# Wrinkling of mode-locked tori in the transition to chaos

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The transition from a mode-locked torus to a chaotic fractal attractor via the generic saddle-node bifurcation is considered. Although superficially the transition seems to be one of the known types of intermittency, it is shown to be in fact discontinuous. All the invariants that characterize chaos show a distinct jump. It is also argued that the power spectra are not sensitive to the discontinuous nature of the transition.

Recently some important progress has been achieved in understanding the transition to chaos from quasiperiodicity with strictly irrational winding numbers.[1–3] It appears, however, that experimental systems "taken off the shelf" will tend to mode lock before going chaotic. In such a situation a Poincaré surface of section reveals a finite number of fixed points. The transition to chaos seems to be related to the reappearance, upon a change of a parameter, of a connected "curve" which is however a fractal, or wrinkled on all length scales. Some of the mechanisms involved in this loss of smoothness have been discussed in detail by McGehee, Arson, Chory, and Hall.[4] Essentially the generic bifurcation that leads to mode locking and then to wrinkling of the mode-locked torus seems to be the saddle-node bifurcation. The wrinkling appears to be due to homoclinic tangencies of the invariant manifold with the strong stable foliation of the saddle node.[2,4,5] A number of clear examples of such a transition have been presented first by Gollub and Benson,[6] and recently by Sano and Sawada.[7] On the basis of a series of measurements of power spectra, Gollub and Benson proposed that the transition is continuous.

The main purpose of this Brief Report is to show that this type of transition can be in fact *discontinuous*, but the power spectra appear not to be sensitive to this. Rather, the invariants that characterize chaos like dimension, metric entropy, etc.,[8,9] all show very clearly a discontinuous jump at the transition. We also claim that the wrinkling phenomenon provides the mechanism for the discontinuity in the invariants.

Although our analysis relies on a simple two-dimensional map that is due to Curry and Yorke,[10] the conclusions pertain of course to the generic saddle-node bifurcation sequence underlying such a transition. We consider a map on $R^2$ which is a composition of the two simpler homeomorphisms $\psi_1$ and $\psi_2$, where $\psi_1$ is defined in polar coordinates

$$\psi_1(\rho,\theta)=(\epsilon\ln(1+\rho),\ \theta+\theta_0),\quad \epsilon\geq 1,\quad \theta_0\geq 0, \tag{1a}$$

and $\psi_2$ in Cartesian coordinates

$$\psi_2(x,y)=(x,\ y+x^2). \tag{1b}$$

$\epsilon$ and $\theta_0$ are parameters. In the following we fix $\theta_0=2$ and consider $\psi_2\psi_1$ as a one-parameter family of maps.

Before we describe the evolution of the attracting set we stress that our numerical experiments suggest that the above map is "dynamically ergodic."[10] This means that for each value of $\epsilon$ in the interval of interest, almost every point in $R^2$ has the same positive limit set. This uniqueness of the attracting set restricts the behavior somewhat in that additional attractors cannot just pop up as the parameter $\epsilon$ is varied.

The transition to chaos runs as follows:[10] At $\epsilon=1$ the origin is a global attractor. As $\epsilon$ increases, the attractor appears to be a closed smooth curve encircling the origin [see Fig. 1(a)]. This persists until $\epsilon=\epsilon_{ML}=1.272\,758\ldots$, where we observe a transition to a real orbit having period 3 [Fig. 1(b)]. As such the system models a transition from an irrational torus (two incommensurate frequencies) to a mode-locked torus (the frequencies are rationally related; $f_1/f_2=1/3$). This mode-locked phase persists until $\epsilon=\epsilon_c=1.395\,358\,1\ldots$, where a connected curve reappears, but it has an infinite set of wrinkles on all length scales (see Fig. 2).

