Quantumlike theory for the nonadiabatic behavior of charged particles in inhomogeneous magnetic fields

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(Received 20 September 1984)

A new derivation is given of the set of Schrödinger-like equations for the nonadiabatic behavior of an ensemble of charged particles in inhomogeneous magnetic fields, starting from the Liouville equation for the ensemble. The nonadiabatic loss of charged particles from magnetic-mirror traps thus appears in the nature of quantum tunneling of the adiabatic potential with the initial value of the first action invariant playing the role of $\hbar$. The equations also predict one-dimensional interference-like effects in periodic magnetic fields. Different equations of the set describe different modes of the nonadiabatic behavior of a "pure" ensemble prepared with δ-function distributions for the controllable (in the sense used by Khinchin) integrals of motion of the system. A new concept of "ensemble modes," as distinct from the collective modes, is thus introduced. These ensemble modes have indeed been observed as manifested through the recently observed multiplicity of lifetimes in the nonadiabatic decay.

I. INTRODUCTION

The motion of charged particles in inhomogeneous magnetic fields is of considerable theoretical and practical importance from the point of view of plasma confinement in various magnetic-field configurations. When the fields vary slowly (that is, the fractional change of the magnetic field over a gyroradius or a gyroperiod is small) an adiabatic action invariant $\mu=\frac{1}{2}mv_\perp^2/\Omega$ $(\Omega=eB/mc)$ for gyromotion (motion perpendicular to the magnetic field) exists. As a consequence, the motion along the field line reduces to a motion in an effective potential $V=\mu\Omega$. The adiabatic equation of motion is given by (see, for instance, Northrop[1])

$$m\frac{dv_{\parallel}}{dt}=-\nabla_{\parallel}(\mu\Omega) \ . \qquad (1)$$

Since the adiabatic potential $\mu\Omega$ is proportional to the strength of the magnetic field, this equation of motion offers the possibility of trapping charged particles in an adiabatic potential well, that is, in a region of weaker magnetic field bounded on either side by regions of stronger magnetic field, provided the energy E of the particle is less than the maximum height of the potential. Such a scheme for trapping charged particles is used in the type of devices for confining the plasma known as "mirror machine."

If the invariance of the action μ (or the so-called "magnetic moment" $e\mu/mc$, as is generally used in plasma physics) were exact, the trapping of the charged particles would be perpetual. Since the invariance is only approximate (that is, adiabatic), departures from it, which will necessarily occur, can, in principle, lead to the escape of particles from the adiabatic potential well. Such an escape has indeed been observed experimentally by Ponomarenko *et al.*[2] and Dubinina *et al.*[3] where the particles injected into the trap with a given energy E and action μ are found to decay from the trap with a certain characteristic decay time. These decay lifetimes are found to increase exponentially with the strength of the magnetic field.

A problem of great practical interest for the plasma fusion devices of the mirror-machine type is the theoretical determination of these lifetimes so that machine designs could be optimized in order to achieve a maximum of lifetimes at an optimum cost. From a theoretical standpoint the determination of lifetimes presents a very interesting problem which has continued to engage the attention of physicists and mathematicians for more than 20 years. (See Chirikov[4] for a bibliography.)

A few years back the present author had given a formulation of the problem[5] along lines somewhat different from the conventional one (see Chirikov[6(a),6(b)] and Bernstein and Rowlands[7]). In this formulation the nonadiabatic escape of particles from the adiabatic potential well turns out to be in the nature of tunneling in quantum mechanics. A set of Schrödinger-like equations were obtained [Eqs. (39) and (48) of Sec. IV] where the initial value (at injection) of the action μ appears in the role of $\hbar$ and the adiabatic potential $\mu\Omega$ appears in the place of the potential in the Schrödinger equation of quantum mechanics. These equations thus bear the same relationship with the adiabatic motion as the Schrödinger equation does with the motion in classical mechanics. The different equations of the set, however, have μ/n, $n=1,2,3,\dots$, in the role of $\hbar$.

As discussed in detail in Sec. V, these equations had made predictions regarding the existence of multiple lifetimes in the decay of particles trapped with a given energy E and initial value μ. Recent experiments[8(a)–8(c)] have, in fact, established the existence of at least two lifetimes which are found to vary with the various parameters in accordance with the predictions of the theory.

The purpose of the present paper is to discuss the essential ensemble nature of the theory in more detail than what could be done earlier in the space available for a letter. We also give here an alternate "deductive" derivation of the equations starting from the Liouville equation for the ensemble representing the experiment. Apart from the nonadiabatic tunneling that these equations describe we also discuss the one-dimensional interference effects that they predict.

II. THE NONADIABATIC MOTION AND THE NATURE OF THE FORMULATION

The motion of a charged particle in an inhomogeneous magnetic field is a strongly nonlinear problem where the initial conditions, in general, play a crucial role. Particles differing slightly in their initial conditions can differ very widely with time in their final positions. In the adiabatic approximation, however, one obtains a reduced motion—the adiabatic motion—which is independent of some of the initial conditions, for instance, the Larmor phase of the motion.

The adiabatic motion, to be sure, is an idealization and departures from it, albeit small, occur even when the conditions for its validity are well satisfied. The nature of these departures, referred to as the nonadiabatic effects, is not yet completely understood. Generally, a nonadiabatic change in the action μ would occur whenever the magnetic field or any of its derivatives goes through a zero and the next derivative has an appropriate sign. The largest change, a zeroth-order nonadiabatic change, of course, occurs when the field itself goes through a zero as in a cusp field. The next largest, a first-order change, occurs when the first derivative goes through a zero and the second derivative has a positive sign, as in the midplane of a mirror machine, and so on.

