Laser beat-wave accelerator and plasma noise

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Plasma noise effects on the beat-wave resonance condition and the subsequent reduction of the accelerating electric field for the laser beat-wave accelerator are investigated. Plasma noise or turbulence causes the accelerating electric field to saturate at a reduced level $\epsilon = L_{\rm eff}/L$ of the ideal coherent accelerating field, where $L_{\rm eff}$ is the effective growth length for the plasma wave and L is the ideal growth length. When the noise parameter $\sigma = \langle(\delta n)^2\rangle/n^2$ is above a threshold value, we find $L_{\rm eff} \simeq 2c/\omega_p^2 \tau_c \sigma$, where τ_c is the correlation time of the noise spectrum along $\omega = kv_p$. Slower noise is less effective in limiting the plasma wave growth. For strong noise $L_{\rm eff}$ becomes shortened to the correlation length in the noise $L_c = v_p \tau_c$. In the case of slowly increasing mismatch in the beat-wave resonance condition, the effective growth length $L_{\rm eff}$ becomes the geometric mean of the density gradient scale length L_n and the collisionless skin depth $(L_n c/\omega_p)^{1/2}$.

I. INTRODUCTION

Collective acceleration of particles to high energies by intense fields is a topic of considerable interest in recent years. This is in part due to an increasing awareness that the conventional method of acceleration, such as by the radio-frequency cavity, yields only a limited strength of the accelerating fields which results in a huge, expensive, and perhaps unmanageable accelerator in order to meet the needs for the next generation of high-energy-physics experiments. The hope is that some novel acceleration ideas may be able to substantially reduce the size, cost, and complexity of future accelerators.

The laser beat-wave accelerator[1,2] presents a possible method for an efficacious collective acceleration that may be applicable to a future ultrahigh-energy accelerator.[3] The method calls for an efficient high-power laser, a resilient method of matching the accelerating fields and particles, and a robust way to ensure the production of accelerating fields in a plasma. Some of the possible problems associated with the laser beat-wave accelerator in high-energy applications have been identified:[2,4] (i) the transverse deterioration of matching of the beat-wave condition and (ii) the longitudinal deterioration of the condition. The former includes the Rayleigh refraction of laser light, the filamentation instability,[5] and the self-focusing and defocusing.[6] The latter involves the mismatch in velocities of the phase of the accelerating wave and particles in the laser beat-wave scheme, the phase instability of particles, and the degree of incoherency of the accelerating fields. In order to overcome the difficulty of the longitudinal mismatch, a few ideas have been advanced. These include the Surfatron[7] and the plasma fiber accelerator.[8] The last element of the longitudinal deterioration, the loss of coherency of the accelerating fields, may arise from the imperfect beat-wave resonance $\omega_0 - \omega_1 \neq \omega_p$, for example, where ω_0 and ω_1 are the laser frequencies associated with the first laser light and the other and ω_p is the local plasma frequency. The contamination of coherent accelerating fields can inflict deleterious effects on accelerating fields for the original beat-wave accelerator scheme as well as for the Surfatron and the plasma fiber accelerator schemes. It is the purpose of the present work to investigate the contamination of the coherent plasma accelerating field by the various forms of plasma noise.

Recent numerical simulations[9,10] and an experiment[11] underline the importance of matching the beat-wave condition and that of effects of plasma noise. The computer simulations[9,10] show that even under the perfect resonance condition (i.e., $\omega_1 - \omega_2 = \omega_p$ originally matched in a uniform plasma) the plasma wave created by the beat of two lasers eventually becomes incoherent in the downstream of the wave train. This may be partly attributed[2] to mismatch of the phase velocities of plasma waves generated by successive forward Raman decays. The phase velocity of the beat plasma wave by lasers ω_0 and ω_1 is

$$v_{\rm ph}^1 \cong c\left[1 - \frac{\omega_p^2}{\omega_1^2}\right]^{1/2} \cong c\left[1 - \frac{1}{2\gamma_p^2}\right],$$

if $\omega_p \ll \omega_0$. Here $v_p = (\omega_0 - \omega_1)/(k_0 - k_1) = c\beta_p$ and $\gamma_p = (1-\beta_p^2)^{-1/2}$. Similarly, the phase velocity of the beat wave produced by laser ω_1 and the decay product ω_2 is

$$v_{\rm ph}^2 \cong c\left[1 - \frac{\omega_p^2}{\omega_2^2}\right]^{1/2}.$$

Although these plasma waves generated by successive forward Raman decays have a similar phase velocity and thus do not destroy the accelerating fields immediately, they eventually interfere with each other and become incoherent. The difference in phase velocity between $v_{\rm ph}^1$ and $v_{\rm ph}^2$ is

$$\Delta v_{\rm ph} = v_{\rm ph}^1 - v_{\rm ph}^2 \cong c\left[\frac{\omega_p}{\omega_1}\right]^3.$$

Therefore, the time for the plasma waves to become incoherent is

$$\Delta t = \frac{c}{\omega_p}\frac{1}{\Delta v_{\rm ph}} \sim \left[\frac{\omega_1}{\omega_p}\right]^3 \frac{1}{\omega_p} = \frac{\gamma_p^3}{\omega_p} ,$$

which is sufficiently long for large γ_p.

Among the other possibilities to explain the simulation result[9] of incoherency, two other plausible explanations are mentioned. One is that as the plasma is excited and also heats electrons, the electrons become relativistic which contributes to the detuning[12] through the frequency mismatch. With more than one forward Raman decay, the above-mentioned phase velocity difference becomes magnified, thus increasing the incoherency. Another mechanism is that as the laser lights cascade by multiple Raman scattering, the interaction among many laser lights and the plasma wave becomes chaotic and incoherent because of the many degrees of freedom involved.

In addition, the experiment[11] indicates that the injection of a single laser causes more than one wave number of plasma waves to grow up because the beating light signal has to grow from noise which takes time. If this is the case, the excitation of many plasma waves can give rise to incoherency, thus retarding the growth of the accelerating fields.

In order to investigate the possible saturation or suppression of the beat plasma wave growth from the mismatch of phase velocities from variations in the plasma density, we derive a general equation for the plasma wave with the beating lasers allowing for the presence of plasma noise. In the previous example of multicascade of photons, the noise effects may enter either through the plasma density fluctuation $\delta\omega_p^2$ or through the source noise δS.[13] In the present discussion we confine our attention to the plasma density fluctuation only and assume that the source term is coherent. The equation for the beat wave in the plasma density fluctuations is analyzed in two cases. The first is for the case of plasma noise which may be regarded as turbulence, and the noise characteristics are given by statistical quantities such as the correlation length and the level of fluctuations of plasma (the variance of plasma density fluctuations). This problem is discussed in Sec. II. The second is for the case in which the plasma is slowly changing its characteristics (not turbulently). This case is analyzed using the WKB theory. We treat this problem in Sec. III. In Sec. IV the accelerating plasma electric field is evaluated for some examples considered in Secs. II and III. In Sec. V we discuss further other plasma-noise effects, summarize our results, and give the conclusions.

