Projection-operator method for the nonlinear three-wave interaction

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A theory of nonlinear three-wave interaction is presented, where a finite bandwidth of the interacting waves is considered ($\Delta\omega \neq 0$). Dissipative processes are neglected, and a Hamiltonian formulation is used. The evolution equations for the wave intensities are obtained with the aid of a projection-operator method, similar to those used in nonequilibrium statistical mechanics, but our formulation is deterministic and no statistical hypothesis is needed. These equations generalize the well-known fixed-phase equations ($\Delta\omega = 0$) and are formally analogous to them, if the ballistic and memory effects are neglected.

I. INTRODUCTION

The nonlinear evolution of a wave in a plasma is governed by two different processes: the wave-particle interactions and the wave-wave interactions.[1] The dominant aspect of the wave-wave interaction is the so-called three-wave interaction. In many situations, where the wave-particle effects are negligible, the wave evolution mainly depends on the three-wave interaction processes. This is relevant, for instance, for the stimulated Brillouin and Raman scattering leading to anomalous reflection and absorption of a laser beam interacting with a plasma.[2]

The study of nonlinear three-wave interaction is usually done using two different approximations. The first one is called the fixed-phase approximation[3] and was first developed in the field of nonlinear optics.[4] Its use can be justified when the spectral width $\Delta\omega$ of the waves involved in the interaction is much less than the inverse of the characteristic time τ for the energy exchanged between them.[5] This approximation deals with coherent waves propagating with a slowly modulated amplitude. The second approximation is known as the random-phase approximation and it was developed in the framework of the weak-turbulence theories.[1,5,6] It deals with a large number of plane waves with random phases and its use can be accepted in the reverse situation, when the wave spectral width $\Delta\omega$ is much larger than τ^{-1}.

When applied to particular problems these two different theories can lead to qualitatively different conclusions. Recently, attention has been given to the instability saturation of waves by subharmonic generation or more generally by three-wave decay into stable waves. This is pertinent to the estimation of drift wave saturation levels.[7] In this case the fixed-phase theory shows that a strange attractor can occur and the amplitude of the unstable wave can behave chaotically.[8,9] But it is easy to show, using the random-phase approximation, that in quite general conditions the unstable wave amplitude tends to a well-defined saturation level and no chaos is observed.[10] Similar qualitative differences between the random- and fixed-phase results have already been noted in the study of stimulated rescattering.[11]

To our knowledge there is no consistent theory for the wave interactions with arbitrary spectral width $\Delta\omega$, including the intermediate case $\Delta\omega \simeq \tau^{-1}$. Most of the time, finite spectral width effects are added to the fixed-phase equations, introducing phenomenological phase fluctuations[12] or considering a finite number of wave triplets.[13] As an exception we refer the work of Nishikawa and Fried[14] which presents a wave-packet formulation of electrostatic turbulence.

In the present work we describe a different, and we hope more consistent, approach to the study of nonlinear three-wave interaction, considering an arbitrary spectral width which includes the case $\Delta\omega \simeq \tau^{-1}$. This approach is based on a projection-operator technique. Projection operators are commonly used in nonequilibrium statistical mechanics in order to derive macroscopic transport equations from the microscopic dynamical equations.[15] The statistical properties of the systems are usually included in the definition of the projection operators, but we show in this work that the same kind of techniques can be useful in the frame of a deterministic description and no statistical hypothesis is needed.

The evolution equations for the wave amplitude and phases obtained with our projection-operator method take the form of generalized Langevin equations. They contain three different terms. The first one, which we call the macroscopic term, is formally analogous to that appearing in the fixed-phase equations and reduces to it in the limit $\Delta\omega \to 0$. However, the two other terms have no equivalent in the fixed-phase equations. One of them is a time integral which is associated with memory effects. Its presence means that the wave-wave interaction in a plasma is not a Markovian process and it cannot be described as instantaneous collisions of wave quanta (plasmons, photons, or phonons) as is usually done in the weak-turbulence theory.[5] The third term is associated with the evolution of the initial perturbations and can be called the ballistic term.

The plan of this paper is the following. In Sec. II we describe our model for the three-wave interaction and show how a suitable projection operator can be defined. The model mainly consists of three waves, each one con-

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taining a large number of monochromatic modes. Each mode is allowed to interact with all the other modes, except those belonging to the same wave. Dissipation is neglected and a Hamiltonian description is used. In Sec. III we derive the evolution equations for the total intensity and phase of the three waves. Section IV is devoted to the discussion of the macroscopic dynamics and to the explicit calculation of the macroscopic terms. In Sec. V we consider the memory effects in detail and discuss the implications of these effects to the overall consistency of the theory. Finally, in Sec. VI we state the conclusions.

