Comment on "Correspondence principle in free-electron lasers"

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We show that the derivation of the small-signal gain given by Friedland [Phys. Rev. A **29**, 1310 (1984)] actually applied for an incoherent stimulating field with amplitude and phase fluctuations obeying a stationary Gaussian process rather than the standard situation where the stimulating field is assumed to be coherent. The gains are identical in either case; however, the photon statistics are different. We also try to rectify some confusing remarks about the quantum limit of the free-electron laser.

In a recent paper[1] [henceforth to be referred to as I; equations designated by (In) refer to Eq. (n) of Ref. 1] Friedland investigates the free-electron laser (FEL) by inferring the photon statistics via the correspondence principle. We feel that this paper as it stands is likely to introduce some confusion about the photon statistics of the FEL as well as the limits of applicability of the classical description. Our comments may be summarized by two points. (1) The assessment of the quantum limit of the FEL is contradictory and confusing. In particular, the condition given explicitly in Eq. (I1) is wrong. (2) The model description of the stimulating laser field introduced in Sec. II of Ref. 1 is different from that usually considered in the derivations of the gain in the FEL: Normally, in classical treatments, the stimulating field is assumed to be a coherent field with fixed amplitude and phase, while in quantum-mechanical treatments, it is described by an eigenstate of photon number or a coherent state. In contrast, the description adopted in I is a classical chaotic field with amplitude and phase fluctuations and an infinite correlation time. This will be demonstrated below. Consequently, the photon statistics derived from this model are different from previous results, while the gain is identical, as it should be. The model of a chaotic field is not without physical interest; in fact, the photon statistics of spontaneous emission of a high current electron beam is thermal.[2] Hence, the conclusion which should have been drawn from Ref. 1 is not that the gain has been rederived by different arguments, but rather that it has been derived for a different physical situation and happens to be identical to the one which is normally considered. In what follows we will consider these two points in more detail.

We will first address the question of the quantum limit. It is well known that one has to discriminate between two different quantum regimes, which may be referred to as the low-intensity and the large-recoil regimes. The former is related to the quantum nature of the electromagnetic field and the latter to that of the electron. The low-intensity quantum regime applies to spontaneous emission of very dilute electron beams where the number of electrons is so small that the spontaneously radiated field remains comparable with the vacuum field fluctuations throughout most of the time that the electrons interact with the wiggler.[3] When a very weak stimulating field is present, the field increase is still dominated by spontaneous emission. However, the principle of detailed balancing, as formulated in Eq. (I27), does not apply whenever spontaneous emission is important. Consequently, the statement made below Eq. (I31) that "this expression describes both the quantum and the classical limits" is meaningless since the low-intensity quantum limit is beyond the scope of Ref. 1. In the large-recoil quantum regime the energy of a photon is large compared with the width of the gain profile. In this case the expansion (I28) is inapplicable.

In Eq. (I1) it is assumed that the criterion for classical behavior is that the average energy *loss* by an electron is very large compared with the energy $\hbar\omega$ of a field quantum. This is incorrect: the multiphoton character of the FEL becomes mainly apparent in the average electron energy *spread*, and it is the latter quantity that must be much larger than $\hbar\omega$ for the classical description to apply. In comparison, the average energy loss is a very small quantity in the small signal regime. If this is taken into account the resulting limiting radiation flux comes out to be lower by many orders of magnitude than the value of 1 W/cm^2 given below Eq. (I1). Actually, later in Ref. 1 it is repeatedly and correctly pointed out that the criterion is given by $\bar{n} >> 1$.

