangle e^{-i(t-t')}+\frac{i}{4}\left\langle r\frac{p_\phi'}{r'}e^{i\phi}e^{-i\phi'}\right\rangle e^{i(t-t')}$$
$$+\frac{i}{4}\left\langle r\frac{p_\phi'}{r'}e^{-i\phi}e^{i\phi'}\right\rangle e^{-i(t-t')} . \tag{4.2}$$

The mean values can be computed using the main generating functional

$$R(t-t')=\frac{1}{4}\left[\frac{\delta^2 Z_0'(J=\delta(t-\tau)-\delta(t'-\tau))}{\delta j(t)\delta k_r(t')}e^{i(t-t')}-\frac{\delta^2 Z_0'(J=-\delta(t-\tau)+\delta(t'-\tau))}{\delta j(t)\delta k_r(t')}e^{-i(t-t')}\right.$$
$$\left.-i\frac{\delta^2 Z_0'(J=\delta(t-\tau)-\delta(t'-\tau))}{\delta j(t)\delta k_\phi(t')}e^{i(t-t')}-i\frac{\delta^2 Z_0'(J=-\delta(t-\tau)+\delta(t'-\tau))}{\delta j(t)\delta k_\phi(t')}e^{-i(t-t')}\right]_{j=k_r=k_\phi=0} . \tag{4.3}$$

The computations are straightforward and keeping terms up to order σ^2 one obtains for the derivatives

$$\left.\frac{\delta^2 Z_0'(J=\pm\delta(t-\tau)\mp\delta(t'-\tau))}{\delta j(t)\delta k_r(t')}\right|_{j=k_r=k_\phi=0}$$
$$=-i\left[G(t-t')+\frac{\sigma^2}{2r_0^2 b}\{1-\exp[-b(t-t')]\}\Theta(t-t')\right]Z_0'(J=\pm\delta(t-\tau)\mp\delta(t'-\tau),j=k_r=k_\phi=0) , \tag{4.4}$$

$$\left.\frac{\delta^2 Z_0'(J=\pm\delta(t-\tau)\mp\delta(t'-\tau))}{\delta j(t)\delta k_\phi(t')}\right|_{j=k_r=k_\phi=0}=-K(t')\left[1-\frac{\sigma^2}{2r_0^2}\Delta(t,t')\right]Z_0'(J=\pm\delta(t-\tau)\mp\delta(t'-\tau),j=k_r=k_\phi=0) , \tag{4.5}$$

with

$$Z_0'(J=\pm\delta(t-\tau)\mp\delta(t'-\tau),j=k_r=k_\phi=0)=\exp\left[-\frac{\sigma^2}{4r_0^2}|t-t'|\right] . \tag{4.6}$$

In the prepoint discretization[5] we find,

$$K(t')=\int_{-\infty}^{t'+0}J(\tau)d\tau=\begin{cases}-\Theta(t-t') & \text{if } J=\delta(t-\tau)-\delta(t'-\tau)\\ \Theta(t-t') & \text{if } J=-\delta(t-\tau)+\delta(t'-\tau) .\end{cases} \tag{4.7}$$

Putting together all these results we obtain for the response function the following expression:

$$R_0'(t-t')=\tfrac{1}{2}\sin(t-t')\exp\left[-\frac{\sigma^2}{4r_0^2}(t-t')\right]\Theta(t-t')\left[1+\exp[-b(t-t')]+\frac{\sigma^2}{2r_0^2 b}\{1-\tfrac{3}{2}\exp[-b(t-t')]\}\right] . \tag{4.8}$$

Let us calculate now the correlation function of the oscillator. In polar coordinates and in terms of the generating functional previously defined, the correlation function takes the form

$$C(t,t')=\langle r\cos\psi\, r'\cos\psi'\rangle$$
$$=\tfrac{1}{4}(\langle rr'e^{i\phi}e^{-i\phi'}\rangle e^{i(t-t')}+\langle rr'e^{-i\phi}e^{i\phi'}\rangle e^{-i(t-t')})$$
$$=-\tfrac{1}{4}\left[\frac{\delta^2 Z(J=\delta(t-\tau)-\delta(t'-\tau))}{\delta j(t)\delta j(t')}e^{i(t-t')}+\frac{\delta^2 Z(J=-\delta(t-\tau)+\delta(t'-\tau))}{\delta j(t)\delta j(t')}e^{-i(t-t')}\right]_{j=k_r=k_\phi=0} . \tag{4.9}$$

Using the main generating functional for the evaluation of the derivatives we obtain,

$$C_0'(t,t')=\tfrac{1}{4}\left[\left[r_0^2+\frac{\sigma^2}{2}\Delta(t,t')\right]e^{i(t-t')}\exp\left[-\frac{\sigma^2}{4r_0^2}|t-t'|\right]+\left[r_0^2+\frac{\sigma^2}{2}\Delta(t,t')\right]e^{-i(t-t')}\exp\left[-\frac{\sigma^2}{4r_0^2}|t-t'|\right]\right]$$
$$=\tfrac{1}{2}r_0^2\cos(t-t')\exp\left[-\frac{\sigma^2}{4r_0^2}|t-t'|\right]\left[1+\frac{\sigma^2}{4r_0^2 b}\exp(-b|t-t'|)\right] . \tag{4.10}$$