II. GROWTH OF THE BEAT WAVE IN PLASMA TURBULENCE

We consider two plane-polarized laser beams of length L incident upon an underdense cold plasma. The laser fields are $E_j\sin(k_jx-\omega_jt+\phi_j)\hat{e}_y$ and $B_j\sin(k_jx-\omega_jt+\phi_j)\hat{e}_z$ with $B_j=(ck_j/\omega_j)E_j$ and satisfy the dispersion relation $\omega^2=c^2k^2+\omega_p^2$ where $\omega_p^2=4\pi Ne^2/m_e$ with the mean density N. The local plasma density $n(\mathbf{x},t)$ contains fluctuations with $n(\mathbf{x},t)=N+\delta n(\mathbf{x},t)$. We characterize the density fluctuations in the case of noise by the dimensionless variance $\sigma=\langle(\delta n)^2\rangle/N^2$ and correlation scales L_c and τ_c defined by the autocorrelation functions. The dimensionless oscillation velocities are $a_j=v_j/c=eE_j/m_ec\omega_j$, and we approximate the transverse relativistic factor $\gamma_\perp=(1+a_1^2/2+a_2^2/2)^{1/2}$ by unity.

The longitudinal motion of the electrons in the plasma rest frame are described by the displacement field $\xi(x,t)$. The longitudinal electric field is given by

$$E_x=4\pi en\xi(x,t) , \tag{1}$$

where $\xi(x_0,t)$ is the displacement of the cold plasma electrons at x_0. The electron dynamics is considered in the linear regime where $k_p\xi<1$. The electron longitudinal acceleration by the beat laser fields is

$$\frac{\partial}{\partial t}\left[\gamma_{\dot\xi}\frac{\partial\xi}{\partial t}\right]=-\frac{e}{m_e}E_x-\frac{e}{m_ec}(v_y^{(1)}B_z^{(2)}+v_y^{(2)}B_z^{(1)}) , \tag{2}$$

where

$$\gamma_{\dot\xi}=(1-\dot\xi^2/c^2)^{-1/2} \text{ and } v_y^{(j)}=\frac{eE_j}{m_e\omega_j}\cos(\vartheta_j) \tag{3}$$

with $\vartheta_j\equiv k_jx-\omega_jt+\phi_j$.

We calculate $\partial_t(\gamma_{\dot\xi}\partial_t\xi)=\gamma_{\dot\xi}^3\partial_t^2\xi$, substituting Eqs. (1) and (3) into Eq. (2) to derive the acceleration equation for plasma oscillations

$$\begin{aligned}\gamma_{\dot\xi}^3\frac{\partial^2\xi}{\partial t^2}+\omega_p^2(x,t)\xi&=\frac{e^2cE_1E_2}{m_e^2\omega_1\omega_2}(k_2\sin\vartheta_2\cos\vartheta_1+k_1\sin\vartheta_1\cos\vartheta_2)\\&=S(x-v_gt)\left[\sin[(k_2-k_1)x-(\omega_2-\omega_1)t+\phi_2-\phi_1]\right.\\&\qquad\left.+\left[\frac{k_2+k_1}{k_2-k_1}\right]\sin[(k_2+k_1)x-(\omega_2+\omega_1)t+\phi_1+\phi_2]\right] ,\end{aligned} \tag{4}$$

where

$$S(x-v_gt)=\frac{e^2\Delta k\,E_1E_2}{2m_e^2\omega_1\omega_2}=2\,\Delta\omega\,c\lambda S_0(x-v_gt) \tag{5}$$

with $\lambda=a_1a_2/4\beta_p$ as defined in Ref. 12 and $\Delta k=k_2-k_1$, $\Delta\omega=\omega_2-\omega_1$. The thermal velocity effects of order v_e^2/c^2 are neglected in Eq. (4). The beat-wave driving acceleration $S(x-v_gt)$ is a square pulse of length L containing

$\Delta k\, L \sim \omega_p L/c \gg 1$ wavelengths of the beat wave. Since $\omega_2-\omega_1 \sim \omega_p \ll \omega_{1,2}$ the speed of the difference wave

$$v=\frac{\omega_2-\omega_1}{k_2-k_1}\simeq v_g=\beta_p c \tag{6}$$

is approximately that of the laser pulse[1] $v_p \simeq v_g = d\omega_k/dk$.

The small nonresonant oscillations driven by the sum frequency are given by $\xi_2(x,t)\cong\xi_{2k,2\omega_1}\sin(2\vartheta_1)$ with $\xi_{2k_1,2\omega_1}\cong e^2E_1^2/3m_e^2\omega_1^3c$ producing a small density oscillation $\delta n/N \sim 2k_1\xi_{2k_1,2\omega_1}$. The $2\vartheta_1$ oscillation is not resonant with particles because the phase velocity of this beat wave is larger than the speed of light.

The growth of the plasma wave in the laboratory frame is given by

$$\gamma_\xi^3\partial_t^2\xi+\omega_p^2\xi=S(x-v_pt)\sin[\Delta k(x-v_pt)+\phi]\ . \tag{7}$$

In the Lorentz frame $x'=\gamma_p(x-v_pt)$ shown in Fig. 1 traveling with the laser pulse, the equation is written as

$$-2\gamma_\xi^3\gamma_p^2v_p\frac{\partial^2\xi(x',t')}{\partial x'\partial t'}+(\gamma_\xi^3\gamma_p^2v_p^2\partial_{x'}^2+\omega_p^2)\xi(x',t')=S(x')\sin(k_wx'+\phi)\ , \tag{8}$$

where $k_w=\Delta k/\gamma_p$. This is the basic equation for the present problem. In writing Eq. (8) a weak time dependence $\omega'=\gamma_p(\omega-kv_p)\ll\omega_p$ is allowed in the wave frame. It is straightforward to use the envelope approximation and the weakly relativistic approximation $\gamma_\xi^3\cong 1+3\dot{\xi}^2/2c^2$ to derive the Rosenbluth-Liu equations from Eq. (8).

There are two kinds of noise that are important: resonant noise propagating with the velocity v_p of the laser pulse and transient noise which passes through the laser pulse. The resonant noise is static in the wave frame $(\partial_{t'}=0)$ and thus can permanently reduce the amplitude of the accelerating electric field

$$E(x')=(m_e\omega_p/e)[\omega_p\xi(x')/c]\ .$$

The effect of resonant noise is given by

$$[\gamma_\xi^3\gamma_p^2v_p^2\partial_{x'}^2+\omega_p^2(x')]\xi(x')=S(x')\sin(k_wx'+\phi) \tag{9}$$

and is analyzed in this work. The transient noise propagates through the plasma wave and has the frequency $\omega'=\gamma_p(\omega-kv_p)$ in the wave frame. The most important effect of transient noise may be on the phase locking of the accelerating particles, especially when ω' resonates with the phase-locking frequency. The two-dimensional problem $\xi(x',t')$ of transient noise will be considered in a future work.

The ideal resonant solution for

$$\Delta\omega=\omega_p=\gamma_pk_wv_p=\Delta k\, v_p$$

with $\xi_0(x')=\partial_{x'}\xi_0=0$ for $x'>0$ and $S(x')=2c\omega_p\lambda\Theta(-x')$ from Eq. (8) is

$$\xi_0(x')=\frac{c\lambda}{\omega_p}\left[-\frac{\omega_px'}{\gamma_pv_p}\cos(k_wx'+\phi)+\cos\phi\sin\left[\frac{\omega_px'}{\gamma_pv_p}\right]\right]\ . \tag{10}$$

Equation (10) gives the resonant secular growth[14] of the ideal accelerator system. From Eq. (10) we see that the nonrelativistic approximation breaks down for $\lambda\omega_px'>\gamma_pv_p$.