The model adopted in Sec. II of Ref. 1 describes the laser field as a superposition of N linearly polarized monochromatic wave trains with identical frequencies, where the polarization vector and phase of each wave are considered as stochastic quantities. Since the magnetic wiggler field (I4) is circularly polarized, only the component of the laser field (I3) with the same circular polarization can be amplified or attenuated, whereas the component with the opposite circular polarization is virtually unaffected. The important component is

$$\vec{E}_L = \frac{1}{\sqrt{2}} \sum_{i=1}^{N} [(\hat{e}_x - i\hat{e}_y)E_{L,i}^{(-)} + (\hat{e}_x + i\hat{e}_y)E_{L,i}^{(+)}]$$
$$= \frac{1}{\sqrt{2}} [(\hat{e}_x - i\hat{e}_y)E_L^{(-)} + (\hat{e}_x + i\hat{e}_y)E_L^{(+)}] , \tag{1}$$

where

$$E_{L,i}^{(\mp)} = (p_{i,x} \pm ip_{i,y}) \frac{E_0}{2\sqrt{2}} e^{\mp i\Phi_i} , \tag{2}$$

and

$$\Phi_i = kz - \omega t + \phi_i . \tag{3}$$

Here, and in what follows we use without further explanation the notation of I. Whereas the individual wave trains $\vec{E}_{L,i}$ have fixed amplitudes, the amplitude of the entire field $\vec{E}_L$ is a stochastic quantity, because the polarization vectors

$\vec{p}_i$ are stochastic. We have ($\vec{p}_i^{\,2}=1$)

$$\langle E_L^{(\pm)}(z,t)E_L^{(\pm)}(z',t')\rangle=0\ , \tag{4}$$

$$\begin{aligned}\langle E_L^{(+)}(z,t)E_L^{(-)}(z',t')\rangle &= \langle E_L^{(-)}(z,t)E_L^{(+)}(z',t')\rangle^* \\ &= \frac{E_0^2}{8}\left\langle \sum_i (p_{i,x}-ip_{i,y})(p_{i,x}+ip_{i,y})\right\rangle e^{+ik(z-z')-i\omega(t-t')} \\ &= \frac{NE_0^2}{8}e^{+ik(z-z')-i\omega(t-t')}\equiv\Delta(z,t;z't')\ ,\end{aligned} \tag{5}$$

and for $N>>1$

$$\left\langle \prod_{i=1}^{n} E_L^{(+)}(z_i,t_i)\prod_{k=1}^{m}E_L^{(-)}(z_k',t_k')\right\rangle=\delta_{n,m}\sum_P\prod_{i=1}^{n}\Delta(z_i,t_i;z_{Pi}',t_{Pi}')\ , \tag{6}$$

where in Eq. (6) the sum is over all n permutations P of the variables (z_k',t_k'). Consequently, the field $\vec{E}_L$ satisfies a stationary Gaussian process with an infinite correlation time.[4] It is well known[4] that in this case there is a simple relation between any quantity A calculated for a coherent field with intensity I_c and the ensemble average of the same quantity calculated for the chaotic field with average intensity I, namely,

$$\langle A\rangle_I=\int_0^\infty d\left(\frac{I_c}{I}\right)e^{-I_c/I}A(I_c)\ . \tag{7}$$

The integral operator in Eq. (7) has the property that

$$\int_0^\infty d\left(\frac{I_c}{I}\right)e^{-I_c/I}I_c^n=n!I^n\ . \tag{8}$$

This explains why the gain (I36) agrees with the standard result:[5] the small signal gain for a coherent stimulating field is just proportional to its intensity I, so that it is reproduced by the integral operator according to Eq. (8). It is noteworthy that this equivalence no longer holds if saturation corrections[6] are considered: the lowest-order correction is proportional to I^2, which is in the incoherent case replaced by $2I^2$. Hence, as long as only the second-order correction needs to be considered, saturation sets in faster for an incoherent than for a coherent field.

In contrast to the small-signal gain, the photon statistics are different in the two cases. The photon statistics for the chaotic field are given by Eq. (I19), viz.,

$$R(l)=e^{-2\bar{n}}I_{|l|}(2\bar{n})\ , \tag{9}$$

where $I_{|l|}$ is a modified Bessel function. In contrast, if the stimulating field is in a photon number state with N_0 photons, they are[7]

$$R_c(l)=\frac{N_0!}{(N_0+l)!}e^{-z}z^l[L_{N_0}^l(z)]^2\ , \tag{10}$$

where $L_{N_0}^l$ is a Laguerre polynomial. The parameter z is essentially the line shape of spontaneous emission; expressed in terms of the notation of I it reads

$$z=\frac{Q\pi}{N_0(\hbar\omega)^2}\left(\frac{ewL}{u}\right)^2F(\theta)\ . \tag{11}$$