This result shows the well-known phenomenon of line broadening in nonlinear oscillators in the presence of noise.[12]

As a final check let us calculate some of the probability densities associated with our problem. According to the definition of generating functional, we have

$$Z_0'(j,J)=\int Dr\,D\phi\exp\left[i\int_{-\infty}^{\infty}(jr+J\phi)\right]P(r(t),\phi(t))$$
$$=\exp\left[ir_0\int_{-\infty}^{\infty}j\right]\exp\left[-\frac{\sigma^2}{4}\int_{-\infty}^{\infty}j(t_1)\Delta(t_1,t_2)j(t_2)\right]\delta\left[\int_{-\infty}^{\infty}J\right]\exp\left[-\frac{\sigma^2}{4r_0^2}\int_{-\infty}^{\infty}K^2\right], \quad (4.11)$$

where we wrote the generating functional as the functional Fourier transform of the density functional. For the main generating functional Eq. (4.11) can be factored into

$$\int Dr\exp\left[i\int_{-\infty}^{\infty}jr\right]P(r(t))=\exp\left[ir_0\int_{-\infty}^{\infty}j\right]\exp\left[-\frac{\sigma^2}{4}\int_{-\infty}^{\infty}j(t_1)\Delta(t_1,t_2)j(t_2)\right] \quad (4.12)$$

and

$$\int D\phi\exp\left[i\int_{-\infty}^{\infty}J\phi\right]P(\phi(t))=\delta\left[\int_{-\infty}^{\infty}J\right]\exp\left[-\frac{\sigma^2}{4r_0^2}\int_{-\infty}^{\infty}K^2\right]. \quad (4.13)$$

The functional densities are obtained inverting the preceding transformations,

$$P(r(t))=\int Dj\exp\left[-i\int_{-\infty}^{\infty}jr\right]\exp\left[ir_0\int_{-\infty}^{\infty}j\right]\exp\left[-\frac{\sigma^2}{4}\int_{-\infty}^{\infty}j(t_1)\Delta(t_1,t_2)j(t_2)\right] \quad (4.14)$$

and

$$P(\phi(t))=\int DJ\exp\left[-i\int_{-\infty}^{\infty}J\phi\right]\delta\left[\int_{-\infty}^{\infty}J\right]\times\exp\left[-\frac{\sigma^2}{4r_0^2}\int_{-\infty}^{\infty}K^2\right]. \quad (4.15)$$

Rather than in the density functional, we are interested in the density function, namely that $r(t)$ takes the value r at a given arbitrary time t. This density function is given by

$$p(r)=\int Dr\,\delta(r(t)-r)P(r(t)). \quad (4.16)$$

The calculation is done using the Fourier decomposition of the δ function. All the integrals are Gaussian and can be computed easily leading us to the result

$$p(r)=\left[\frac{2b}{\sigma^2\pi}\right]^{1/2}\exp\left[-\frac{2b}{\sigma^2}(r-r_0)^2\right]. \quad (4.17)$$

We find here a well-known characteristic of the probability density for the radial coordinate: The shift towards increasing values of the radius of the distribution maximum with increasing noise strength and the simultaneous spreading of the distribution curve.[12]

In an analogous way we obtain

$$p(\phi)=\int D\phi\,\delta(\phi(t)-\phi)P(\phi(t))=(2\pi)^{-1} \quad (4.18)$$

and

$$P(\phi(t),\phi(0))=2(2\pi)^{-3/2}(\sigma^2 t)^{-1/2}\times\exp\left[-\frac{2}{\sigma^2 t}[\phi(t)-\phi(0)]^2\right] \quad (4.19)$$

for the density function of the phase. This result shows another well-known phenomenon: the Brownian phase diffusion.[13,14]

As a more complicated example, the density function for the x coordinate is found from our previous results as

$$p(x)=\int dr\,d\phi\,\delta(r\cos\psi-x)p(r,\phi)$$
$$=\left[\frac{2b}{\sigma^2\pi}\right]^{1/2}(2\pi)^{-1}\int dr\,d\phi(d\lambda/2\pi)\exp[i\lambda(r\cos\psi-x)]\exp\left[-\frac{2b}{\sigma^2}(r-r_0)^2\right]. \quad (4.20)$$

Using the Bessel function expansion[15]

$$\exp(i\lambda r\cos\psi)=\sum_{k=-\infty}^{\infty}i^kJ_k(\lambda r)e^{ik\psi} \quad (4.21)$$

(only the term $k=0$ will survive when integrating over the phase), Eq. (4.20) can be put in the form

$$p(x)=\pi^{-1}\left[\frac{2b}{\sigma^2\pi}\right]^{1/2}\int_{|x|}^{\infty}dr(r^2-x^2)^{-1/2}\times\exp\left[-\frac{2b}{\sigma^2}(r-r_0)^2\right]. \quad (4.22)$$

Since the density function of an harmonic oscillator is proportional to $(r_0^2-x^2)^{-1/2}$, we see that Eq. (4.22) can be interpreted as a Gaussian fluctuation around the harmonic motion.