For low laser power $\lambda\ll1$ or short pulse length L the plasma wave grows secularly by Eq. (10) for $\lambda\omega_pL/v_p<1$. For high laser power or long laser pulse length the secular growth ends before $\omega_px'/\gamma_pv_p=1/\lambda$ from the relativistic detuning[12] from $\omega_p/\gamma_\xi^{3/2}\neq\Delta\omega$ in Eq. (9).

From relativistic detuning growth stops when the displacement $\xi_0(x')$ reaches the Rosenbluth-Liu value[12]

$$\xi_{\rm RL}=\left[\frac{16}{3}\right]^{1/3}\frac{c}{\omega_p}\left[\frac{eE_1}{m\omega_1c}\frac{eE_2}{m\omega_2c}\right]^{1/3}=\frac{4}{3^{1/3}}\frac{c\lambda^{1/3}}{\omega_p}\ . \tag{11}$$

In this second case of high laser power the ideal growth length L at which the secular growth of the beat wave terminates is obtained from Eq. (10) and by relating Eq. (11) with Eq. (10) as

$$L\simeq\frac{4}{3^{1/3}}\lambda^{-2/3}\left[\frac{c}{\omega_p}\right]\ , \tag{12}$$

where $a_1=eE_1/m\omega_1c$ and $a_2=eE_2/m\omega_2c$. In the third case in which the laser amplitude a_i is of the order of unity, the wavebreaking or trapping condition[1,2] determines the ideal growth length L as

$$L\simeq\pi\left[\frac{c}{\omega_p}\right]\ . \tag{13}$$

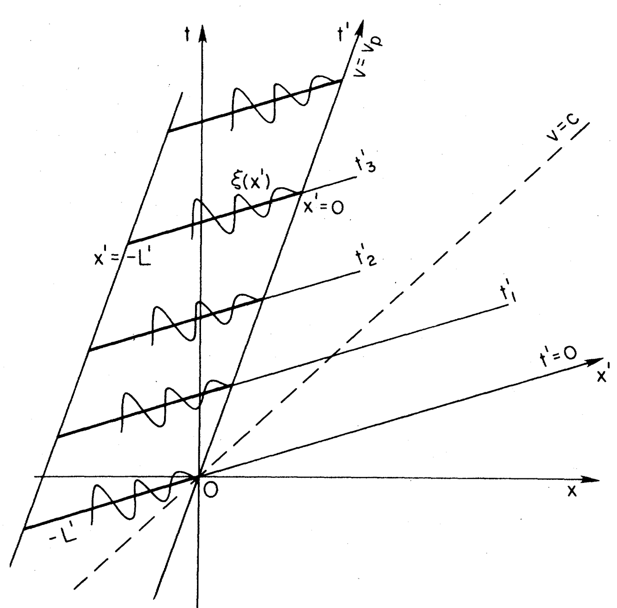

FIG. 1. Lorentz rest frames of the thermal plasma x,t and the laser pulse x',t' used in the calculations.

Now we investigate the modification of the ideal solution $\xi_0(x')$ by plasma turbulence in the form of density fluctuations. The effect of the scattering of energy away from $\xi_0(x')$ by the action of the density fluctuations is shown clearly by writing Eq. (7) in $k\omega$ space. Representing $\xi(x,t)$ and $\delta n(x,t)$ by their Fourier transforms with

$$\xi(x,t)=\int \frac{dk\,d\omega}{(2\pi)^2}\xi(k,\omega)e^{ikx-i\omega t}\,, \tag{14}$$

we have from Eq. (7)

$$(-\omega^2+\omega_p^2)\xi_{k\omega}+\int \frac{dk'\,d\omega'}{(2\pi)^2}\delta\omega_p^2(k',\omega')\xi(k-k',\omega-\omega') = \mathscr{S}(k,\omega,v)=2\pi\delta(\omega-kv)S(k)\;. \tag{15}$$

Note here that the noise effects under consideration are due to the plasma noise or plasma nonuniformity that come in through the second term on the left-hand side of Eq. (4) that is proportional to the squared "noise plasma frequency" $\delta\omega_p^2$. The presence of the Dirac δ function $\delta(\omega-kv_p)$ on the right-hand side (the source term) stems from our assumption that the laser beams are coherent and do not contain noise. Thus, the source term produces a sharp (δ-function) beat velocity at the photon group velocity in Eq. (15).

In order to solve Eq. (15) for the longitudinal electron displacement $\xi_{k\omega}$, we have to invert the integral operator on the left-hand side of Eq. (15). An approximate method to accomplish this is to iterate the inversion process by starting from the unperturbed propagator $L_0(k,\omega) = -\omega^2+\omega_p^2$ for Langmuir waves.

We obtain the dominant effect of the stochastic density fluctuations by summing to all orders the most secular terms in the perturbation expansion

$$\xi_{k\omega}=(L_0+\delta L)^{-1}\mathscr{S}_{k\omega} = L_0^{-1}(1-\delta L\,L_0^{-1}+\delta L\,L_0^{-1}\delta L\,L_0^{-1}+\cdots)\mathscr{S}_{k\omega}$$

of Eq. (15). The secular terms contain the highest-degree multiple pole in $L_0(k,\omega)\equiv 0$ in each order of the perturbation expansion, and their contribution can be summed to all orders as given by Horton and Choi.[15] The resulting renormalized propagator, $L(k,\omega)=L_0-\langle \delta L\,L_0^{-1}\delta L\rangle$, is given by[15,16]

$$L(k,\omega)\xi_{k\omega}=\{-\omega^2+\omega_p{}^2+[(\Delta\omega_p)^2]_k-i\omega\nu_k\}\xi_{k\omega} = \mathscr{S}(k,\omega,v)\,, \tag{16}$$

where the frequency shift $[(\Delta\omega_p)^2]_k$ and the stochastic damping ν_k are given by

$$[(\Delta\omega_p)^2]_k-i\omega\nu_k=\frac{-\omega_p^4}{N^2}\int \frac{dk_1 d\omega_1\,|\delta n(k_1,\omega_1)|^2}{-(\omega-\omega_1)^2+\omega_p^2+[(\Delta\omega_p)^2]_{k-k_1}-i(\omega-\omega_1)\nu_{k-k_1}}\;. \tag{17}$$

Evaluating the denominator of the right-hand side of Eq. (17) at $\omega=\omega_p$ leads to resonances of the interaction term at $\omega_1(2\omega_p-\omega_1)\simeq 0$. The $\omega_1=2\omega_p$ is a parametric resonance that leads to backscatter in $\xi_0(x,t)$. Although we do not expect fluctuations at $2\omega_p$, we show the effect of noise at $2\omega_p=2\Delta k v_p$ in Sec. IV. Thus, in the following we consider contributions from the low-frequency resonance. Reducing Eq. (17) for $\omega_1\lesssim\omega_p$ and defining $I(\omega) = \int (dk/2\pi)\,|\delta n(k,\omega)|^2/N^2$ we obtain

$$(\Delta\omega_p)^2-i\omega_p\nu=-\frac{\omega_p^3}{2}\int_{-\infty}^{+\infty}\frac{d\omega_1 I(\omega_1)}{\omega_1+(\Delta\omega_p)^2/2\omega_p-i\nu/2}\;. \tag{18}$$

For a broad frequency spectrum centered about $\omega_1/\omega_p\sim 0$ we obtain

$$(\Delta\omega_p)^2=-\omega_p^3\mathrm{P}\int_{-\infty}^{+\infty}d\omega_1 I(\omega_1)/[\omega_1+(\Delta\omega_p)^2/2\omega_p]\simeq 0\;.$$