II. PROJECTION OPERATORS

Let us consider three waves, each one containing a large number of modes, $n \gg 1$ (see Fig. 1). Each mode is monochromatic, having a frequency ω_{jk} where $j=1,2,3$ and $k=1,2,\ldots,n$. Its evolution is described with the aid of two variables: the photon number, or wave action, N_{jk} and the phase θ_{jk}. Actually, the modes (j,k) are only nearly monochromatic, in the sense that their amplitude and phase are allowed to slowly vary in time. They are very narrow wave packets, as defined by Nishikawa and Fried.[14] The spectral width of the three waves $\Delta\omega_j \simeq \omega_{j1} - \omega_{jn}$ is assumed arbitrary but small enough to prevent nonlinear three-wave interactions inside the spectrum of one wave. Then, each mode (j,k) exchanges energy via the three-wave coupling process with all the other modes $(i,1),(i,2),\ldots,(i,n)$ (with $i \neq j$). If dissipation is neglected, the evolution of such a system can be described with the aid of the following Hamiltonian:[3]

$$H(N_{jk},\theta_{jk}) = \sum_{j=1}^{3}\sum_{k=1}^{n} \omega_{jk}N_{jk} + 2w\sum_{p=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \sqrt{N_{1p}n_{2k}N_{3l}}\sin\theta_{pkl} , \quad (1)$$

where

$$\theta_{pkl} = \theta_{1p} - \theta_{2k} - \theta_{3l} . \quad (2)$$

Here we have assumed for simplicity that the nonlinear coupling coefficient w is a constant and does not depend

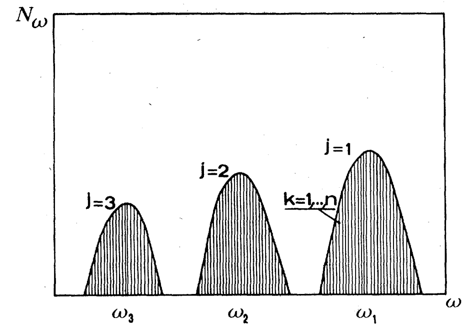

FIG. 1. Model for the three-wave nonlinear interaction. The frequency spectrum is composed by three distinct waves ($j=1,2,3$) with a finite bandwidth. Each wave contains $n \gg 1$ internal modes.

on the frequencies of each mode triplet $(1p,2k,3l)$. This is only approximately true and requires that $\Delta\omega_j$ are very small. However, this assumption is not a limitation to the theory because the generalization to the case where $w \equiv w(\omega_{1p},\omega_{2k},\omega_{3l})$ is straightforward.

When the nonlinear coupling is absent ($w=0$) the equations of motion reduce to

$$N_{jk} = \text{const}, \quad \theta_{jk} = \omega_{jk}t + \phi_{jk} , \quad (3)$$

where $\phi_{jk} = \text{const}$. Clearly, N_{jk} and θ_{jk} are the action-angle variables describing an ensemble of $3n$ independent harmonic oscillators. In the general case ($w \neq 0$), however, the canonical equation of motion for the Hamiltonian (1) became quite complicated and can be written as

$$\frac{dN_{1p'}}{dt} = -2w\sum_{k=1}^{n}\sum_{l=1}^{n} \sqrt{N_{1p'}N_{2k}N_{3l}}\cos\theta_{p'kl} ,$$

$$\frac{dN_{2k'}}{dt} = +2w\sum_{p=1}^{n}\sum_{l=1}^{n} \sqrt{N_{1p}N_{2k'}N_{3l}}\cos\theta_{pk'l} , \quad (4)$$

$$\frac{dN_{3l'}}{dt} = +2w\sum_{p=1}^{n}\sum_{k=1}^{n} \sqrt{N_{1p}N_{2k}N_{3l'}}\cos\theta_{pkl'} ,$$

and

$$\frac{d\theta_{p'k'l'}}{dt} = \Delta\omega_{p'k'l'} + w\left[\sum_{k=1}^{n}\sum_{l=1}^{n}\left[\frac{N_{2k}N_{3l}}{N_{1p'}}\right]^{1/2}\sin\theta_{p'kl} - \sum_{p=1}^{n}\sum_{l=1}^{n}\left[\frac{N_{1p}N_{3l}}{N_{2k'}}\right]^{1/2}\sin\theta_{pk'l} - \sum_{p=1}^{n}\sum_{k=1}^{n}\left[\frac{N_{1p}N_{2k}}{N_{3l'}}\right]^{1/2}\sin\theta_{pkl'}\right] . \quad (5)$$

Here we have used the frequency mismatch

$$\Delta\omega_{pkl} = \omega_{1p} - \omega_{2k} - \omega_{3l} . \quad (6)$$

Obviously, Eqs. (4) and (5) could be obtained directly from the Maxwell's equations and the (kinetic or hydrodynamic) plasma equations. But the Hamiltonian derivation used here is more convenient for our purposes. The wave action N_{jk} is related to the electric field E_{jk} of the plasma eigenmode (j,k) through the expression

$$N_{jk}=\frac{1}{4}\left[\frac{\partial D}{\partial\omega}\right]_{\omega_{jk}}|E_{jk}|^2\ , \tag{7}$$

where $D\equiv D(\omega,k)=0$ is the dispersion relation pertinent for this mode.