Very close to the onset of chaos $\epsilon\gtrsim\epsilon_c$ the motion seems intermittent in the sense that the orbit sits for fairly long times in the vicinity of the nearly periodic orbit, with rapid bursts to fill the attractor between the previous fixed points. In between bursts the orbit returns to the same region of the nearly periodic orbit and is trapped there for relatively long periods. We remark here that in our numerous numerical experiments we kept verifying the uniqueness of the attracting set. This includes the check of trajectories starting near the tangent bifurcation, at the time of tangent bifurcation. These trajectories follow the above motion and they do not reach any other attractor. The corresponding return maps appear to have the usual "bottlenecks" typical to tangent bifurcations[11,12] (see Fig. 3). The fact is that this

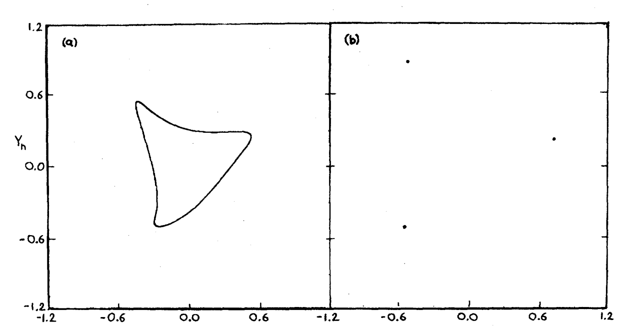


FIG. 1. (a) The smooth attractor of the map Eqs. (1) for $\epsilon=1.20$. The iterates cover the attractor in a clockwise motion. (b) The mode-locked state at $\epsilon=1.395$.

 

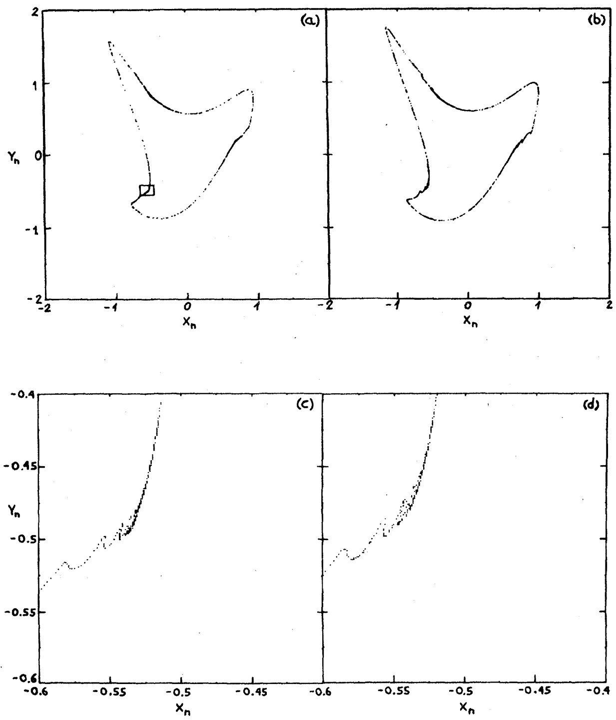


FIG. 2. (a) The attractor at $\epsilon = 1.39536$. The curve here is wrinkled on the small lengthscales. (b) The attractor at $\epsilon = 1.42$. The wrinkles are more apparent now. (c) An enlargement of the corresponding square of Fig. 2(a) to show the wrinkles. (d) An enlargement at $\epsilon = 1.40$.

motion has nothing to do with tangent bifurcations. The reason is that the maximal Lyapunov exponent $\lambda_1$, the information dimension $D_1$, and the average laminarity length $\langle l \rangle$, all show a discontinuous jump at $\epsilon = \epsilon_c$, and do not rise smoothly like $(\epsilon - \epsilon_c)^\beta$ beyond onset as expected from true intermittent transition. By a discontinuous jump we mean that these invariants remain essentially constant over a range of 4–5 orders of magnitude in $(\epsilon - \epsilon_c)$. $\langle l \rangle$ was obtained by counting how many iterates fall within a narrow gate around the previous fixed points,[12] whereas $\lambda_1$ and $D_1$ were calculated with the help of algorithms that were