The damping rate obtained is

$$\nu=\frac{\pi\omega_p^2}{2}\int_{-\infty}^{+\infty}d\omega_1\delta(\omega_1)I(\omega_1)=\tfrac{1}{2}\omega_p^2\tau_c\sigma\;. \tag{19}$$

The damping of the mean wave is due to the diffusion of the wave phase by the density fluctuations. In rewriting Eq. (19) we have used the definitions of the dimensionless variance σ^2 and the autocorrelation time τ_c of the density fluctuations

$$\tau_c\sigma=\int_{-\infty}^{+\infty}d\tau\frac{\langle\delta n(x,t+\tau)\delta n(x,t)\rangle}{N^2} = \int\frac{dk_1 d\omega_1}{(2\pi)^2}\frac{|\delta n(k_1,\omega_1)|^2}{N^2}\delta(\omega_1)\;. \tag{20}$$

Since the source $\mathscr{S}(k,\omega,v)$ is peaked along $\omega=kv_p$ the decay rate ν defines the effective growth length $L_{\rm eff}=\gamma_p v_p/\nu(\sigma)$ in the beat-wave frame.

The growth of the plasma wave limited by the scattering rate ν follows from Eq. (16) and is given by

$$\xi(x,t)=\xi(x-v_pt) = -\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{e^{ik(x-v_pt)}S(k)}{(kv_p)^2+i\nu kv_p-\omega_p^2}\,, \tag{21}$$

where $S(k)=\int_{-\infty}^{+\infty}dx\,e^{-ikx}S(x)\sin(\Delta k\,x+\alpha)$. For $S(x)=0$ for $x>0$, $S(k)$ is analytic in the upper half of the k plane, and $\xi(x')=0$ for $x'>0$. For a box of length L for the laser envelope function and $\nu=0$ the integral (21) reproduces the ideal solution (9). For finite ν the plasma poles are displaced to

$$k_\pm=\pm\frac{\omega_p}{v_p}-\frac{i\nu}{2v_p} \tag{22}$$

and the resonance $\Delta kv=\omega_p$ becomes broadened. The details of the response function $\xi(x)$ are complicated and depend on the choice of the envelope function. The formulas given here apply when the rise time τ_r of the laser pulse is short compared with the growth length, e.g.,

$\omega_p \tau_r \ll 3/\lambda^{2/3}$ from Eq. (12). Some examples are evaluated in Sec. IV.

The general features of the plasma response are that for $|x'| < L'_{\rm eff}(\sigma) \equiv \gamma_p v_p / \nu(\sigma)$ the growth of the Langmuir wave continues as

$$\xi(x') \simeq \frac{c\,|x'|\,\lambda}{v_p \gamma_p} \cos(k_w x' + \phi) \tag{23}$$

for $\gamma_p v_p/\omega_p \ll |x'| < L'_{\rm eff}(\sigma)$. The noise limits the growth when $L_{\rm eff}(\sigma) < L'$ which occurs for the regime

$$\sigma = \left\langle \frac{\delta n^2}{N^2} \right\rangle > \frac{4 v_p^2}{\omega_p^2 L_c L} , \tag{24}$$

where $L_c = v_p \tau_c$. For $L'_{\rm eff}(\sigma) < |x'| < L'$ the amplitude is saturated at the value

$$\xi_L = \frac{c L_{\rm eff}(\sigma) \lambda}{v_p} . \tag{25}$$

In this regime the laser-driving force balances the scattering of the plasma wave by the density fluctuations into secondary plasma waves. For $|x'| > L'$ in the regime of Eq. (24), the Langmuir wave decays due to the continued scattering of the longitudinal plasma wave by density fluctuations.

The dependence of the effective growth length $L_{\rm eff}$ on the noise is a complicated question in general. Here, we give two approximate formulas, one for the weak-noise limit and one for the strong-noise limit. For weak noise the phase becomes stochastic and diffuses with the coefficient

$$D = \int_{-\infty}^{+\infty} \langle \delta k(x') \delta k(0) \rangle dx' = \langle (\delta k)^2 \rangle L_c = \frac{\omega_p^2 \tau_c}{4 v_p} \sigma \tag{26}$$

consistent with Eq. (19). The effective growth length is then

$$L_{\rm eff} = \frac{v_p}{\nu} = \frac{1}{\langle (\delta k)^2 \rangle L_c} = \frac{4 v_p^2}{\omega_p^2 L_c \sigma} . \tag{27}$$

For strong noise the growth length for $\xi(x)$ can be estimated from renormalized turbulence theory using Eq. (18) with $\nu\omega_p > \Delta\omega_p^2$, and it becomes limited to the correlation length L_c of the density fluctuations. The transition occurs at the critical noise level σ_c obtained from the limit $L_{\rm eff} = L_c$ in Eq. (27) which determines

$$\sigma_c = \frac{4 v_p^2}{\omega_p^2 L_c^2} \tag{28}$$

as the critical noise level above which the growth is limited to

$$\xi_{L_c} = \frac{S L_c}{v_p \omega_p} = \frac{2 c \lambda L_c}{v_p} . \tag{29}$$

In the extreme limit of strong Langmuir turbulence the correlation length L_c may be determined by the driver at ω_p, $k_p = \omega_p / v_p$ and would limit $L_{\rm eff}$ to $L_c = v_p/\omega_p$ with no significant growth of the plasma wave ($\xi_L \simeq S/\omega_p^2 = 2\lambda c/\omega_p$).

With regard to the nature of the driven Langmuir turbulence the character of the spectrum will change according to the relationship of the driven wave number $k_p = \omega_p / v_p \simeq \omega_p / c$ and the characteristic wave number $k_* = (m_e/m_i)^{1/2} (\omega_{pe}/v_e)$. For $k_p < k_*$ there is a strong interaction of the Langmuir turbulence with the ion-acoustic waves. Once the typical turbulence wave number reaches k_* the turbulence can produce a broad spectrum of ion-acoustic and Langmuir turbulence. The Zhakharov equations[17] can describe this process including the modulational instability of the Langmuir waves and the excitation of ion-acoustic waves during the collapsing stage of the nonlinear Langmuir turbulence.[18]

For a hydrogen plasma the condition $\omega_p = k_* v_p$ defines a critical electron temperature of $T_c = 280$ eV. For $T_e > T_c$ the threshold wave number for the modulational instability lies below the source wave number k_p. In this case the source-generated plasma turbulence must spread in k down from k_p before reaching k_*. The usual mechanism for this downward spread is induced scattering from ions. The energy density available to drive the modulational instability at the Rosenbluth-Lui nonlinear limit is

$$\frac{W}{8\pi n T_e} = \frac{E_L^2}{4\pi n T_e} = \left[\frac{16}{3} \right]^{2/3} \frac{\left[\dfrac{m \omega_p c}{e} \right]^2 (a_1 a_2)^{2/3}}{4\pi n m_e v_e^2} = \left[\frac{16}{3} \right]^{2/3} \frac{m_e c^2}{T_e} (a_1 a_2)^{2/3} ,$$

which is typically much greater than unity. The time scale for growth of the modulational instability is sufficiently rapid that it may be required to maintain $k_p \gg k_*$ to retard the development of the modulational instability. In this respect it is desirable to have the growth to $\omega_p \xi / c \sim 1$ in a laser pulse length $L \lesssim c/\omega_{pi}$.

