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Electron distribution function in the potential electric field of a high-frequency monochromatic plasma wave with an arbitrarily large amplitude

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An analytic solution of the Vlasov equation in the one-dimensional case is presented for electrons in the potential electric field of a high-frequency monochromatic plasma wave with an arbitrarily large amplitude. The phase velocity of this wave is assumed to be appreciably higher than the electron thermal velocity. The ambipolar potential due to charge separation is taken into account and the form of the electron distribution function on the boundary need not be Maxwellian.

In laser-pellet experiments and rf heating in tokamaks, large-amplitude high-frequency monochromatic plasma waves are generated in a plasma. If the phase velocity of these spatially modulated plasma waves is appreciably higher than the electron thermal velocity, then resonant effects (Landau damping) are negligible. Modulation of the amplitude is crucial in changing the plasma equilibrium. In many problems of interest, the collisionless regime is appropriate, and so the Vlasov equation can be used to determine the electron distribution function.

For a considerable time only a "pure" quasilinear solution of this equation was known. This solution remained Maxwellian with a modified temperature and density. Some years ago it was demonstrated for the first time[1] that even in the frame of quasilinear theory, the electron distribution function, with the phase-space dependence modified in a nontrivial manner, can be found. Hence, from the practical point of view it became very important to find a solution of the Vlasov equation beyond the limits of quasilinear theory.

The accurate analytic solution of the Vlasov equation for electrons in a homogeneous plasma, with an externally driven, nonmodulated, high-frequency monochromatic electric field of arbitrarily large amplitude, is easy to find, as is the time-independent component of this function. The complete solution is in the form of a displaced Maxwellian distribution. However, until recently there had been no success in finding an exact solution of the Vlasov equation for more general conditions, even in the one-dimensional case.

The first successful attempt was reported[2] for the simplest one-dimensional case of the potential electric field of a high-frequency monochromatic standing plasma wave of arbitrarily large amplitude. Even though the importance of taking the ambipolar potential into account was mentioned in that paper, the method used to generate the solution made its inclusion difficult. In a subsequent paper,[3] a more powerful method of solving the Vlasov equation was used which allowed the limitation of a standing plasma wave to be removed and the ambipolar potential to be taken into account.

All the solutions mentioned above were based on the assumption that, where the high-frequency electric field was zero, the electron distribution function remained Maxwellian. Although this assumption may not be far removed from reality (since we are not dealing with resonant velocities), a more general solution which does not require this Maxwellian boundary condition can, in principle, be found. In the frame of a quasilinear approximation, such a solution for the case of a simple traveling plasma wave whose amplitude varies slowly in time, is known.[4] In this Rapid Communication a solution of this type for the general conditions assumed in Ref. 3 is found, using the method developed in that paper.

We shall analyze the one-dimensional Vlasov equation

$$\frac{\partial f}{\partial t}+v\nabla f+\frac{e}{m_e}\nabla\varphi\cdot\nabla_v f-\frac{e}{m_e}E\nabla_v f=0 \ , \qquad (1)$$

allowing for the boundary condition

$$E(0,t)=0,\ \ \varphi(0)=0,\ \ f(0,v,t)=f_0^{(0)}(v) \ . \qquad (2)$$

Here $-e$ and m_e are the electron charge and mass; $\varphi(x)$ is the ambipolar potential; $E(x,t)$ is the potential electric field of a high-frequency monochromatic plasma wave with the angular frequency ω [subsequently, $E(x,t)$ will sometimes be denoted simply by E]; and ∇ and ∇_v denote $\partial/\partial x$ and $\partial/\partial v$, respectively. We shall assume that the phase velocity of the plasma wave is appreciably higher than the electron thermal velocity, as can be expressed in the form

$$\frac{v_T}{\omega}\left|\frac{\nabla E}{E}\right| << 1 \ , \qquad (3)$$

where v_T is the electron thermal velocity defined as

$$v_T^2\int_{-\infty}^{+\infty} f_0^{(0)}\,dv=2\int_{-\infty}^{+\infty} v^2 f_0^{(0)}\,dv \ . \qquad (4)$$