TABLE I. Values of the maximal Lyapunov exponent $\lambda_1$, the information dimension $D_1$, and the average laminar region $\langle l \rangle$ for map equation (1).

| $\log_{10}(\epsilon - \epsilon_c)$[a] | $\lambda_1$[b] | $D_1$[c] | $\langle l \rangle$[d] |
|---|---|---|---|
| −3.0 | 0.118 | 1.14 | |
| −4.0 | 0.117 | 1.14 | 20.6 |
| −5.0 | 0.116 | 1.14 | 20.9 |
| −6.0 | 0.116 | 1.14 | 20.7 |
| −7.0 | 0.115 | 1.14 | 20.8 |
| −8.0 | 0.114 | 1.14 | 20.8 |

[a] $\epsilon_c = 1.395358\ldots$

[b] The values of $\lambda_1$ are obtained from $10^6$ threefold iterations of map (1).

[c] For the computation of $D_1$ we took 800 000 threefold iterations of map (1) and used the algorithm of Ref. 8 with averaging over 500 points.

[d] For the computation of $\langle l \rangle$ we used a run of $2\times 10^6$ threefold iterations of map (1), and a gate of 0.045. The number of laminar regions observed within this run was around $10^4$.

described in detail elsewhere.[8,9] The results are summarized in Table I.

Nevertheless, we still observed a continuous change in the power spectra in the vicinity of $\epsilon_c$, in the sense that the noise grew smoothly with $\epsilon$ above $\epsilon_c$. The reason is that all important changes are occurring on small length scales, to which the power spectra appear to be insensitive.

The apparent discontinuity of the transition is due to the fact that the orbits are not really trapped for long times in the bottlenecks which appear in the return maps. Since the attractor is strongly wrinkled in these regions, iterates can skip the bottleneck (see Fig. 4). Consequently the invariants are determined by the motion far away from the bottleneck no less than by the motion near them, contrary to the case of tangent bifurcations in intermittency. To test this assertion we considered a piece-wise linear map which attempts to model the return map in the neighborhood of the saddlepoint. It consists of the line $x' = x - \alpha_1 x + \epsilon$ for $x < 0$, and the line $x' = x + \alpha_2 x + \epsilon$ for $x \geqslant 0$. Defining a gate $(-x_0, x_0)$ we ran this map and found that $\langle l \rangle$ depended on $\epsilon$. We mimicked the real situation of Fig. 4 by adding a random jitter $r$ from the interval $(0,R)$ to this map every time the iterate was reinjected into the region $x < 0$. After

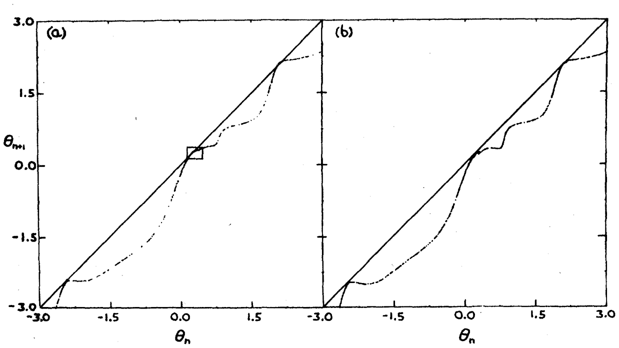


FIG. 3. (a) The return map $\theta_{n+1}$ vs $\theta_n$ at $\epsilon = 1.39536$. The motion is from the upper-right corner to the lower-left corner. The appearance of this map can give the superficial impression of a tangent bifurcation. (b) Return map for $\epsilon = 1.42$.