Detailed calculations, both numerical and analytical, have been carried out (Garren *et al.*[9], Hastie *et al.*[10], Howard[11], Cohen *et al.*[12]) for the determination of a single nonadiabatic change in the value of μ as, for instance, for a particle crossing the midplane of a mirror machine, in an attempt to understand the nonadiabatic escape of particles and to calculate the lifetime for the escape. In the standard approach to the determination of lifetimes,[6(a),6(b),7] the nonadiabatic escape is regarded as a random walk of the particle in the μ space and into the loss cone with the individual nonadiabatic change $\Delta\mu$ constituting the step for the random walk. We do not wish to give a critique of this approach here. Suffice it to say that the existence of multiple lifetimes with the characteristics observed experimentally has not yet been understood in terms of this approach.

It may be pointed out that the lifetimes, as determined experimentally, refer to the decay of an ensemble of (noninteracting) particles injected into the trap with the same energy E and action μ but with different Larmor phases ϕ. The decay of particles from the trap with the observed characteristics is thus to be regarded as an ensemble property determined by the dispersion in the initial conditions of its members and the lifetime, as an essentially ensemble concept. Each particle, a member of the ensemble, as labeled by its initial conditions, is different, follows its own distinct trajectory, and gets out of the trap, when it does, as its own distinct time.

Since the adiabatic motion as described by the equation of motion (1) is an idealization, no real motions of particles in the magnetic field are described by it. However, the projections of the above-mentioned ensemble of exact motions onto the "parallel" coordinate (parallel to the field line) would lie in the neighborhood of the corresponding adiabatic motion in the function space. The nonadiabatic behavior of the ensemble of exact trajectories is thus sought in this formulation, as a statistical property of the neighborhood of the adiabatic motion, without having to worry about the details of any particular trajectory. These considerations are somewhat in the spirit of "general dynamics" in the sense used by Birkhoff, as discussed by Khinchin.[13]

III. MOTION IN AN AXISYMMETRIC MAGNETIC FIELD AND THE ADIABATIC LIMIT

The Lagrangian for the motion of a charged particle in a magnetic field is

$$L=\tfrac{1}{2}m(\dot{x}_{\parallel}^{2}+\dot{x}_{\perp}^{2}+r^2\dot{\theta}^2)+\frac{e}{c}r\dot{\theta}A_\theta \ , \tag{2}$$

where $x_{\parallel}$ is the coordinate along the field line with the unit vector $\hat{e}$, and $x_\perp$ is the coordinate perpendicular both to $\hat{e}$ and $\hat{e}_\theta$, the unit vector in the θ direction at a point.

When the magnetic field is axisymmetric, the vector potential A_θ is independent of θ, and the canonical angular motion P_θ,

$$P_\theta=\frac{\partial L}{\partial\dot{\theta}}=mr^2\dot{\theta}+\frac{e}{c}rA_\theta=M \ , \tag{3}$$

is a constant of motion. Using this fact, one can obtain a reduced Lagrangian $\bar{L}$ which describes the effective motion in the $(x_{\parallel},x_\perp)$ plane after the θ motion has been eliminated. The reduced Lagrangian $\bar{L}$ is essentially given by the Routhian R

$$\bar{L}=R=L-P_\theta\dot{\theta} \ , \tag{4}$$

where $\dot{\theta}$ in R is to be substituted from (3). This yields

$$\bar{L}=\tfrac{1}{2}m(\dot{x}_{\parallel}^{2}+\dot{x}_{\perp}^{2})-\frac{1}{2mr^2}\left[M-\frac{e}{c}rA_\theta\right]^2 \ , \tag{5}$$

where r in expression (5) is to be expression in terms of $x_{\parallel}$ and $x_\perp$. Note that the last term in (5) then appears as an effective potential for the motion in the $(x_{\parallel},x_\perp)$ plane.

Two cases arise. The first case corresponds to $M<0$ (with $e<0$); that is, the expression (3) for the canonical angular momentum is dominated by the term $(e/c)rA_\theta$. In that case we can expand $(e/c)rA_\theta$ around the value M:

$$\frac{e}{c}rA_\theta=M+(x_\perp-x_{\perp 0})\frac{\partial}{\partial x_\perp}\left[\frac{e}{c}rA_\theta\right]\Bigg|_{x_\perp=x_{\perp 0}}+\cdots \ , \tag{6}$$

where $x_{\perp 0}$ is defined by the relation

$$M=\frac{e}{c}rA_\theta\Big|_{x_\perp=x_{\perp 0}} . \qquad (6')$$

Then we have

$$\bar{L}=\tfrac{1}{2}m(\dot{x}_{\parallel}^2+\dot{x}_{\perp}^2)-\tfrac{1}{2}m\Omega^2(x_\perp-x_{\perp 0})^2+\cdots , \qquad (7)$$

where

$$\Omega(x_{\parallel},x_{\perp 0})=\frac{e}{mcr}\frac{\partial}{\partial x_\perp}(rA_\theta)\Big|_{x_\perp=x_{\perp 0}} . \qquad (8)$$

To the lowest order in $x_\perp-x_{\perp 0}$, the terms $x_\perp$, $\dot{x}_\perp$ in the Lagrangian (7) describe a harmonic oscillator at any given $x_{\parallel}$ with a circular frequency Ω and centered around $x_{\perp 0}$. The Hamiltonian corresponding to $\bar{L}$ is given by

$$\hat{H}=\frac{p_{\parallel}^2}{2m}+\frac{1}{2m}p_\perp^2+\tfrac{1}{2}m\Omega^2(x_\perp-x_{\perp 0})^2 . \qquad (9)$$

Transforming $(p_\perp,x_\perp)$ to action-angle variables (μ,ϕ) at any given $x_{\parallel}$, defined by

$$p_\perp=\sqrt{2m\mu\Omega}\cos\phi , \quad x_\perp-x_{\perp 0}=\left[\frac{2\mu}{m\Omega}\right]^{1/2}\sin\phi , \qquad (10)$$

we get

$$\hat{H}=\frac{p_{\parallel}^2}{2m}+\mu\Omega(x_{\parallel}) , \qquad (11)$$

when μ is the action corresponding to the lowest-order oscillatory motion of the $x_\perp$ coordinate. μ is an "adiabatic invariant" if Ω is a slowly varying function of $x_{\parallel}$. The reduced Hamiltonian $\hat{H}$ of (11) then describes the "adiabatic motion" of the charged particle given by Eq. (1).