When we make $n=1$ in Eqs. (4) and (5) we obtain the fixed-phase equations, which have already been studied in detail.[3] In the general case, $n\gg 1$, the system of $3n$ modes evolves in a $6n$-dimensional phase space. We will call it the microscopic phase space, because it contains all the information about the internal structure of the three groups of n modes each.

We now look for the representation of the motion in a reduced phase space with six dimensions, where we try to represent the evolution of the three waves as a whole. This space will be called the macroscopic phase space, and the corresponding six macroscopic variables are defined as

$$N_j=\sum_{k=1}^{n}N_{jk},\quad \theta_j=\sum_{k=1}^{n}\theta_{jk} \tag{8}$$

with $j=1,2,3$. It is obvious that N_j and θ_j are linearly independent variables and can be used as a basis in the macroscopic phase space, which is a subspace of the microscopic one. N_j is the total number of photons, or the total action, associated with the wave j and is related to the total wave intensity by $I_j=\bar{\omega}_j N_j$, where the mean frequency $\bar{\omega}_j$ is defined by

$$\bar{\omega}_j=\frac{1}{N_j}\sum_{k=1}^{n}\omega_{jk}N_{jk}\ . \tag{9}$$

The wave intensity I_j, or the mean number of photons $\bar{N}_j=N_j/n$, could also be used to replace N_j as macroscopic variables. The physical meaning of θ_j is somewhat less clear than that of N_j, but we can call it the total phase for the wave j. It could also be replaced by the mean phase: $\bar{\theta}_j=\theta_j/n$.

In order to obtain a subdynamics in this six-dimensional phase space we have to define a suitable projection operator P. Let $u(x)$ be an arbitrary dynamical variable defined on the microscopic phase space, denoting by x the ensemble of the $6n$ canonical variables N_{jk} and θ_{jk}. The projection operator P acts on every $u(x)$, reducing it to a function of the six macroscopic variables, $u(\mathbf{N},\boldsymbol{\theta})$, where $\mathbf{N}\equiv(N_1,N_2,N_3)$ and $\boldsymbol{\theta}\equiv(\theta_1,\theta_2,\theta_3)$:

$$u(\mathbf{N},\boldsymbol{\theta})=Pu(x)\ . \tag{10}$$

We define P in the following way:[16,15]

$$Pu(x)=\frac{1}{S(\mathbf{N},\boldsymbol{\theta})}\int u(x)\delta(\mathbf{N}(x)-\mathbf{N})\delta(\boldsymbol{\theta}(x)-\boldsymbol{\theta})dx\ , \tag{11}$$

where

$$dx\equiv\prod_{j=1}^{3}\prod_{k=1}^{n}dN_{jk}d\theta_{jk}\ . \tag{12}$$

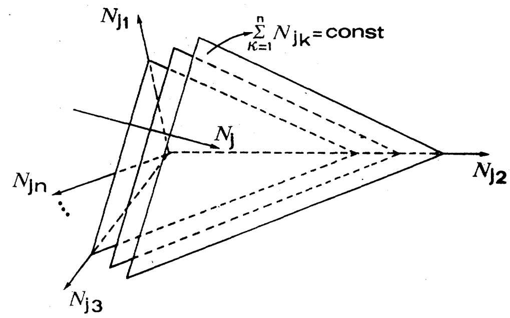

FIG. 2. Schematic representation of the microscopic $(N_{j1},N_{j2},\ldots,N_{jn})$ and macroscopic (N_j) variables. The projection operator P integrates each dynamic variable u over the plane $\sum_k N_{jk}=N_j=$const represented here.

$S(\mathbf{N},\boldsymbol{\theta})$ is a normalization coefficient defined as

$$S(\mathbf{N},\boldsymbol{\theta})=\int\delta(\mathbf{N}(x)-\mathbf{N})\delta(\boldsymbol{\theta}(x)-\boldsymbol{\theta})dx\ . \tag{13}$$