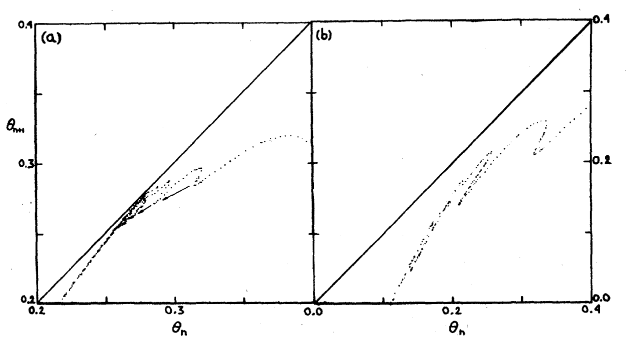


FIG. 4. (a) An enlargement of a "bottleneck" in the square of Fig. 3(a). The arrows show how an iterate skips the bottleneck. (b) A similar enlargement at $\epsilon = 1.42$.

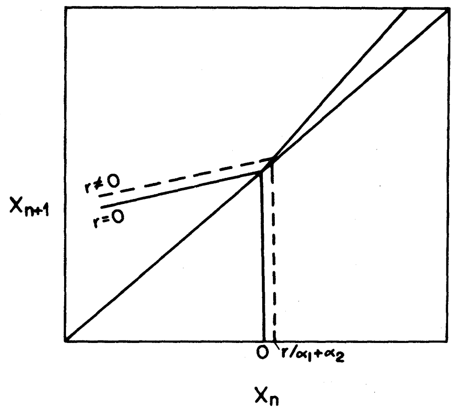


FIG. 5. A piece-wise linear model of Fig. 4. The quantity $r$ is a random jitter of the reinjected orbit. $(1-\alpha_1)$ and $(1+\alpha_2)$ are the "left" and "right" slopes at $x=0$.

the jitter the iteration continued under the dynamics (see Fig. 5):

$$x'=\begin{cases} x-\alpha_1 x+\epsilon, & x<r/(\alpha_1+\alpha_2), \quad (2a)\\ x+\alpha_2 x+\epsilon, & x\geq r/(\alpha_1+\alpha_2). \quad (2b)\end{cases}$$

In this way we allow a random spread of the incoming orbits that can avoid the bottleneck as in Fig. 4. The results of this procedure are shown in Table II, and prove that a small random jitter is sufficient to cause an average intermission length that is essentailly independent of $\epsilon$. Needless to say, in this example one can estimate $\langle l\rangle$ analytically and find $\langle l\rangle \rightarrow$ const as $\epsilon\rightarrow 0$.

TABLE II. Values of the average laminar region $\langle l\rangle$ for the piece-wise linear model, with various choices of $\alpha_1$, $\alpha_2$, and $R$.

| $\epsilon-\epsilon_c$ [a] | $\langle l\rangle$ [b] | $\langle l\rangle$ [c] |
|---|---|---|
| $10^{-5}$ | 25.8 | 18.0 |
| $10^{-6}$ | 26.3 | 18.4 |
| $10^{-7}$ | 26.4 | 18.5 |
| $10^{-8}$ | 26.5 | 18.5 |
| $10^{-9}$ | 26.5 | 18.5 |

[a] $\epsilon_c=0$.
[b] $\alpha_1=0.5$, $\alpha_2=0.2$, $R=0.001$.
[c] $\alpha_1=0.7$, $\alpha_2=0.1$, $R=0.01$. For the computation of $\langle l\rangle$ we used a run of $10^6$ iterations and a gate of 0.02. The number of laminar regions observed within this run was around $2\times 10^4$.

Once the uniqueness of the attractor is verified for a one-parameter system in $R^2$, the lack of scaling and the presence of the discontinuity in the invariants, close to the onset of chaos, can serve the experimentalist in testing whether the mechanism discussed in Ref. 4 underlies the transition to chaos from a mode-locked torus. If the invariants remain essentially constant after the transition, it should be seen as a good indication for the scenario discussed in Ref. 4.


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