The second case corresponds to $M>0$ (with $e<0$). In this case the value of M in (3) is dominated by the term $mr^2\dot{\theta}$ which must always be greater than $(e/c)rA_\theta$ and must always carry the same (positive) sign. This means that such particles, always having the same sign for $\dot{\theta}$, must encircle the axis of symmetry. As is well known, such particles find themselves exactly trapped if M is sufficiently large positive. Such particles are not of interest from the point of view of nonadiabatic escape. We shall therefore not consider this case here.

We note that $\bar{L}$ of Eq. (5) is the exact reduced Lagrangian for the particle motion in the axisymmetric field. The corresponding Hamiltonian is given by

$$\bar{H}=\frac{p_{\parallel}^2}{2m}+\frac{p_\perp^2}{2m}+\frac{1}{2mr^2}\left[M-\frac{e}{c}rA_\theta\right]^2 . \qquad (12)$$

The action-angle transformation (10) can be used in (12) to yield again the reduced Hamiltonian which, still exact, would then be a function of the canonical variables $(p_{\parallel},x_{\parallel})$ and (μ,ϕ).

IV. THE LIOUVILLE EQUATION FOR THE EVOLUTION OF THE ENSEMBLE

Consider now the ensemble of particles which correspond to the experimental conditions where a large number of them are injected into a magnetic-mirror trap at a given space coordinate $\mathbf{x}_0$ and with a specified energy E_0 and action μ but with a distribution in the Larmor phase angle ϕ_0. Though the formulation developed here has a more general applicability, we consider here, for simplicity, the case of an axisymmetric mirror trap. The average density of the injected particles in the trap is assumed to be so low that interparticle collisions can be neglected. This then constitutes an ensemble of independent particles.

The motion of particles in a magnetic field is governed by the Lagrangian (2) or the corresponding Hamiltonian. Since we consider an axisymmetric magnetic-mirror field, the motion may be described equivalently by the effective Hamiltonian $\bar{H}$ of (12) in which the dynamical variables (p_θ,θ) stand eliminated and which then is a function of the remaining canonical variables $(x_{\parallel},p_{\parallel},x_\perp,p_\perp)$.

If f represents the Liouville density for the ensemble in the phase space of the canonical variables $(x_{\parallel},p_{\parallel},x_\perp,p_\perp)$, then the evolution of f is governed by the Liouville equation

$$\frac{\partial f}{\partial t}+\frac{p_{\parallel}}{m}\frac{\partial f}{\partial x_{\parallel}}+\frac{p_\perp}{m}\frac{\partial f}{\partial x_\perp}+\dot{p}_{\parallel}\frac{\partial f}{\partial p_{\parallel}}+\dot{p}_\perp\frac{\partial f}{\partial p_\perp}=0 . \qquad (13)$$

This expresses the conservation of probability along trajectories described by the Hamiltonian (12). The initial form f_0 of the distribution function f is determined by the "state preparation." A state may thus correspond to δ-function distributions for certain conserved quantities such as the energy or canonical angular momentum and a distribution in the values of the others. In our case we take the initial distribution f_0, in accordance with the conditions of injection, to be δ functions in the energy E, the canonical angular momentum P_θ, and also a δ function in the initial value of the action μ while an *uncontrollable* distribution $g(\phi_0)$ in the initial values of the phase angle. (We use the term "uncontrollable" in the sense discussed by Khinchin.[13] This means here that because of the rapid gyrowinding of the trajectory it is difficult to prepare an initial state with a *preassigned* distribution in the initial Larmor phase. On the other hand, the same circumstance, namely, the rapid gyromotion, leads to the distribution being close to a uniform one, though not prepared to some preassigned specification, in its details.)

Now if $S=S(x_{\parallel},x_\perp,t)$ represents Hamilton's principal function, then the Hamilton-Jacobi equation for the system is

$$\frac{\partial S}{\partial t}+\bar{H}\left[\frac{\partial S}{\partial x_{\parallel}},\frac{\partial S}{\partial x_\perp},X_{\parallel},x_\perp,t\right]=0 \qquad (14)$$

with the Hamiltonian $\bar{H}$,

$$\overline{H}=\frac{1}{2m}\left[\frac{\partial S}{\partial x_{\parallel}}\right]^2+\frac{1}{2m}\left[\frac{\partial S}{\partial x_{\perp}}\right]^2+\frac{1}{2mr^2}\left[M-\frac{e}{c}rA_\theta\right]^2 . \quad (15)$$

Let S be a complete solution of the Hamilton-Jacobi equation (14) and thus a function of the constants of motion α_i, which are the momenta. In particular, these could also be the *initial* values of the appropriately chosen momenta:

$$S=S(x_{\parallel},x_{\perp},t;\alpha_i) . \quad (16a)$$

If β_i are the initial values of the corresponding canonical coordinates, then we have the relations

$$\beta_i=\frac{\partial S}{\partial \alpha_i} \quad (16b)$$

which, on inversion, provide, in principle, the solution to the problem.