Its presence in Eq. (11) is necessary to assure that P is a true projection operator: $P^2=P$. Let us explain in more detail the meaning of this projection operator. The product of the two Dirac δ functions appearing in Eq. (11) represents in fact a product of six δ functions of the form $\delta(N_j(x)-N_j)$, or $\delta(\theta_j(x)-\theta_j)$, with $j=1,2,3$. By $N_j(x)$, or $\theta_j(x)$, in these expressions we want to represent N_j, or θ_j, as functions of the microscopic variables, as defined by Eqs. (8). And by N_j, or θ_j, we represent the actual value of the macroscopic variables. Then, each of the six δ functions defines a hyperplane in the microscopic space, as represented in Fig. 2. The macroscopic variables are then the coordinates which parametrize these hyperplanes. And the projection of the dynamic variable $u(x)$ is the amount of $u(x)$ contained on the hyperplane defined by the macroscopic variables $(\mathbf{N},\boldsymbol{\theta})$. On the other hand, the normalization coefficient $S(\mathbf{N},\boldsymbol{\theta})$ can be seen as the area of this hypersurface.[16]

A simpler projection operator P' could be defined, which is a mere integration over $6(n-1)$ of the $6n$ microscopic variables, if we make a suitable orthogonal canonical transformation over the Hamiltonian (1), in such a way that the variables N_j and θ_j defined by Eq. (8) appear now as microscopic variables (see Appendix A). But this new form of projection operator is less suited for the explicit calculations described in Secs. IV and V, because the domains of integration on the hyperplanes are not so clearly defined. For this reason we conserve the form (11).

III. EVOLUTION EQUATIONS

Using well-established results of the Hamiltonian dynamics we can now derive an evolution equation for the dynamical variable $u(x)$. If $u(x)$ does not depend explicitly on time we can write[17]

$$\frac{d}{dt}u(x)=iLu(x)\ , \tag{14}$$

where L is the Liouvillian operator:

$$iL=-\{H,\ \}\ . \tag{15}$$

Equation (14) can be formally integrated to give

$$u(x)=e^{itL}u(0)\ , \tag{16}$$

where $u(0)\equiv u(x(t=0))$ is the initial value of $u(x)$. Taking the total derivative of Eq. (16) with respect to time we obtain a different evolution equation, which is equivalent to (14):

$$\frac{d}{dt}u(x)=\left[\frac{d}{dt}e^{itL}\right]u(0)\ . \tag{17}$$

Now we can use the following operator identity:[18]

$$\frac{d}{dt}e^{itL}=e^{itL}iL_0+\int_0^t e^{i(t-s)L}iL_0e^{is(L-L_0)}i(L-L_0)ds+e^{it(L-L_0)}i(L-L_0)\ , \tag{18}$$

where L_0 is an arbitrary operator. In our problem it is convenient to define L_0 as the projection of the Liouvillian operator on the macroscopic phase space:

$$L_0=PL\ . \tag{19}$$

Using Eqs. (18) and (19) in Eq. (17) and replacing $u(x)$ by N_j and θ_j, we obtain the evolution equations for the total number of photons and the total phase of the three interacting waves:

$$\begin{aligned}\frac{dN_j}{dt}&=P\frac{dN_j}{dt}+\int_0^t M(s)N_j(t-s)ds+B(t)N_j(0)\ ,\\ \frac{d\theta_j}{dt}&=P\frac{d\theta_j}{dt}+\int_0^t M(s)\theta_j(t-s)ds+B(t)\theta_j(0)\ ,\end{aligned} \tag{20}$$

with $j=1,2,3$. There we have used two new operators, $M(t)$ and $B(t)$, which depend on P and L:

$$\begin{aligned}M(t)&=PLB(t)\ ,\\ B(t)&=e^{it(1-P)L}i(1-P)L\ .\end{aligned} \tag{21}$$

Let us make some comments on Eqs. (20). These equations show that the total evolution of N_j and θ_j is determined by three different terms. The first term is just the projection of such an evolution on the macroscopic phase space. We can call it the macroscopic term. The second one describes the influence of the past on the actual value of N_j and for this reason is called the memory term. The third term shows that the actual value of N_j also depends explicity on the free streaming of the initial conditions. In a certain sense it is also a memory term because it contains the memory of the initial perturbation. But it is not a cumulative term as the previous one and we can call it the ballistic term. For the same reasons we refer to $M(t)$ and $B(t)$ as the memory and the ballistic operators, respectively.