V. CONCLUSIONS

As we have pointed out, naive attempts to evaluate the Green's functions of self-sustained oscillating systems fail in the loop expansion due to the presence of a zero mode which must be extracted. We focused our attention on oscillators governed by an equation of the form (2.2), but as we already mentioned this is not a major restriction. A complication arises from the fact that limit-cycle solutions are not generally known in closed form but as an expansion in the nonlinearity parameter ϵ and so a two-parameter expansion must be undertaken. As far as the zero mode is concerned its appearance does not depend on whether one has a closed expression for the cycle or a series expansion. Self-sustained oscillators, being described by a second-order differential equation, possess in their solution two integration constants. What characterizes this type of oscillators is that when the steady-state regime is reached one of these constants still remains (as a phaselike term in periodic functions), regardless of the analytical expression for the limit cycle. The other constant vanishes. It is precisely the former one (that which fixes the departure point for traversing the cycle), the collective variable that generates a whole family of steady-state solutions and that is responsible for the zero mode. The technique we employed for isolating this mode is quite general and can be applied to any self-sustained oscillator. We restore the possibility of transversal displacements to the cycle, make the remaining constant a new coordinate in the problem, and impose a fixing condition in order to restore to one the number of degrees of freedom. Although the remaining developments are not immediate they will be free from divergences originating in this zero mode.

The resulting generating functional Z has a complicated structure because the Lagrangian is nonpolynomic. In order to extract an unperturbed generating functional we take the nonoscillating part of the Lagrangian and form with it the main generating functional Z_0 whose angular part can be computed exactly. The resulting integral for the radial part cannot be computed in closed form. Since for small σ^2 the main contribution to this integral comes from the stable solution of the nonoscillating radial equation, we expand the integrand around this solution and retain only the linear terms.

As we have seen, simple first-order calculations employing this technique reveal that we can regain an amount of well established results that characterize this type of oscillators in the quasiharmonic regime. For the radial probability distribution is obtained a Gaussian approximation which shows the outstanding characteristics of the exact one corresponding to the nonoscillating equations. A more precise approximation to this probability distribution would require an expansion of the nonlinear terms contained in the Lagrangian of Z_0. For corrections introduced at higher frequencies the fluctuating terms contained in V will have to be taken into account.

ACKNOWLEDGMENTS

The authors take pleasure in acknowledging many helpful discussions with Carlos A. García Canal from Laboratorio de Física Teórica U.N.L.P. This work has been made possible by a grant in aid from Consejo Nacional de Investigaciones Científicas y Técnicas, República Argentina.

[1]Lord Rayleigh, Philos. Mag. **15**, 229 (1983); *The Theory of Sound* (Dover, New York, 1945), Vols. I and II.

[2]B. van der Pol, Philos. Mag. **43**, 700 (1922); 2, 978 (1926); 3, 65 (1927).

[3]H. Haken, *Synergetics*, 2nd ed. (Springer, Berlin, 1978).

[4]G. Parisi and Wu Yongshi, Sci. Sin. **24**, 483 (1981).

[5]F. Langouche, D. Roekaerts, and E. Tirapegui, Phys. **95A**, 252 (1979); Prog. Theor. Phys. **61**, 1617 (1979).

[6]L. N. Epele, H. Fanchiotti, and H. Vucetich, presentation at the XIV International Conference on Thermodynamics and Statistical Mechanics, Edmonton, Alberta, Canada (1980) (unpublished).

[7]L. N. Epele, H. Fanchiotti, O. Piro, and H. Vucetich, in *Stochastic Processes Applied to Physics and Related Fields*, Proceedings of the Escuela Latinoamericana de Física (World Scientific, Singapore, 1983).

[8]Akio Hosoya and Keiji Kikkawa, Nucl. Phys. **B101**, 271 (1975).

[9]N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Gordon and Breach, New York, 1961).

[10]J. R. Cary, Phys. Rep. **79**, 129 (1981).

[11]H. Fanchiotti, C. A. García Canal, H. O. Girotti, and H. Vucetich, Lett. Nuovo Cimento **2**, 1001 (1971).

[12]R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967), Vol. II.

[13]K. Tomita, T. Ohta, and H. Tomita, Prog. Theor. Phys. **52**, 1744 (1974).

[14]N. G. van Kampen, in *Advances in Chemical Physics 34*, edited by I. Prigogine and S. A. Rice (Wiley, New York, 1976).

[15]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980).