III. EFFECT OF A WEAKLY NONUNIFORM PLASMA

We now consider the reduction of the beat-wave amplitude due to coherent plasma density variations. In the case in which the background plasma has slowly varying inhomogeneities compared with the scale c/ω_p of the plasma wavelength, we return to Eq. (8) and use the WKB approximation to solve for the driven plasma wave.

In the wave frame the equation for $\xi(x)$ is [Eq. (9)]

$$[\gamma_p^2 v_p^2 \partial_x^2 + \omega_p^2(x)] \xi(x) = S(x) \sin(k_w x + \phi) ,$$

where the primes have been dropped. For the scale length over which the plasma density changes $L_n = |\omega_p^2 / d\omega_p^2/dx| \gg \gamma_p v_p / \omega_p$ in the wave frame, the WKB solutions of the homogeneous equation are

$$E_c(x) = \left[\frac{k_w^0}{k_w(x)} \right]^{1/2} \cos \left[\int_0^x k_w(x') dx' \right] , \tag{30}$$

$$E_s(x)=\left[\frac{k_w^0}{k_w(x)}\right]^{1/2}\sin\left[\int_0^x k_w(x')dx'\right], \quad (31)$$

where

$$k_w(x)=\frac{\omega_p(x)}{v_p\gamma_p} \quad (32)$$

and k_w^0 is the reference mean value of $k_w(x)$. By the method of Green's function we obtain the solution of the driven equation as

$$\xi(x)=\int_0^x dx'\frac{S(x')\sin(\Delta k\, x'+\phi)}{v_p^2 W(E_s,E_c)} \times[E_c(x')E_s(x)-E_s(x')E_c(x)] \quad (33)$$

with the boundary condition $\xi(x=0)=0$. The Wronskian $W(E_s,E_c)$ is constant in the WKB approximation with $W=k_w^0$. The driven wave reduces to

$$\xi(x)=\frac{E_s(x)}{v_p^2(k_w^0)^{1/2}}\int_0^x \frac{dx'S(x')}{\sqrt{k_L(x')}} \times\sin(\Delta k\, x'+\phi)\cos\left[\int_0^{x'}k_w dx'\right] -\frac{E_c(x)}{v_p^2(k_w^0)^{1/2}}\int_0^x dx'\frac{S(x')\sin(\Delta k\, x'+\phi)}{\sqrt{k_L(x')}} \times\sin\left[\int_0^{x'}k_w dx''\right]. \quad (34)$$

For $\Delta k\simeq k_w$ the resonant contribution

$$\xi(x)\simeq-\frac{E_c(x)}{2v_p^2(k_w^0)^{1/2}}\int_0^x\frac{dx'S(x')}{\{[k_w(x')]\}^{1/2}} \times\cos\left[\int_0^{x'}[k_w(x')-\Delta k]dx''\right]. \quad (35)$$

The dominant contribution to $\xi(x)$ arises from the neighborhood resonant point(s) x_r defined by the condition $k_w(x_r)=\Delta k$. Near and around the resonant point(s), we evaluate the integral as

$$\int_{x_r}^x[k_w(x')-\Delta k]dx'=\frac{1}{2}\left.\frac{dk_w}{dx}\right|_{x=x_r}(x-x_r)^2. \quad (36)$$

Performing the x' integral in Eq. (35) determines the effective growth length of the plasma beat wave L_{eff} for the resonance as

$$L_{\text{eff}}=\left[2\pi\left.\frac{dx}{dk_w}\right|_{k_w=k_w(x_r)}\right]^{1/2}=\left[\frac{2\pi}{k_w}L_n\right]^{1/2}. \quad (37)$$

This evaluation of Eq. (35) is valid for regime $k_w^{-1}\ll L_{\text{eff}}\ll L$.

For a single resonance the plasma wave growth is expressed in terms of the displacement

$$\xi(x)=\frac{S_0E_c(x)L_{\text{eff}}}{2v_p^2k_w^0}\cos\left[\int_0^{x_r}[k_w(x')-\Delta k]dx'+\frac{\pi}{4}\right]. \quad (38)$$

For N isolated resonances, we define $\vartheta_j=\int_{x_r(j-1)}^{x_r(j)}[k_w(x')-\Delta k]dx'$ with $x_r(-1)=0$ and $j=1$ to N. We similarly obtain the electron displacement as a measure of the plasma wave growth in this case as

$$\xi(x)=\frac{S_0E_c(x)}{2v_p^2k_w^0}\sum_{j=1}^N L_{\text{eff}}(x_r(j))\cos\vartheta_j, \quad (39)$$

where $L_{\text{eff}}(x_r(j))$ is the effective growth length for the jth resonance point $x_r(j)$. Depending on the distribution of the resonances $k_w(x_r)=\Delta k=(\omega_2-\omega_1)/c$ and magnitude of the phase differences ϑ_j, the summation varies between $\bar{L}_{\text{eff}}N^{1/2}$ and $\bar{L}_{\text{eff}}N$ with $\bar{L}_{\text{eff}}$ being a mean resonance length of order $(\lambda_pL_n)^{1/2}$, where $\lambda_p=c/\omega_p$.

The reduction of the beat plasma wave amplitude due to inhomogeneity of the plasma density is by a factor of the ratio of the geometrical mean of the plasma collisionless skin depth λ_p and the density scale length L_n to the ideal beat-wave growth length.

IV. NUMERICAL EVALUATION OF PLASMA BEAT WAVE

In this section several examples of the effect on the plasma wave of the turbulent damping ν and the mean density variation $\langle n(x)\rangle$ are evaluated by numerical integration.

By virtue of the Lorentz transformation from the cold plasma rest frame to the beat-wave frame with $v_p=\Delta\omega/\Delta k$, the equation for the Lagrangian displacement $\xi(x,t)$ at fixed x becomes time independent with the independent variable $x'=-\gamma_pv_pt$. See Fig. 1 for this transformation. The driving force transforms according to $S(\Delta k(x-v_pt))=S(k_wx')$ with $k_w=\Delta k/\gamma_p$. We choose to leave the displacement ξ defined in the cold plasma frame rather than introducing $\xi'=\gamma_p\xi$ the same displacement observed in the wave frame. We note that the wave-breaking condition $\Delta k\xi=k_w\xi'=\pi$ is a Lorentz scalar.

In the following figures we use the dimensionless independent variable $\omega_pt=-\omega_px'/v_p\gamma_p$ for the distance from the head of the laser pulse. We take $\xi=\dot{\xi}=0$ for $x'\geq 0$ and a sharp turn-on of the beat-wave strength to $S_0=2c\omega_p\lambda$ for $x'\leq 0$. The value of ξ is measured in units of c/ω_p in the plasma rest frame. With these units the value of $\hat{\xi}_m=\max(\omega_p\xi/c)$ gives the reduction factor of the accelerating field E_m compared with the field $E_0=m_e\omega_{pe}c/e\cong[N(\text{particles/cm}^3)]^{1/2}$ V/cm of the ideal accelerator.