We shall write the electric field in the form

$$E=E(x,t)=E(x)e^{-i\omega t}+E^*(x)e^{i\omega t} \ . \qquad (5)$$

We shall introduce the dimensionless potentials

$$\psi=\frac{e^2\langle E^2(x,t)\rangle}{m_e^2\omega^2 v_T^2},\ \ \phi=\frac{2e\varphi}{m_e v_T^2} \ , \qquad (6)$$

where the brackets $\langle\cdots\rangle$ denote time average over the period of the oscillations. These potentials are the normalized ponderomotive potential and the normalized ambipolar potential, respectively. The square root of the first of these is the ratio of the average electron oscillation velocity to the electron thermal velocity and will be further used as a formal parameter.

The electron distribution function may be written in the form of a sum of two terms—one slowly varying and the other oscillating rapidly as a function of time:

$$f(x,v,t)=f_0(x,v,t)+\tilde{f}(x,v,t) \quad . \tag{7}$$

We shall be primarily interested in the solution of $f_0=\langle f\rangle$.

We shall express the solution (7) in the form of an infinite series of terms of order n with respect to the formal parameter defined above:

$$f_0=\sum_{n=0}^{+\infty} f_0^{(n)}, \quad \tilde{f}=\sum_{n=1}^{+\infty} \tilde{f}^{(n)} \quad . \tag{8}$$

Since each of the terms $\tilde{f}^{(n)}$ can contain higher harmonics with respect to ω, we write

$$\tilde{f}^{(n)}=\sum_{m=1}^{+\infty}(f_m^{(n)}e^{-im\omega t}+f_m^{(n)*}e^{im\omega t}) \quad . \tag{9}$$

Bearing in mind that the function f_0 varies slowly with time, we can confine our interest to velocities for which the following condition is satisfied:

$$\left|\frac{\partial f_0^{(n)}}{\partial t}\right| << |v\nabla f_0^{(n)}| \quad . \tag{10}$$

The procedure leading to the recurrent relations for the nonzero terms $f_m^{(n)}$ does not depend on the form of the boundary condition $f_0^{(0)}$. Therefore, we can omit all the lengthy derivations and start our analysis just from these relations [see Eq. (32) in Ref. 3]:

$$v\nabla f_0^{(2n)}=\frac{e}{m_e}\{-\nabla\varphi\cdot\nabla_v f_0^{(2n-2)}+\nabla_v[E(x)f_1^{(2n-1)*}+E^*(x)f_1^{(2n-1)}]\} \quad ,$$

$$f_n^{(n)}=\frac{ie}{nm_e\omega}\left(1-\frac{iv\nabla}{n\omega}\right)E(x)\nabla_v f_{n-1}^{(n-1)} \quad , \tag{11}$$

$$f_m^{(n)}=\frac{ie}{mm_e\omega}\left(1-\frac{iv\nabla}{m\omega}\right)[-\nabla\varphi\cdot\nabla_v f_m^{(n-2)}+E(x)\nabla_v f_{m-1}^{(n-1)}+E^*(x)\nabla_v f_{m+1}^{(n-1)}], \quad n\geq 1, \quad m=n-2k\geq 1, \quad k\geq 1 \quad .$$

Owing to the fact that

$$f_0^{(2n-1)}=0, \quad n\geq 1 \quad , \tag{12}$$

it follows from (8) that we need to determine a general formula for the term $f_0^{(2n)}$.

The main advantage of the system of recurrent relations (11) is that it can be solved in a straightforward manner, even if with increasing n the number of mathematical operations needed to find the form of $f_0^{(2n)}$ grows very quickly. Bearing in mind that

$$f_0^{(2n)}(0,v,t)=0, \quad n\geq 1 \quad , \tag{13}$$

and taking into account the condition (3), we have found the form of terms $f_0^{(2)}$, $f_0^{(4)}$, $f_0^{(6)}$, and $f_0^{(8)}$. It is then apparent that introducing a dimensionless velocity $w=v/v_T$ and defining a differential operator

$$\mathbf{M}=-\frac{1}{2w}\nabla_w \quad , \tag{14}$$

where $\nabla_w=\partial/\partial w$, these terms fit the general formula

$$f_0^{(2n)}=\sum_{l=0}^{n}\sum_{k=0}^{l}\frac{\psi^k}{(k!)^2}\left(\tfrac{1}{2}\nabla_w^2\right)^k \times\frac{(-\psi\mathbf{M})^{l-k}}{(l-k)!}\frac{(\phi\mathbf{M})^{n-l}}{(n-l)!}f_0^{(0)} \quad . \tag{15}$$