If we want to have a simplified description of Eqs. (20) we can assume, for a moment, that N_j do not significantly change during the characteristic time for the nonlinear wave interaction:

$$N_j(t-s)\simeq N_j(t)\quad \text{for } s\lesssim\tau\ .$$

This is equivalent to assuming a Markovian process, and the memory reduces to

$$\int_0^t M(s)N_j(t-s)ds\simeq\gamma(t)N_j(t)\ , \tag{22}$$

where $\gamma(t)$ is some macroscopic dissipation. In a similar attempt to simplify we can see that $N_j(0)$ depend on a great number of microscopic initial values which will evolve independently from each other, and we can imagine that the free streaming of these initial conditions corresponds to very complicated oscillations. We can then eventually replace $B(t)N_j(0)$ by some random variable $\widetilde{R}(t)$. Then, Eqs. (20) are replaced by evolution equations of the form

$$\frac{dN_j}{dt}=P\frac{dN_j}{dt}+\gamma(t)N_j+\widetilde{R}(t)\ . \tag{23}$$

This is analogous to the usual Langevin equation of motion for a Brownian particle in a fluid. The deterministic term, or the macroscopic projection, represents now the external forces acting on the particle, for instance, gravity or the interacting forces due to other Brownian particles. The memory effect is here the viscous damping due to the mean friction of the fluid on the particle. And the ballistic term is the random force due to the collisions with the molecules of the fluid.

In our problem the Brownian particle is the wave j and the molecules are the microscopic modes (i,k), with $k=1,2,\ldots,n$ and $i\neq j$. But in general the nonlinear wave interaction is not Markovian, because by definition N_j change significantly during a time interval of the order of τ, and (22) is not valid. On the other hand, it is not obvious that the ballistic term $B(t)N_j(0)$ can be simply assimilated to a random force.

IV. MACROSCOPIC DYNAMICS

We now present explicit calculations of the macroscopic terms appearing in Eqs. (20). Replacing N_j by (8) in the arguments of the Dirac δ functions which are present in the definition of P, and after some rearrangements, we can write the following expression for the macroscopic terms of N_1:

$$P\frac{dN_1}{dt}=-\frac{2w}{S}\sum_{i=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}A_{jst}(\mathbf{N})B_{jst}(\boldsymbol{\theta})\ , \tag{24}$$

where we define

$$\begin{aligned}A_{jst}(\mathbf{N})&=\sum_{\lambda=1}^{3}A_\lambda\ ,\\ B_{jst}(\boldsymbol{\theta})&=\int\left[\prod_{\lambda=1}^{3}B_\lambda\right]\cos\theta_{jst}d\theta_{1j}d\theta_{2s}d\theta_{3t}\ ,\end{aligned} \tag{25}$$

and

$$A_1=\int\sqrt{N_{1j}}\,\delta(N_1(x)-N_1)\prod_{k=1}^{n}dN_{1k}\ . \tag{26}$$

The expressions for A_2 and A_3 can be obtained by replacing in this expression the subscripts 1 and j by $(2,s)$ and $(3,t)$, respectively. The coefficients B_λ are defined by

$$B_\lambda = \int \delta(\theta_\lambda(x)-\theta_\lambda) \prod_{\substack{k=1 \\ k\neq j}}^{n} d\theta_{\lambda k} \ . \tag{27}$$

Taking into account the hyperplanes determined by (8) and schematized in Fig. 2 we can write explicitly the limits of integration for the integrals appearing in Eqs. (26) and (27). We only retain in the integration that portion of the hyperplanes corresponding to positive values of N_j and θ_j. Equation (26) then becomes

$$A_1 = \int_0^{N_1} dN_{1n} \int_0^{N_1-N_{1n}} dN_{1,n-1} \cdots \int_0^{N_1 - \sum_{\alpha=2, \alpha\neq j}^{n} N_{1\alpha}} dN_{11} \left[N_1 - \sum_{\substack{\alpha=1 \\ \alpha\neq j}} N_{1\alpha} \right]^{1/2} , \tag{28}$$

with similar expressions for A_2 and A_3. In the same way we get from Eq. (27)

$$B_\lambda = \int_0^{\theta_\lambda} d\theta_{\lambda n} \int_0^{\theta_\lambda-\theta_{\lambda n}} d\theta_{\lambda,n-1} \cdots \int_0^{\theta_\lambda - \sum_{\alpha=2, \alpha\neq j}^{n} \theta_{\lambda\alpha}} d\theta_{\lambda 2} \ . \tag{29}$$

After carrying out the $(n-1)$ integrations for A_λ and the $(n-2)$ integrations for B_λ, we obtain

$$A_\lambda = \frac{2^{n-1}}{\prod_{p=1}^{n-1}(2p+1)} N_\lambda^{n-1/2}, \quad B_\lambda = \frac{(\theta_\lambda - \theta_{\lambda j})^{n-2}}{(n-2)!} \ . \tag{30}$$

Replacing these results in Eqs. (25) we can write

$$A_{jst}(\mathbf{N}) = \left[\frac{2^{n-1}}{\prod_{p=1}^{n-1}(2p+1)} \right]^3 (N_1 N_2 N_3)^{n-1/2} \tag{31}$$

and

$$B_{jst}(\boldsymbol{\theta}) = \frac{1}{[(n-2)!]^3} \times \int_{\theta_1}^{0} dx \int_{\theta_2}^{0} dy \int_{\theta_3}^{0} dz \cos(\theta - x + y + z) \times (xyz)^{n-2} , \tag{32}$$

where we have defined a macroscopic phase difference θ and used auxiliary variables x, y, and z such that

$$\theta = \theta_1 - \theta_2 - \theta_3, \quad x = \theta_1 - \theta_{1j} ,$$
$$y = \theta_2 - \theta_{2s}, \quad z = \theta_3 - \theta_{3t} \ . \tag{33}$$