Any arbitrary function $f(\alpha_i,\beta_i)$ of the constants of motion α_i and β_i is a solution of the Liouville equation (13). In our problem we choose for α_i the following conserved quantities: the total energy E, the canonical angular momentum P_θ, and the initial value of the action μ for the injected particles with a δ-function distribution in each of them.

We now carry out an important transformation from β_i to $(x_{\parallel},\phi)$, where ϕ is the gyrophase at the time t defined by

$$\phi=\phi_0-\int_0^t \Omega\, dt' . \quad (17)$$

This leads from (α_i,β_i) to a "mixed" representation of variables $(x_{\parallel},\phi,t;\alpha_i)$, as a function of which the distribution function $\hat{f}(x_{\parallel},\phi,t;\alpha_i)$ has the meaning of a probability very close to that in quantum mechanics. As we noted above, one or more of the α_i would correspond to the conserved quantities (momenta) in which the ensemble may be prepared with δ-function distributions. In the problem under consideration, these are E, P_θ, and μ. The function $\hat{f}(x_{\parallel},\phi,t;\alpha_i)$ then has the meaning of a probability of finding a particle at $(x_{\parallel},\phi)$ at time t, if it initially has the momenta α_i $(E=E_0,P_\theta=M,P_\phi\equiv\mu=\mu_0)$.

The transformation to the mixed representation is obviously not a canonical transformation. Hence

$$\hat{f}(x_{\parallel},\phi,t;\alpha_i)=\left|\left|\frac{\partial^2 S}{\partial x_\mu\,\partial\alpha_\nu}\right|\right| f(\alpha_i,\beta_i)\prod_i \delta\left[\beta_i-\frac{\partial S}{\partial \alpha_i}\right] , \quad (18)$$

where $||\partial^2 S/\partial x_\mu\,\partial\alpha_\nu||$ is the Jacobian of the transformation. Thus the probability density at $(x_{\parallel},\phi)$ is given by

$$\hat{G}(x_{\parallel},\phi,t)=\int \prod_i d\alpha_i \left|\left|\frac{\partial^2 S}{\partial x_\mu\,\partial\alpha_\nu}\right|\right| f(\alpha_i,\beta_i=\partial S/\partial\alpha_i) . \quad (19)$$

One of the α_i, namely, the initial value of μ, has a δ-function distribution by virtue of the state preparation. Hence $\hat{f}$ has the form

$$\hat{f}=f(x_{\parallel},\phi,t;\alpha_i',\bar{\mu})\delta(\mu^{(0)}-\bar{\mu}) . \quad (20)$$

The Liouville equation governing the evolution of $\hat{f}$ is now

$$\frac{\partial\hat{f}}{\partial t}+v_{\parallel}\frac{\partial\hat{f}}{\partial x_{\parallel}}+\dot{\phi}\frac{\partial\hat{f}}{\partial\phi}=0 , \quad (21)$$

where $v_{\parallel}$ and $\dot{\phi}$ are to be regarded as functions of $(x_{\parallel},\phi;\alpha_i)$.

A. Equations for the probability amplitudes for the ensemble

The Liouville equation (13) or its transformed form (21) contains, in principle, all the information about the time evolution of the ensemble. We are, however, interested in determining that part of the information which concerns the departures of the trajectories from the idealized adiabatic one. To this end we introduce in place of the Larmor phase ϕ the variable Φ:

$$\Phi=\phi+\frac{1}{\mu}\int_0^t dt(\tfrac{1}{2}mv_{\parallel}^2) , \quad (22)$$

where the time integration is carried out along exact trajectories. Note that the time derivative of (22) gives

$$\dot{\Phi}=\dot{\phi}+\tfrac{1}{2}mv_{\parallel}^2/\mu=\frac{1}{\mu}(\tfrac{1}{2}mv_{\parallel}^2-\mu\Omega)=L/\mu , \quad (23)$$

where $L=\frac{1}{2}mv_{\parallel}^2-\mu\Omega$ is the adiabatic Lagrangian which generates the adiabatic equation of motion as the Euler-Lagrange equation. $\mu\Phi$ is thus an action which is the time integral of the adiabatic Lagrangian along exact trajectories. As we shall see later, the variable Φ is a functional variable and has the important property that it defines the neighborhood of the adiabatic motion through its stationarity for the latter. This neighborhood is, by definition, the region of nonadiabaticity, and Φ is thus the proper variable in terms of which the ensemble modes of motion in this neighborhood can be analyzed. It may be noted that μ, as a common initial value for all the members of the ensemble, is an exact constant of motion for the ensemble.

Carrying out the transformation of variables from ϕ to Φ, the Liouville equation (21) takes the form

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+\frac{L}{\mu}\frac{\partial f}{\partial\Phi}=0 , \quad (24)$$

where we have dropped all subscripts.

Since f defines a probability, we would like it to be, by construction, such that it remains positive definite at all space-time points. This can be taken care of simply if we write f as

$$f=\psi^2\ , \tag{25}$$

where ψ is real. A finite time integral form of Eq. (24) is given by

$$f(x,\Phi,t)=f\left[x-\int_{t'}^{t}dt''\,v,\ \Phi-\int_{t'}^{t}dt''\,L/\mu,\ t'\right] . \tag{26}$$

Taking the square root of (26) using (25) one gets (taking the positive sign)

$$\psi(x,\Phi,t)=\psi\left[x-\int_{t'}^{t}dt''\,v,\ \Phi-\int_{t'}^{t}dt''L/\mu,\ t\right] . \tag{27}$$

The distribution function f and hence ψ to be single valued must be periodic in the Larmor phase ϕ as also in Φ since it is related to ϕ additively. Hence we introduce a Fourier series expression of ψ with respect to the phase Φ:

$$\psi(x,\Phi,t)=\sum_n \hat{\Psi}(x,n,t)e^{in\Phi} . \tag{28}$$

Then the Fourier transformation of Eq. (27) gives

$$\hat{\Psi}(x,n,t)=\exp\left[-in\int_{t'}^{t}dt''L/\mu\right]\hat{\Psi}\left[x-\int_{t'}^{t}v\,dt'',\,n,\,t'\right] , \tag{29}$$

where in this and all the foregoing equations we have suppressed the momenta α_i as the arguments of the functions ψ, f, or Ψ, etc., and shall continue to do so except when explicit reference to them is required.