Eventually the function f_0 can be found in the form

$$f_0=\mathbf{L}f_0^{(0)} \quad , \tag{16}$$

where $\mathbf{L}$ is the differential operator in velocity space which can be expressed formally as

$$\mathbf{L}=J_0(i\sqrt{2\psi}\nabla_w)e^{(\phi-\psi)\mathbf{M}} \quad , \tag{17}$$

and J_0 is a zero-order Bessel function. Slow time dependence of the function f_0 may be incorporated using the slow time dependence of the ponderomotive and ambipolar potentials, respectively.

Special requirements for the boundary condition $f_0^{(0)}$ have not been mentioned so far; however, only a function for which either a finite or a convergent infinite series is generated can be treated by this method. In the case of odd powers of w this means that a condition $w^2>\psi-\phi$ must be satisfied. This condition expresses the physical fact that only those electrons from the boundary which are energetic enough not to be back reflected by the potential barrier $\psi-\phi$ can cause an asymmetry in the form of the boundary condition $f_0^{(0)}$. More precisely, the maximum value of this potential barrier must be taken into account.

To illustrate the use of the operator $\mathbf{L}$ let us suppose that the boundary condition $f_0^{(0)}$ is an even function of velocity, i.e., $f_0^{(0)}=f_0^{(0)}(w^2)$. Owing to the fact that

$$e^{(\phi-\psi)\mathbf{M}}w^{2n}=(w^2+\psi-\phi)^n \quad , \tag{18}$$

it is easy to show that the following expression is valid (after the Fourier transformation in velocity space has been performed):

$$f_0=\frac{2}{\pi}\int_0^{+\infty}\cos(pw)J_0(p\sqrt{2\psi})F(p,\psi-\phi)\,dp \quad , \tag{19}$$

where

$$F(p,\zeta)=\int_0^{+\infty}\cos(pw)f_0^{(0)}(w^2+\zeta)\,dw \quad . \tag{20}$$

As one can readily see for the case of the Maxwellian boundary condition

$$f_M=\frac{n_0}{\sqrt{\pi}v_T}e^{-w^2} \quad , \tag{21}$$

where n_0 is the corresponding electron density at the boundary, our result (19) is in full agreement with the result obtained in Ref. 3.

In the case in which we are particularly interested in the moments of the distribution function f_0 it is more convenient to use the following formula:

$$f_0 = J_0(i\sqrt{2\psi}\nabla_w) f_0^{(0)}(w^2+\psi-\phi) \quad . \tag{22}$$

Then, for example, we can write expressions for the electron density and the average quadratic velocity of the electrons as

$$n(x) = \int_{-\infty}^{+\infty} f_0^{(0)}(w^2+\psi-\phi)\, dv \tag{23}$$

and

$$v_T'^2 = \left[\frac{2}{n(x)} \int_{-\infty}^{+\infty} w^2 f_0^{(0)}(w^2+\psi-\phi)\, dv + 2\psi\right] v_T^2 \quad , \tag{24}$$

respectively. Again, for the Maxwellian boundary condition, our results agree with the results obtained in Ref. 3.

Finally, we would like to mention the fact that an inverse differential operator $\mathbf{L}^{-1}$ can, in principle, be found,[5] which permits the reverse procedure

$$f_0^{(0)} = \mathbf{L}^{-1} f_0 \tag{25}$$

to be performed. This inverse operator can be formally expressed as

$$\mathbf{L}^{-1} = e^{(\psi-\phi)\mathbf{M}} J_0^{-1}(i\sqrt{2\psi}\nabla_w) \quad , \tag{26}$$

where $J_0^{-1}(x)$ is an infinite series with coefficients such that the condition $J_0(x) * J_0^{-1}(x) = 1$ is satisfied for each value x for which $J_0(x) \neq 0$.

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