It is important to notice that z is independent of N_0 since Q is the radiation energy density. Equation (10) holds for all N_0; for $N_0>>1$ when spontaneous emission is negligible its limit is

$$R_c(l)=[J_l(2\sqrt{N_0z}\,)]^2\ , \tag{12}$$

where J_l is a Bessel function. This has now to be compared with $R(l)$ from Eq. (9). Owing to the integral[8]

$$\int_0^\infty dx\,e^{-\alpha x}[J_n(2\beta\sqrt{x}\,)]^2=\frac{1}{\alpha}e^{-(2\beta^2/\alpha)}I_{|n|}\left(\frac{2\beta^2}{\alpha}\right)\ ,$$

we have in fact

$$\int_0^\infty d\left(\frac{N_0}{\bar{n}/z}\right)e^{-N_0/(\bar{n}/z)}[J_l(2\sqrt{N_0z}\,)]^2=e^{-2\bar{n}}I_{|l|}(2\bar{n}) \tag{13}$$

with $\bar{n}/z$ the average radiation density in the incoherent case. Hence, $R(l)$ and $R_c(l)$ are connected by the general relation (7) as was to be expected.

A brief comparison of the two photon distribution functions $R_c(l)$ and $R(l)$ for the two cases of the coherent and incoherent stimulating fields, respectively, is instructive. The function $J_l^2(x)$ for fixed x undergoes extremely rapid oscillations as a function of l. If these are averaged over, since they are completely unobservable quantum features, we observe the limit

$$\lim_{\hbar\to 0}R_c(l)=\begin{cases}0, & |l|>2\sqrt{N_0z}\ ,\\ \dfrac{1}{\pi}(4N_0z-l^2)^{-1/2}, & |l|<2\sqrt{N_0z}\ .\end{cases} \tag{14}$$

This limit agrees with the classical distribution of the electron energy after the interaction with a coherent laser field inside the wiggler, as derived, e.g., from the pendulum equation. In the classical limit the energy exchange between the electron and the laser field is limited by the maximum of the ponderomotive potential $2e^2\vec{A}_L\vec{A}_W$, where $\vec{A}_L$ and $\vec{A}_W$ are the vector potential of the laser and the wiggler field, respectively. For a coherent field with no amplitude fluctuations this maximum has a well defined value. This is the origin of the sharp cutoff exhibited in Eq. (14). In contrast, for the incoherent field with amplitude fluctuations no corresponding maximum exists, and this results in the smooth distribution function (9).

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[1]L. Friedland, Phys. Rev. A **29**, 1310 (1984).

[2]W. Becker and J. K. McIver, Phys. Rev. A **27**, 1030 (1983); **28**, 1838 (1983); see also W. Becker, J. K. McIver, M. O. Scully, and M. S. Zubairy, in *Free-Electron Generators of Coherent Radiation*, edited by C. A. Brau, S. F. Jacobs, and M. O. Scully [Proc. Soc. Photo Opt. Instrum. Eng. **453**, 306 (1984)].

[3]It should be emphasized that even in this case the power spectrum of the radiation remains classical. A quantum-mechanical description of the field is only mandatory for the photon statistics and very small quantum effects like antibunching and squeezing. It

should also be noticed that a correct treatment of the FEL photon statistics of spontaneous emission requires a many electron model; see Ref. 2.

[4]R. J. Glauber, in *Quantum Optics and Electronics,* edited by C. de Witt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, New York, 1965), p. 63.

[5]On the way from Eq. (I25) to (I36) a factor of −2 got lost, so that the gain (I36) should be multiplied by −2.

[6]A. Bambini, M. Lindberg, and S. Stenholm, Phys. Scr. **22**, 397 (1980).

[7]W. Becker, Opt. Commun. **33**, 69 (1980); **36**, 64 (1981); the derivation of the gain given in these papers is almost congruent with that of Ref. 1, except that it is done in the context of a correct model.

[8]I. S. Gradstein and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965).