Note that the integral $\int_{t'}^{t}L\,dt''$ is evaluated along the projections of the exact three-dimenisonal trajectories on the one-dimenisonal coordinate parallel to the magnetic-field line. We are thus still dealing with exact trajectories and no assumptions of any sort have been introduced which may cause any loss of information about them.

Equation (29) represents an interesting result, according to which the amplitude $\hat{\Psi}(x',n,t')$ at the space-time point (x',t') is propagated to the space-time point (x,t) (connected by the trajectory $x=x'+\int_{t'}^{t}v\,dt''$) through the propagator $\exp(-in\int_{t'}^{t}L\,dt''/\mu)$ which is a functional of the trajectory. In the limit $\mu\rightarrow 0$ (that is, the adiabatic limit taken via $B\rightarrow\infty$ and not via the pitch angle $\theta\rightarrow 0$) propagation is nonzero only along trajectories which extremize the action

$$\delta\int_{t'}^{t}dt''\,L=0\ .$$

This, by definition, is the adiabatic trajectory since L is the adiabatic Lagrangian. When μ is finite, however, all trajectories in the neighborhood of the adiabatic trajectory (that is, the nonadiabatic trajectories) give a nonvanishing contribution to their respective amplitudes at later times. These considerations may appear to be somewhat reminiscent of the Feynman path integral formulation of quantum mechanics. But as will be seen below there is a difference between the meaning of our Eq. (29) and the Feynman interpretation.

Note that the quantities $L=\frac{1}{2}mv^2-\mu\Omega$ and v under the trajectory integral in Eq. (29) are functions of x; the trajectory values of x between the times t' and t are to be used in evaluating these trajectory integrals. That is, we have

$$\int_{t'}^{t}dt''L(x(t''))=\int_{t'}^{t}dt''L\left[x(t')+\int_{t'}^{t''}v(x(t_1))dt_1\right] .$$

Expressed in this fashion, the trajectory integrals are explicit functions of the position $x(t')$ at the time t' and not of $x(t)$ at time t. Note also that the x appearing in the argument of $\hat{\Psi}$ in Eq. (29) refers to the position at time t.

We now introduce a Fourier transform of $\hat{\Psi}(x,n,t)$ with respect to x [that is, the position at time $t,x(t)$]:

$$\tilde{\Psi}(k,n,t)=\int dx\,e^{+ikx}\hat{\Psi}(x,n,t) . \tag{30}$$

The Fourier transformation of Eq. (29) with respect to x then yields

$$\tilde{\Psi}(k,n,t;\alpha_i')=\exp\left[-i\int_{t'}^{t}dt''\left[\frac{nL}{\mu}-kv\right]\right]\tilde{\Psi}(k,n,t';\alpha_i') . \tag{31}$$

The functions under the trajectory integral are not functions of $x(t)$ and are therefore not involved in the Fourier transformation.

Now the integrand in Eq. (31) $[(nL/\mu)-kv]$ may be written as

$$\frac{nL}{\mu}-kv=\frac{n}{\mu}(\tfrac{1}{2}mv^2-\mu\Omega)-kv$$
$$=\frac{1}{2}\frac{nm}{\mu}\left[v-\frac{\mu k}{nm}\right]^2-\frac{n}{\mu}\left[\frac{1}{2m}\left[\frac{\mu k}{n}\right]^2+\mu\Omega\right] . \tag{32}$$

If we next define a function

$$\overline{\Psi}(k,n,t)=\tilde{\Psi}^*(k,n,t)\exp\left[-\frac{1}{2}i\frac{nm}{\mu}\int_{t_0}^{t}dt\left[v-\frac{\mu k}{mn}\right]^2\right] , \tag{33}$$

where t_0 is some arbitrary initial time, we obtain Eq. (31) in terms of $\overline{\Psi}(k,n,t)$ as

$$\overline{\Psi}(k,n,t;\alpha_i')=\exp\left\{-\frac{in}{\mu}\int_{t'}^{t}dt''\left[\frac{1}{2m}\left[\frac{\mu k}{n}\right]^2+\mu\Omega\right]\right\}$$
$$\times\overline{\Psi}(k,n,t';\alpha_i') . \tag{34}$$

We may now take the inverse Fourier transform of Eq. (34) with respect to k which will give back the dependence of functions on $x(t)$ (the position at time t). We thus have

$$\Psi(x,n,t;\alpha_i')=\exp\left\{-\frac{in}{\mu}\int_{t'}^{t}dt''\left[-\left[\frac{\mu}{n}\right]^2\frac{1}{2m}\frac{\partial^2}{\partial x^2}+\mu\Omega(x(t''))\right]\right\}\times\Psi(x,n,t';\alpha_i') , \tag{35}$$

where $\Psi(x,n,t;\alpha_i')$ is the inverse Fourier transform of $\overline{\Psi}(kn,n,t;\alpha_i')$ and is given by

$$\Psi(x,n,t;\alpha_i')=\int\frac{dk}{2\pi}\overline{\Psi}(k,n,t;\alpha_i')e^{ikx}=\int\frac{dk}{2\pi}e^{ikx}\times\exp\left[-\frac{1}{2}\frac{inm}{\mu}\int_{t_0}^{t}dt''\left[v-\frac{\mu k}{mn}\right]^2\right]\times\widetilde{\Psi}^*(k,n,t;\alpha_i') . \tag{36}$$