In the first case the relativistic detuning limit[12] of the ideal accelerator is shown. The Rosenbluth-Liu solution $\xi_{\text{RL}}(x')$ uses the approximation $\gamma_{\xi}^3\simeq1+\frac{3}{2}\xi^2/c^2$ in Eq. (4) and solves the resulting harmonic oscillator equation for exact resonance $\Delta\omega=\omega_p$ in a small λ expansion. We show in Fig. 2(a) (1) the exact numerical solution with γ_{ξ}^3, (2) the numerical solution with the approximation $\gamma_{\xi}^3=1+\frac{3}{2}\xi^2/c^2$, and (3) the Rosenbluth-Liu limiting amplitude $\xi_{\text{RL}}=4(\lambda/3)^{1/3}(c/\omega_p)$ given in Eq. (11). In making this comparison we define the amplitude as the maximum $\xi(x')$ over x' and also show the value of $x'_{\max}$ where $\xi(x')$ is maximum in Fig. 2(b).

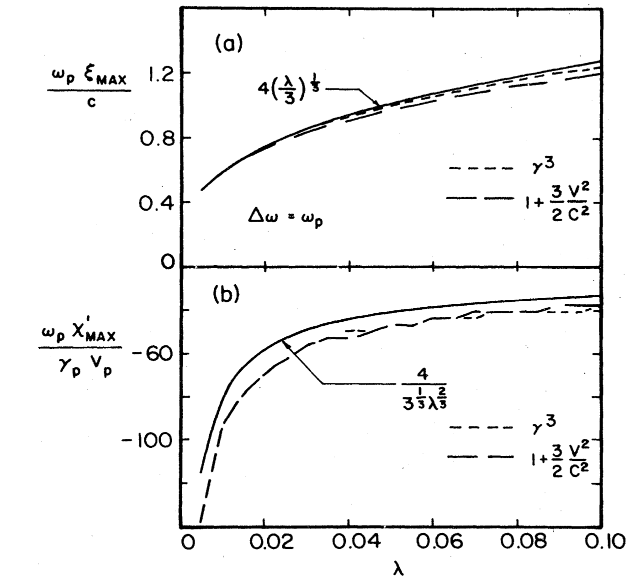

FIG. 2. Comparison of the maximum amplitude $\xi_{\max}$ in (a) and the location $x_{\max}$ of the maximum in (b) as a function of $\lambda = a_1 a_2 / 4\beta_p$ for (1) the exact γ^3 equation, (2) the $\gamma^3 = 1 + \frac{3}{2}(\xi^2/c^2)$ weak relativistic approximation, and (3) the Rosenbluth-Liu formulas.

In these studies we assume the shortest laser pulse length is $\omega_p T = L\omega_p / c = 50$ and allow the maximum pulse length to increase with small $a_1 a_2$ such that the laser power is fixed by $\omega_p T\lambda = \omega_p L\lambda / c \le 5$. The length of the pulse required to reach the maximum of $\xi(x')$ is given by Fig. 2(b) and follows well Eq. (12).

In Fig. 3 we vary the driving frequency $\Delta\omega/\omega_p = k_w v_p \gamma_p / \omega_p = \Delta k\, v_p / \omega_p$ through the resonant region at fixed λ. The peak of the resonance curve is shifted below $\Delta\omega/\omega_p = 1$ by an amount $\lambda^{2/3}$ as also reported by Tang *et al.*[19]

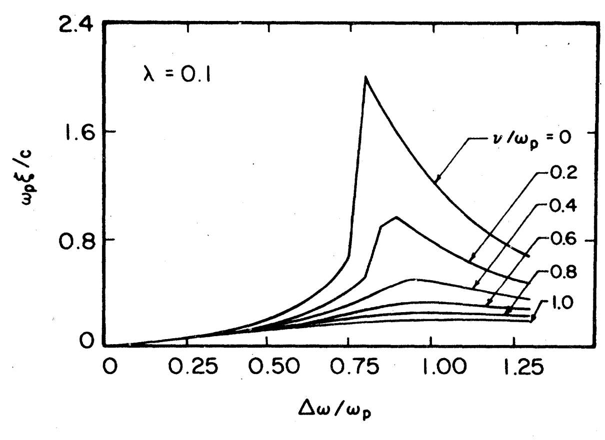

FIG. 3. Effect of turbulent damping on the maximum amplitude is shown as a function of $\Delta\omega/\omega_p$ for fixed λ and increasing ν/ω_p.

Now we consider the effect of the turbulent damping ν/ω_p on the maximum amplitude of the plasma wave. In Fig. 3 we show the effect of repeating the calculation taking into account increasing values of ν/ω_p. For this relatively strong value of λ the resonance is effectively lost for $\nu > \omega_p/5$. For smaller values of λ the condition on the maximum allowable ν/ω_p becomes progressively more severe. Analysis of the effect of ν/ω_p on the weakly relativistic oscillation is given by Lee *et al.*[20]

In Figs. 4 and 5 we show the ideal plasma wave growth ($\nu = 0$) for a strongly driven system ($\lambda = 0.1$) first at linear resonance $\Delta\omega = \omega_p$ and then near the maximum of the relativistic resonance. The relativistic velocity $v_x = \dot{\xi}$ of the thermal plasma is also shown and determines the detuning of the resonance. The detuning appears in Fig. 4 as the lengthening of the wavelength in the region near $\xi_{\max}$. Comparison of Figs. 4 and 5 shows that there is a gain in the maximum amplitude by increasing $\langle n \rangle$ and that the location of the maximum is shifted by three wavelengths toward the tail of the laser pulse.

Now we consider the case of nonuniform density variations in the wave frame. First we consider linearly increasing and decreasing density variations with $n(x') = N(1 \pm x'/L_n')$ for $|x'/L_n'| < 1$. In the laboratory frame the density disturbance is a pulse with scale $L_n = L_n'/\gamma_p$ and rise time L_n/v_p.

Figure 6 shows an example in which $n(x')/\langle n \rangle$ increases from $\frac{1}{2}$ at the head to $\frac{3}{2}$ at the tail of the laser beam with $L_n = 50c/\omega_p$. A weak maximum appears after

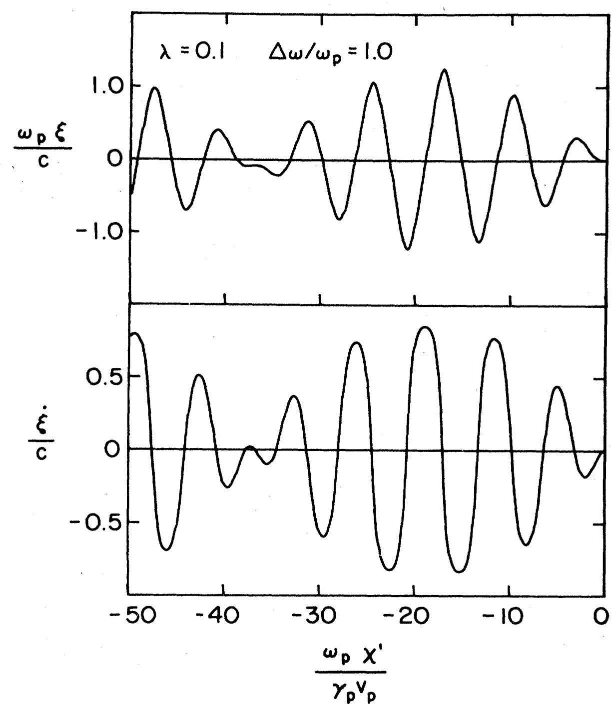

FIG. 4. Ideal relativistic plasma wave for exact resonance $\Delta\omega = \omega_p = \text{const}$.