The phase difference θ is not to be confused with the vector $\boldsymbol{\theta} \equiv (\theta_1, \theta_2, \theta_3)$. Developing the cosine in (32) we can write $B_{jst}(\boldsymbol{\theta})$ in a more appropriate form:

$$B_{jst}(\boldsymbol{\theta}) = \frac{1}{[(n-2)!]^3} [M(\boldsymbol{\theta}) \sin\theta - N(\boldsymbol{\theta}) \cos\theta] , \tag{34}$$

where the expressions for $M(\boldsymbol{\theta})$ and $N(\boldsymbol{\theta})$ are

$$M(\boldsymbol{\theta}) = g_1(f_2 f_3 - g_2 g_3) - f_1(g_2 f_3 + g_3 f_2) ,$$
$$N(\boldsymbol{\theta}) = f_1(f_2 f_3 - g_2 g_3) + g_1(g_2 f_3 + g_2 f_2) \ . \tag{35}$$

f_λ and g_λ are well-known integrals defined by

$$\left. \begin{matrix} f_\lambda \\ g_\lambda \end{matrix} \right\} = \int_0^{\theta_\lambda} u^{n-2} \times \left\{ \begin{matrix} \cos u \\ \sin u \end{matrix} \right\} \times du \ . \tag{36}$$

We see from Eqs. (32) and (34) that the functions $A_{jst}(\mathbf{N})$ and $B_{jst}(\boldsymbol{\theta})$, which determine the action and angle dependence of the macroscopic projection of the time derivative of N_λ, are independent of the subscripts. This means that we only have to calculate the normalization factor S in order to evaluate the macroscopic terms of (20). Rearranging the integrals of Eq. (13) we obtain

$$S = \prod_{\lambda=1}^{3} D(\theta_\lambda) D(N_\lambda) , \tag{37}$$

where

$$D(N_\lambda) = \int d(N_\lambda(x) - N_\lambda) \prod_{k=1}^{n} dN_{\lambda k} \tag{38}$$

and $D(\theta_\lambda) \equiv D(N_\lambda \to \theta_\lambda)$. Using the same limits of integration as before, we get

$$D(N_\lambda) = \int_0^{N_\lambda} dN_{\lambda n} \int_0^{N_\lambda - N_{\lambda n}} dN_{\lambda, n-1} \cdots \times \int_0^{N_\lambda - \sum_{\alpha(\neq 1)} N_{\lambda\alpha}} dN_{\lambda 1} = \frac{N_\lambda^{n-1}}{(n-1)!} \ . \tag{39}$$

The normalization factor then becomes

$$S = \frac{1}{[(n-1)!]^6} \prod_{\lambda=1}^{3} N_\lambda^{n-1} \theta_\lambda^{n-1} \ . \tag{40}$$

Now, using Eqs. (31), (34), and (40) we can write (14) in the final desired form,

$$P \frac{dN_1}{dt} = -2w \sqrt{N_1 N_2 N_3} F(\boldsymbol{\theta}) , \tag{41}$$

where $F(\boldsymbol{\theta})$ contains all the phase dependence:

$$F(\theta)=\frac{F(n)}{(\theta_1\theta_2\theta_3)^{n-1}}[M(\theta)\sin\theta-N(\theta)\cos\theta] ,$$
$$F(n)=\left[\frac{2^{n-1}n^2}{\prod_{p=1}^{n-1}(2p+1)}(n-1)!\right]^3 . \quad (42)$$

Similar calculations can be made for the macroscopic time evolution of N_2 and N_3. The results are

$$P\frac{dN_2}{dt}=P\frac{dN_3}{dt}=-P\frac{dN_1}{dt} . \quad (43)$$

We can also make the same kind of calculation for θ_1, θ_2, and θ_3. The integrals are somewhat different but the procedure is straightforward (see Appendix B). The final result is

$$P\frac{d\theta}{dt}=\Delta\omega+w\left[\left[\frac{N_2N_3}{N_1}\right]^{1/2}-\left[\frac{N_1N_3}{N_2}\right]^{1/2}-\left[\frac{N_1N_2}{N_3}\right]^{1/2}\right]G(\theta) , \quad (44)$$

where we have defined a new angular function:

$$G(\theta)=\frac{F(n)}{(\theta_1\theta_2\theta_3)^{n-1}}[M(\theta)\cos\theta-N(\theta)\sin\theta] . \quad (45)$$

The macroscopic frequency mismatch appearing in Eq. (44) is determined by

$$\Delta\omega=\frac{1}{n}\sum_{k=1}^{n}(\omega_{1k}-\omega_{2k}-\omega_{3k}) . \quad (46)$$

Equations (41)—(44) give the macroscopic evolution equation for the three interacting waves in closed form. They give a rough description of the nonlinear interaction. A complete description is contained in Eqs. (20), which are equivalent to the microscopic evolution equations (4) and (5).