We now let $t=t'+\tau$ with τ as an infinitesimal time. Expanding both sides of (35) around t' and dropping the prime on t we get

$$\left[1+\tau\frac{\partial}{\partial t}\right]\Psi(x,n,t;\alpha_i')=\left\{1-\frac{in}{\mu}\left[-\tau\left[\frac{\mu}{n}\right]^2\frac{1}{2m}\frac{\partial^2}{\partial x^2}+\int_t^{t+\tau}dt''[\mu\Omega(x(t''))]\right]+\cdots\right\}\Psi(x,n,t;\alpha_i') . \tag{37}$$

In the limit $\tau\to 0$ we can write

$$\lim_{\tau\to 0}\int_t^{t+\tau}dt''\Omega(x(t''))=\Omega(x)\tau . \tag{38}$$

Equation (37) thus yields the following differential equations in the limit $\tau\to 0$:

$$\frac{i\mu}{n}\frac{\partial\Psi(n)}{\partial t}=-\left[\frac{\mu}{n}\right]^2\frac{1}{2m}\frac{\partial^2\Psi(n)}{\partial x^2}+(\mu\Omega)\Psi(n) ,\quad n=1,2,3,\ldots \tag{39}$$

for the functions $\Psi(n)$ defined by Eq. (36).

We have to next determine the meaning and interpretation of these functions $\Psi(n)$ and their relationship with the probability density function $\hat{f}$ of (20). Recall that in the mixed representation the function $\hat{f}$ of Eq. (20), of the variables (x,Φ) and the initial momenta α_i, was interpreted as the probability of finding a particle at (x,Φ) at time t, if it initially had the momenta α_i. Note first of all that if $f_0(\alpha_i,\beta_i)$ is the initial distribution—a function of the initial values (α_i,β_i)—then the function $f(x,\Phi,t;\alpha_i)$ governed by Eq. (24) is given by

$$f(x,\Phi,t;\alpha_i)=\left|\left|\frac{\partial^2 S}{\partial x_\mu\,\partial\alpha_\nu}\right|\right|f_0(\alpha_i;x_0,\phi_0)\times\delta\left[\Phi-\int_0^t L\,dt'/\mu-\phi_0\right]\times\delta\left[x-\int_0^t v\,dt'-x_0\right] , \tag{40}$$

where $\{\beta_i\}=(x_0,\phi_0)$.

Though it has not been indicated explicitly, the function f defined by (40) is still a function of (x_0,ϕ_0) by virtue of their being the arguments of f_0 and also their occurring in the trajectory integrals in (40). While its x_0 dependence is a δ function $\delta(x_0-x_0^{(0)})$ by the conditions of state preparation, its ϕ_0 dependence is arbitrary though (as discussed earlier) close to a uniform one, as results from the state preparation. The probability integrated over the initial distribution in x_0 and ϕ_0 is given by

$$F(x,\Phi,t;\alpha_i)=\int dx_0\,d\phi_0 f(x,\Phi,t;\alpha_i,\phi_0,x_0) . \tag{41}$$

Next, since the action phase Φ is not measurable in the experiment, the probability integrated over Φ is given by

$$G(x,t;\alpha')=\int d\Phi f\,dx_0\,d\phi_0=\sum_n\int d\phi_0\hat{\Psi}^*(x,n,t;\alpha_i,\phi_0)\hat{\Psi}(x,n,t;\alpha_i,\phi_0) , \tag{42}$$

where use has been made of Eqs. (25) and (28) in obtaining (42) and also of the fact that the distribution in x_0 is $\delta(x_0-x_0^{(0)})$.

We note, however, that it is the functions $\Psi(n)$ defined by Eq. (36) and not $\hat{\Psi}(n)$ which obey the generalized Schrödinger-like equations (39). The functions $\hat{\Psi}(n)$, by virtue of (30), are given by the inverse transform:

$$\hat{\Psi}(x,n,t)=\int\frac{dk}{2\pi}e^{-ikx}\widetilde{\Psi}(k,n,t) . \tag{43}$$

The definition (36) of $\Psi(x,n,t)$ thus differs from that of $\widetilde{\Psi}$ in the presence of an additional factor

$$\exp\left[-\frac{1}{2}\frac{inm}{\mu}\int_{t_0}^{t}dt''(v-\mu k/nm)^2\right]$$

in the integrand. First, Fourier decomposing $\hat{\Psi}$ in (42) according to (43) and then expressing $\widetilde{\Psi}(k,n,t)$ in terms of $\overline{\Psi}$ using (33) we obtain

$$G(x,t;\alpha_i)=\sum_n\int d\phi_0\frac{dk\,dk'}{(2\pi)^2}e^{i(k-k')x}\overline{\Psi}(k,n,t)\overline{\Psi}^*(k',n,t)\times\exp\left\{\frac{1}{2}\frac{inm}{\mu}\int_{t_0}^{t}dt'\left[\left[v-\frac{\mu k}{mn}\right]^2-\left[v-\frac{\mu k'}{mn}\right]^2\right]\right\} , \tag{44}$$

where v under the trajectory integral in the exponential factor is the trajectory value $v=v(\alpha_i,\phi_0,x_0,t)$, a function of the initial values (α_i,ϕ_0,x_0) and time t. Introducing a change of variables $\kappa=k-k'$, and $K=\frac{1}{2}(k+k')$, we get

$$G(x,t;\alpha_i)=\sum_n \int d\phi_0\, d\kappa\, dK \frac{1}{(2\pi)^2} e^{i\kappa x} \times\overline{\Psi}(K+\tfrac{1}{2}\kappa,n,t;\alpha_i',\phi_0) \times\overline{\Psi}^*(K-\tfrac{1}{2}\kappa,n,t;\alpha_i',\phi_0) \times\exp\left[i\kappa\int_{t_0}^{t}dt'\left[v-\frac{\mu K}{mn}\right]\right] . \tag{45}$$