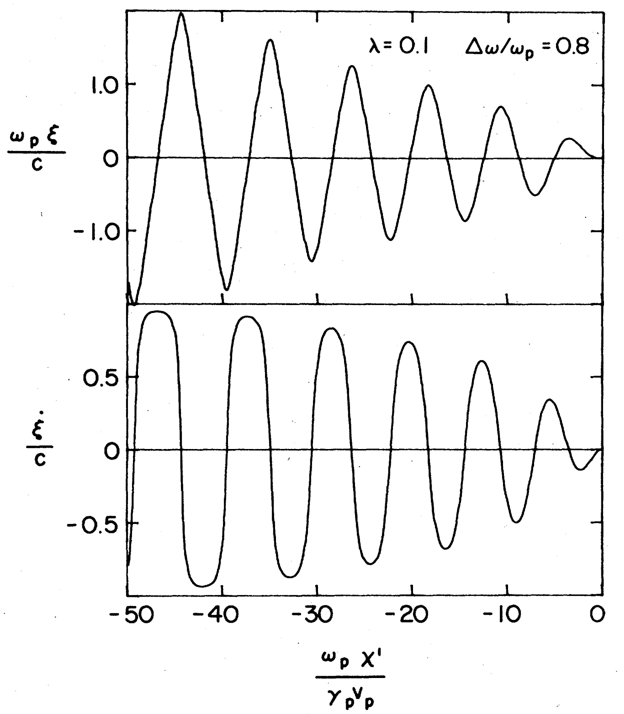

FIG. 5. Ideal relativistic plasma wave driven at $\Delta\omega = 0.8\omega_p = \text{const}$ to increase $\xi_{\max}$.

one wavelength, but the primary maximum does not occur for seven wavelengths. For the case in which the density decreases toward the tail the effect of the inhomogeneity is less severe.

In Fig. 7 the linear variation of $n(x)/\langle n \rangle$ starts at $\frac{3}{2}$ and decreases to $\frac{1}{2}$ at the tail of the laser pulse. The maximum amplitude is now 10% larger and occurs after four wavelengths. In both cases, however, the resonant growth region is limited to a region of order $L'_{\text{eff}} = (v_p \gamma_p L'_n / \omega_p)^{1/2}$ or $L_{\text{eff}} = (cL_n/\omega_p)^{1/2}$ about the local resonance condition. The maximum amplitude is found to vary approximately as $\xi_n = \lambda L_{\text{eff}}/v_p$.

Finally, we show the case for a modulation of the density $n(x') = N[1 + \epsilon \sin(q_w x' + \phi)]$. For $q_w \ll k_w$ the previous formula for the linear gradient applies with $L_n \sim 1/\epsilon q_w$. For $q_w \gg k_w$ the oscillation of $n(x')$ averages out with little net effect on $\xi(x')$. Finally, we note that for $q_w = 2k_w$ the equation becomes the driven Mathieu equation with unstable bands in which the q_w modulation resonantly transfers energy between the plasma wave and the modulation. We do not expect this last regime to be of importance since the Mathieu resonance conditions are unlikely to be satisfied in practice.

Figures 8(a) and 8(b) show the resonant case with $q_w = 2k_w$ and $\epsilon = 0.5$. In case (a) $\phi = 0$ and case (b) $\phi = \pi$. The maximum amplitudes reached are nearly identical in the two cases; however, the number of wavelengths required to reach $\xi_{\max}$ depends on ϕ and is much shorter in case (b).

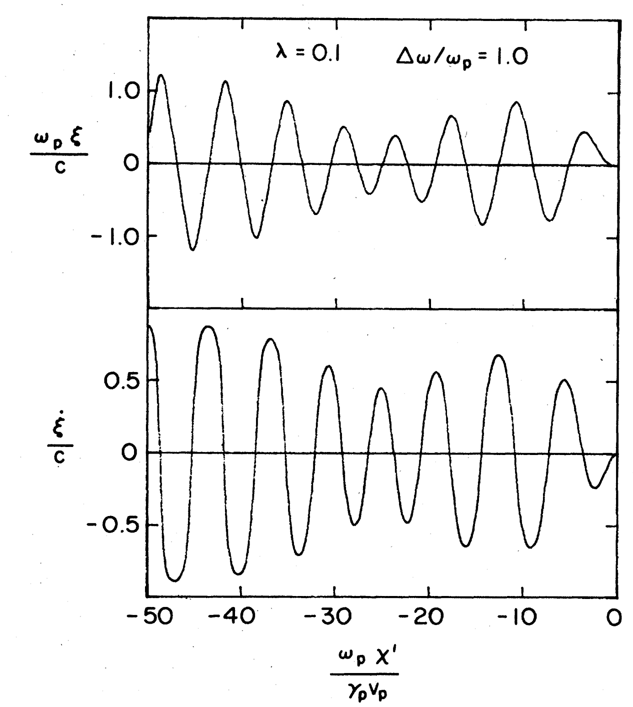

FIG. 6. Relativistic plasma wave with $L_n = 50c/\omega_p$ with the density increasing toward the tail of the beam.

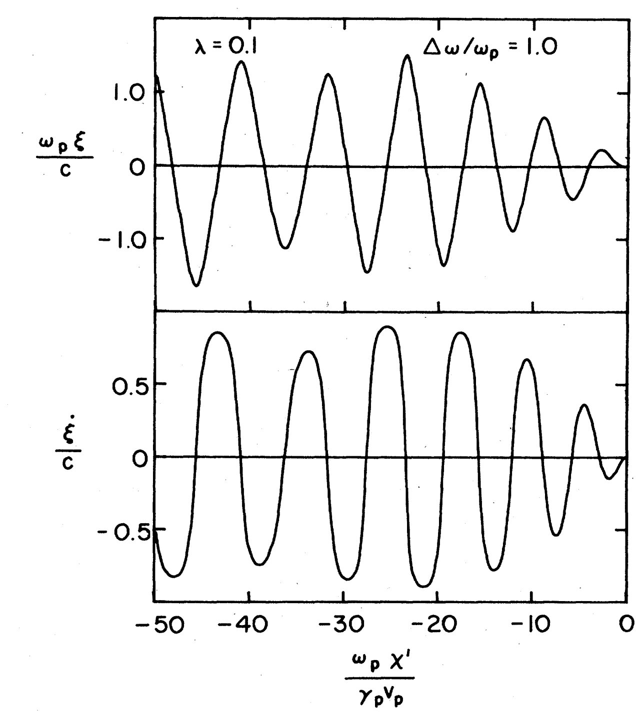

FIG. 7. Relativistic plasma wave with density increasing toward the head of the beam.