It is interesting to compare Eqs. (41)—(44) with the so-called fixed-phase equations, which correspond to the case $n=1$ in Eqs. (4) and (5). Such comparison leads us to the striking conclusions that $P(dN_\lambda/dt)$ for the general case ($n\gg1$) and (dN_λ/dt) for the fixed-phase case ($n=1$) are described by the same equations, excepting the phase dependence. The $\cos\theta$ and $\sin\theta$ appearing in the fixed-phase equations[3,5] are replaced by $F(\theta)$ and $G(\theta)$ in the general case. Of course, when we make $n=1$ in Eqs. (41)—(44) we obtain $F(\theta)=\cos\theta$ and $G(\theta)=\sin\theta$, as it should be. This means that Eqs. (41)—(44) contain the fixed-phase equations as a limiting case.

In the opposite limit of very large n, we can see from (42) and (45) that $F(\theta)$ and $G(\theta)$ become very rapidly oscillating functions of θ. This suggests that the mean values of $F(\theta)$ and $G(\theta)$ tend to zero, even for short time intervals, and the nonlinear interaction becomes less effective than it was for small n. This is also the expected behavior in the limit of large $\Delta\omega_j$ (or large n), when the random-phase approximation becomes valid. Again, we note that the random-phase equations require a second-order correction to the Hamiltonian (1), with nonlinear terms proportional to N_j instead of $\sqrt{N_j}$.

V. MEMORY EFFECTS

Let us now discuss in detail the memory terms contained in the evolution equation (20). These terms describe (with the ballistic ones) the subdynamics on the $6(n-1)$ space which is complementary to the macroscopic phase space. But, contrary to the ballistic terms which cannot be described in macroscopic "language," the memory terms can be written as functions of the macroscopic variables. This is a great advantage of the method because it allows us to keep in the macroscopic phase space part of the evolution of the system which occurs outside of this space.

In order to obtain simple expressions for the memory terms, we develop the operator $M(t)$ defined in Eq. (21) as a series expansion in powers of the Liouvillian operator L and retain only the first term in the expansion. The operator L is equivalent to a total time differentiation and the neglect of the higher-order terms corresponds to the neglect of the high-frequency oscillations.[16] The memory operator is now reduced to

$$M(t)\simeq iPL^2-PLPL . \quad (47)$$

Making the explicit operations associated with PL^2, we obtain, after carrying out lengthy but straightforward calculations,

$$iPL^2N_1=-2w^2[N_1N_2F_3(\theta)+N_1N_3F_2(\theta)-N_2N_3F_1(\theta)] -2w\Delta G_1(\theta)\sqrt{N_1N_2N_3} , \quad (48)$$

where the angular functions $G_1(\theta)$ and $F_j(\theta)$, with $j=1,2,3$, contain a large number of terms similar to $G(\theta)$ and $F(\theta)$. For simplicity their explicit definition is omitted here.

When the frequency mismatch Δ is nonzero and the coupling coefficient is very small, $w\ll1$, we can eventually neglect the first term on the right-hand side of Eq. (48). In this case the first memory contribution to the evolution of N_1 is formally analogous to the macroscopic term, apart from the time integral. But, in the general case we have to retain the first term in (48). It is remarkable that this term is formally analogous to those appearing in the random-phase equations, excepting again the time integral. This can be seen more clearly when we assume that $F_1(\theta)\simeq F_2(\theta)\simeq F_3(\theta)$. Assuming further that $w/\Delta\gg1$, we can write Eq. (48) in the approximate form

$$iPL^2N_1=-2w^2F_1(\theta)(N_1N_2+N_1N_3-N_2N_3) . \quad (49)$$

The expression inside the parentheses represents the typical behavior of the nonlinear interaction in the random-phase approximation.[1,5] This is a remarkable result because we know that the Hamiltonian (1) from which we started is not able to describe the nonlinear wave interaction in the random-phase approximation. We can then conclude that the first contribution to the memory effects, described by the time integral of Eq. (48), consists on two terms, one of the random-phase type and the other of the fixed-phase type.