If we now define an average velocity $\overline{v}$ by

$$\overline{v}=(t-t_0)^{-1}\int_{t_0}^{t} v\, dt' ,$$

where t_0 may be taken as the initial time, then

$$\int_{t_0}^{t} dt\left[v-\frac{\mu K}{mn}\right]=(t-t_0)(\overline{v}-\mu K/mn) . \tag{46}$$

In the limit of large times $(t-t_0)$ the exponential factor under integral in (45) will oscillate rapidly and will give a vanishing contribution to the integral unless

$$m\overline{v}=\mu K/n . \tag{47}$$

This relation identifies, as in quantum mechanics, $\mu K/n$ with the average momentum $m\overline{v}$ of the particle. Thus carrying out the integration over ϕ_0 in Eq. (45) yields

$$G(x,t;\alpha_i')=\sum_n \int d\kappa\, dK \frac{1}{(2\pi)^2} e^{i\kappa x}\overline{\Psi}(K+\tfrac{1}{2}\kappa,n,t;\alpha_i') \times\overline{\Psi}^*(K-\tfrac{1}{2}\kappa,n,t;\alpha_i') =\sum_n \Psi^*(x,n,t;\alpha_i')\Psi(x,n,t;\alpha_i') , \tag{48}$$

where the last step follows from Eq. (36), $\overline{\Psi}(k,n,t)$ being the Fourier transform of the function $\Psi(x,n,t)$ which satisfies the Schrödinger-like equation (39). Equation (48) gives the interesting result that the total probability density $G(x,t;\alpha_i')$ is given as a sum of $\Psi^*(n)\Psi(n)$ over all the modes $n=1,2,\ldots$.

The set of equations (39) for the functions $\Psi(x,n,t)$, along with the connection (48) with the probability density $G(x,t)$, is the same set of equations as obtained earlier[5] but derived now using a systematic deductive procedure starting from the Liouville equation for the ensemble under consideration. This set of equations, wherein μ/n $(n=1,2,3,\ldots)$, appears in the various equations in the role of $\hbar$ and the adiabatic potential $(\mu\Omega)$ in the place of the potential in the Schrödinger equation bears the same relationship with the adiabatic motion as the Schrödinger equation does with the classical mechanical motion. The nonadiabatic effects thus appear in the nature of quantum effects. This set thus constitutes a close analog of the Schrödinger formalism of quantum mechanics, with the important difference that we have here a set of infinite equations for the functions $\Psi(x,n,t)$ $(n=1,2,3,\ldots)$. We have thus here a generalization of the formalism of quantum mechanics.

V. THE ENSEMBLE MODES

The equations (39) for the functions $\Psi(n)$ along with the connection (48) with the probability density describe what we may term as the "ensemble modes" of the system. The ensemble in question is prepared in a particular fashion corresponding to the experimental conditions. The modes are infinite in number and are all independent and must be distinguished from collective modes since the particles are noninteracting.

A. Nonadiabatic decay of particles: The quantumlike tunneling

From the Schrödinger-like form of the equations (39) and (48) one can now predict quantumlike effects for the nonadiabatic ensemble behavior of particles. In particular, these equations describe the quantumlike tunneling of particles of an energy less than the maximum height of the adiabatic potential. This is identified with the nonadiabatic decay of particles from magnetic-mirror traps.

One may calculate the lifetimes of particles injected in a magnetic trap with a certain energy E and action μ using the standard techniques of quantum mechanics. Different equations of the set (39) then predict different modes of decay corresponding to $n=1,2,3,\ldots$. Different fractions of the particles would then decay with different lifetimes corresponding to these different modes. Since the equations (39) are all uncoupled, these fractions relate to the *uncontrollable* initial distribution of particles in the Larmor phase and cannot therefore be estimated. One can, however, make some general remarks about the relative order of magnitudes of these fractions.

To do so consider the Fourier expansion (28). If we compare it with the small Larmor radius expansion for the distribution function f of the form (see, for instance, Rosenbluth and Varma[14])

$$f=\sum \epsilon^n f_m^{(n)} e^{im\phi} , \tag{49}$$

where ϵ is the adiabaticity parameter, then one may assign corresponding magnitudes to the various terms in the series (28) and rewrite it as

$$\psi=\sum_{n=1}^{\infty}\epsilon^{(n-1)/2}\hat{\Psi}(x,n,t)e^{in\Phi} . \tag{50}$$