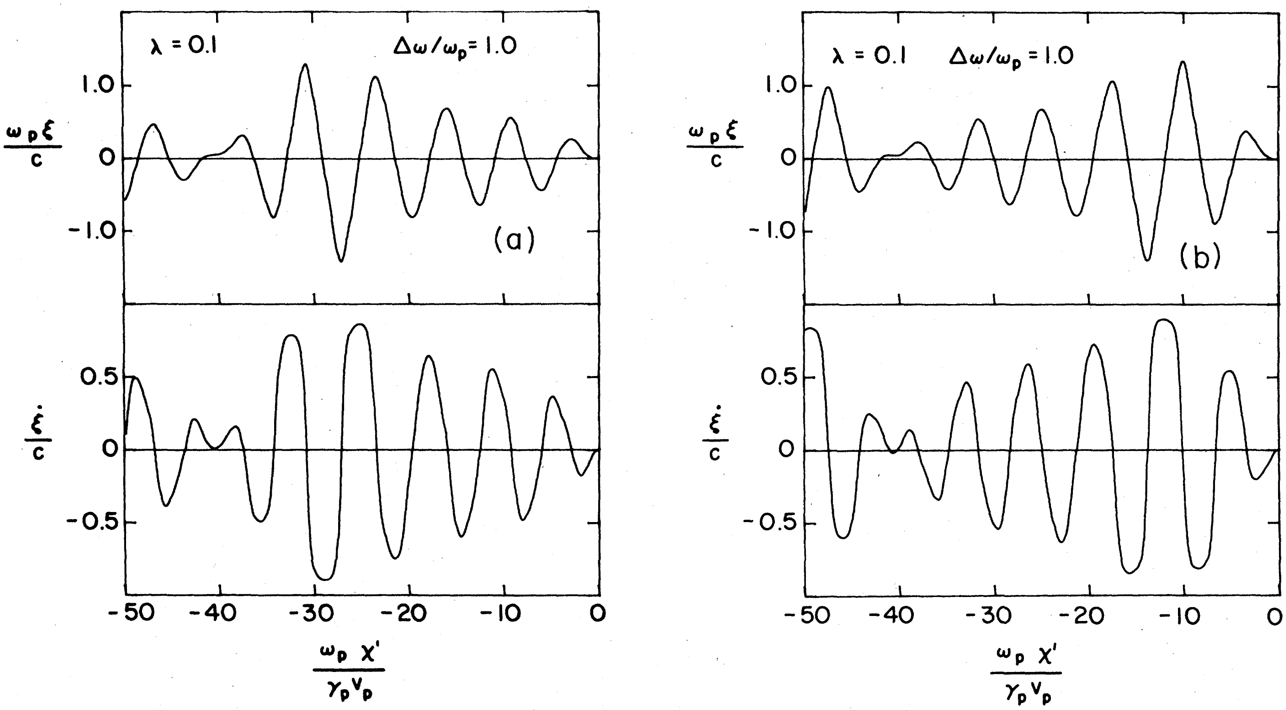

FIG. 8. Relativistic plasma waves with sinusoidal variations $[1\pm0.5\sin(2k_w x')]$ of the density with the + sign in (a) and the − sign in (b).

V. CONCLUSIONS

We have analyzed the effect of variations of the background plasma density on the growth of the plasma wave in the beat-wave accelerator. In general the results of the variations are to limit the length over which the ponderomotive force is effective in driving the plasma wave. We call this effective growth length $L_{\rm eff}$ and give several approximate formulas for $L_{\rm eff}$ in different regimes for both noisy and weakly varying plasma density cases. The results are summarized in Table I.

The noise effect of limiting the growth of the beat-wave amplitude may play an important role in explaining the experiment.[11] In the experiment[11] a single-frequency-laser light was shone on a plasma with various densities, yielding no evidence of high-energy electrons. The laser HELIOS with a single-frequency arm with a rise time from 300 to 1000 ns was injected into a preformed plasma. The laser power density is such that the factor a, the ratio of the quivering velocity to the speed of light, is of order unity. Subsequent numerical simulations by Kindel and Forslund[9] of the experimental conditions confirmed the experimental observation. They carried out two-dimensional electromagnetic particle simulation of both a single-frequency laser and two-frequency lasers injections. The simulation shows that when a single-frequency laser is injected into a plasma, few relativistic electrons are produced. On the other hand, when laser beams with two

TABLE I. Several approximate formulas for $L_{\rm eff}$ in different regimes for both noisy and weakly varying plasma density cases.

	Accel. field $\dfrac{eE_s}{m\omega_p c}=\dfrac{\omega_p\xi_s}{c}$	Condition $\lambda=\dfrac{a_1a_2}{4\beta p}$	Growth length $L_{\rm eff}$
Breaking limit	1	$\lambda>0.05$ $(a_1a_2>0.2)$	$\dfrac{c}{\omega_p\lambda}\le 20\dfrac{c}{\omega_p}$
Relativistic detuning	$2.8\lambda^{1/3}$	$\lambda<0.05$	$2.8\dfrac{c}{\omega_p\lambda^{2/3}}$
Plasma noise	$\dfrac{\lambda\omega_p}{\nu}$	$\dfrac{\nu}{\omega_p}>0.36\lambda^{2/3}$	$\dfrac{c}{\nu}$
Inhomogeneity	$\lambda\left[\dfrac{cL_n}{\omega_p}\right]^{1/2}$	$\dfrac{\omega_p L_n}{c}<\dfrac{7.8}{\lambda^{4/3}}$	$\left[\dfrac{cL_n}{\omega_p}\right]^{1/2}$

frequencies are injected, a substantial number of relativistic electrons are produced. The process of acceleration is mitigated by a slower rise time. However, even in cases with a slower rise time the overall qualitative features remain the same.

These findings may be interpreted in light of our present theory. When the input laser consists of only one frequency, the other light waves have to rise from the noise as a result of the forward Raman instability. This situation allows for many different modes to grow simultaneously and leads to a noisy plasma. In light of the present theory, when the condition Eq. (24) is satisfied, noise of the plasma thus generated can limit the amplitude of the beat plasma wave below the trapping level (or the wave-breaking limit).[1] If it does, the laser light would fail to trap electrons in the bulk and to accelerate to relativistic energies.

The application of the formulas given in Table I requires knowledge of the character of the plasma turbulence or plasma density variations. There would appear to be many possible regimes of plasma noise and density variations. Here, we only mention the one case which would appear to be dangerous in view of the very large energy densities in the driven plasma wave $E_L^2 \lesssim nmc^2$. If the laser power is enough so that the beat wave grows to the wave-breaking limit (trapping level) before the laser generates plasma noise that retards the plasma wave growth, we predict a strong plasma wave near the head of laser light. Even in this case, it is possible to have noise behind the coherent region. In addition to this direct excitation of noise by laser light, it is possible to have secondary plasma noise generated through the modulational instability.[18] A second condition for coherent wave growth is that the driven plasma avoids undergoing the modulational instability. We estimate that the modulational instability can be avoided provided that $T_e \gg 280$ eV so that the driven plasma wave has $k_p \cong \omega_p/c \gg k_* = (m_e/m_i)^{1/2}\omega_{pe}/v_e$.

In the absence of the modulational instability we may estimate that the correlation length L_c of generic plasma noise is characterized by c/ω_p and the correlation time by $v_p/L_c \sim \omega_p$ in applying the formulas in Table I. For these fluctuation scales the weak stochasticity limit would apply for density fluctuations with $\sigma \equiv \langle \delta n^2 \rangle / N^2 \ll 1$. The effective interaction length is then given by $L_{\text{eff}} \simeq 2c/\omega_p^2 \tau_c \sigma \simeq (2c/\omega_p)(N^2/\langle \delta n^2 \rangle)$. It should not impose a serious limit to the accelerator provided $\langle \delta n^2 \rangle / N^2 < 2c/\omega_p L$. Such a condition may be realized if two resonant lasers are applied and if they are strong enough.

When the density varies smoothly on the scale c/ω_p but contains only one resonance point where $ck_w(x) = \omega_2 - \omega_1$, then the effective growth length is limited to the geometric mean of the density gradient scale length L_n and the plasma wavelength c/ω_p. Sufficient growth may occur in this regime provided L_n is sufficiently large.

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