This leads to the following expression for $G(x,t)$:

$$G(x,t)=\sum_{n=1}^{\infty}\epsilon^{n-1}\Psi^*(x,n,t)\Psi(x,n,t) . \tag{51}$$

Comparing this expression with the phase-averaged part of f of Eq. (49), namely,

$$f_0=\sum_{n=0}^{\infty}\epsilon^{|n|}f_0^{(n)} , \tag{52}$$

we find that the contribution in the various n modes corresponds to the $f_0^{(n)}$, the phase-averaged parts in the various orders of the adiabaticity parameter ϵ. One would therefore expect that the fractions of particles decaying according to the higher modes $n=2,3,\ldots$ would be larger for larger values of the adiabaticity parameter ϵ and vice versa. The same conclusion follows from the exponential factor $\exp(-in\int L\,dt/\mu)$ in Eq. (29): For larger values of μ (that is, the adiabaticity parameter ϵ) a nonvanishing contribution can come for values of $n>1$, though the contribution becomes progressively smaller for larger values of n. Thus as the magnetic field is increased or the adiabaticity parameter is decreased in any other way, the fraction of particles decaying according to the $n=1$ mode would increase at the expense of the fractions corresponding to $n=2$ and higher modes of decay. This is what is indeed observed experimentally.[8(b),8(c)]

If we consider these different modes of decay from the point of view of the motion of different particles constituting the ensemble, which correspond to the different initial values of Larmor phase, it is clear that these particles would behave very differently depending on their initial conditions. But what seems remarkable is that a continuous variation over the initial phases should get mapped into distinct discrete macroscopic ensemble modes of decay ($n=1,2,\ldots$) as have been experimentally observed. This reveals an interesting property of the mapping represented by the classical equation of motion in a magnetic field in the neighborhood of the adiabatic motion.

B. One-dimensional interference effects

Another ensemble mode of behavior that Eqs. (39) and (48) predict is one-dimensional interference effects, analogous, for example, to the quantum interference effects arising in the interaction of electrons with the periodic potential of a crystal lattice. One may, likewise, consider here the motion of an ensemble of particles in a periodic magnetic field as that of a multimirror system (a large number of mirror traps joined end to end).

Consider thus an ensemble of particles of a given energy E and action μ injected at one end of a multimirror system, such that the pitch angle of particles at injection is somewhat less than the loss cone angle. Particles with different initial values of the Larmor phase would then, in general, behave differently. The particles may first get trapped in one of the mirror traps and may eventually get reflected, or they may get transmitted. According to the $n=1$ mode of Eq. (27) a reflection of particles, analogous to the Bragg reflection, would occur if

$$2\int_0^L p_{\parallel}dx = l\mu\ ,\quad l=1,2,3,\ldots \tag{53}$$

where $p_{\parallel}$ is the momentum of particles parallel to the magnetic field and where L is the periodic length of the multimirror system, that is, the distance between consecutive mirror traps.

If we consider the modes $n=2,3,\ldots$ then fainter reflections are also expected to occur corresponding to these modes. We are presently planning experiments to check these predictions. If these interference effects are indeed observed then this theory would constitute a complete analog, in a generalized form, of the Schrödinger formalism of quantum mechanics.

VI. CONCLUSIONS, DISCUSSION, AND COMMENTS

We have given an alternative derivation of the set of Schrödinger-like equations for the ensemble nonadiabatic behavior of particles in inhomogeneous magnetic fields starting from the Liouville equation for the ensemble. An earlier brief and somewhat heuristic derivation of these equations[5] perhaps left some questions unanswered. We have furthermore tried to clarify here the essential ensemble nature of the theory and introduced the concept of ensemble modes that our equations describe. As we have already discussed in detail in Sec. V, some of the predictions of these equations, which were somewhat unusual and could not have been otherwise foreseen, have recently been verified experimentally.[8(a)–8(c)]

To be sure, this theory has its limitations. It is not applicable to situations where an adiabatic motion is not defined, as, for instance, in a cusp field. Furthermore, as given here the theory is applicable to an axisymmetric magnetic field, though in principle it is possible to generalize it to nonaxisymmetric situations. We are presently trying to do so.

From the point of view of the mirror-type fusion devices a very useful result of the theory is that for a given value of the parameters, namely, the energy E and the pitch angle θ, the lifetime of particles in the mirror trap increases exponentially with the scale length of the magnetic-field hump even if the loss cone angle remains the same. Accordingly, the lifetimes of particles can be increased exponentially by simply increasing the scale length of the magnetic hump, without increasing its height (that is, without decreasing the loss cone angle). This result which has been verified experimentally in cases investigated so far[8(b),8(c)] appears to be in contradiction with the random-walk diffusion approach. For in the latter approach[6(b)] the lifetime of a particle is determined by the departure of its initial pitch angle from the loss cone angle regardless of the scale length of the magnetic-field hump.

The theory has so far been verified insofar as the prediction of the behavior of lifetimes is concerned. We have yet to see the experimental or numerical results on the one-dimensional interference effects that the theory predicts. However, the experimental verification (even if limited) of a quantum-mechanical-like theory of a classical process (governed by classical equations of motion), though useful in itself, is very interesting from a conceptual viewpoint. It shows that there exists a situation in classical mechanics (a deterministic theory, to be sure) which can be meaningfully described in terms of probability amplitudes, with the probability density being given, as in quantum mechanics, by the modulus squared of the amplitudes. These results are doubtless of interest for the hidden variable theories of quantum mechanics which try to derive it from a deterministic substructure. One such theory has in fact been given by the author[15,16] where a generalized set of Schrödinger equations similar to (39) and (48) have been obtained for quantum mechanics and some new quantum effects corresponding to $n=2$, and higher modes have been predicted.

ACKNOWLEDGMENTS

A major part of this work was performed while the author was visiting at the Institute for Fusion Theory, Hiroshima University. He would like to express his gratitude for the support by the Japanese Ministry of Education and for gracious hospitality accorded him by Professor Kyoji Niskikawa, Director, Institute for Fusion Theory, during his stay at Hiroshima. He would also like to thank him for his continued interest in this work and illuminating discussions. Thanks are also due to Professor T. Watanabe and Dr. H. Irie for discussions and for their criticism during the progress of this investigation.

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