Let us now return to Eq. (47) and calculate the second contribution to the memory effects. We obtain

$$iPLPLN_1 = -2iw\left[\left[N_1N_2\frac{\partial F_3}{\partial\theta_3}+N_1N_3\frac{\partial F_2}{\partial\theta_2}-N_2N_3\frac{\partial F_1}{\partial\theta_1}\right]G(\boldsymbol{\theta})-\sqrt{N_1N_2N_3}\left[\omega_1\frac{\partial F_1}{\partial\theta_1}+\omega_2\frac{\partial F_2}{\partial\theta_2}+\omega_3\frac{\partial F_3}{\partial\theta_3}\right]\right] . \quad (50)$$

It is clear that the same qualitative dependence on the action variables N_j appears. The first term is of the random-phase type and the second one is of the fixed-phase type. Integration of (48) and (50) in time gives the memory contribution to the evolution of N_1, to first order in the operator L. Similar expressions can be obtained for the evolution of N_2 and N_3.

Let us now consider the case of θ_j. If we apply (47) to the second equations (20) and proceed as before we obtain a singularity. Such singular behavior of the angular memory terms is associated with that which is already present in the microscopic equations of motion. As we can see from Eqs. (5) the time derivative of θ_{pkl} goes to infinity when N_{jk} tends to zero, for $j=1,2,3$. This leads to an infinite contribution to the memory effects associated with the macroscopic variables θ_j. If we add to the action variables appearing in the denominators of (5) a phenomenological constant ϵ the memory term goes like ϵ^{-1}. Instead of looking for some *ad hoc* explanation of such phenomenological correction it is perhaps more convenient to investigate other Hamilonians more consistent than the usually assumed Hamiltonian (1). This will be done in a future work.

VI. CONCLUSIONS

We have shown in this work how a projection-operator method can be used in order to obtain general equations for the nonlinear interaction of waves. These equations contain three kinds of terms: the macroscopic, the memory, and the ballistic terms. If we neglect the second and the third terms, we get an approximate picture of the interaction, which corresponds to the projection of the dynamics into a six-dimensional subspace of the entire phase space. In this approximation, Eqs. (41)–(44) can be considered as a generalization of the fixed-phase equations,[3] for the case of interacting waves with a finite bandwidth (number of modes $n \gg 1$, or $\Delta\omega \neq 0$). The well-known fixed-phase equations are obtained as a limit of Eqs. (41)–(44), when the number of modes tends to 1 ($n \to 1$ or $\Delta\omega \to 0$).

On the other hand, for n large, the nonlinear interaction becomes less efficient due to the rapid oscillations of the phase factors $F(\boldsymbol{\theta})$ and $G(\boldsymbol{\theta})$. This is compatible with the fact that in the random-phase approximation ($n \to \infty$ or $\Delta\omega \to \infty$) there is no three-wave interaction process, as long as the Hamiltonian (1) is considered.

But, in order to get a more precise description of the three-wave interactions we must also retain the memory effects. They describe the influence of the unwanted $6(n-1)$ variables on the retained six macroscopic ones. The analysis of Sec. V has shown that these effects can be explicitly calculated. It has also shown that the Hamiltonian (1) is not appropriate to describe these effects, because the singularities appearing in the microscopic equations of motion (5) give an infinite contribution to the angular memory terms of Eq. (20).

In a future work we intend to develop the present theory to more complete and appropriate Hamiltonians, in order to obtain nondivergent memory terms for the angular variables and to include other nonlinear interaction processes. The transition from the present discrete theory (where the wave spectrum is constructed with a finite number of monochromatic modes) to a continuum theory where the wave spectrum is a superposition of an infinite number of modes will also be considered. Detailed numerical calculations for particular situations and small numbers of modes are also necessary to understand the precise meaning and the typical behavior of the ballistic terms appearing in the evolution equations.

Finally, we recall that we have not made any restriction to the interacting waves or to the nonlinear medium where they propagate. This means that our calculations are not restricted to a plasma and remain valid for other nonlinear media as well.

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APPENDIX A

We show in this appendix how to construct a projection operator P', equivalent to (11), with the aid of an orthogonal canonical transformation. Let us consider a canonical transformation from the $6n$ microscopic variables (N_{jk},θ_{jk}) to other $6n$ variables called P_j and Q_j, with $j=1,2,\ldots,3n$ defined in such a way that the first P_j coincide with (or are proportional to) the macroscopic action variables N_j, and the first Q_j coincide with the macroscopic angular variables θ_j:

$$\{P_j\}=\{N_1,N_2,N_3,P_4,P_5,\ldots,P_{3n}\} ,$$
$$\{Q_j\}=\{\theta_1,\theta_2,\theta_3,Q_4,Q_5,\ldots,Q_{3n}\} . \quad (A1)$$

If the canonical transformation is orthogonal, we can write[17]

$$Q_j=\sum_{k=1}^{3n}a_{jk}q_k, \quad P_j=\sum_{k=1}^{3n}a_{jk}p_k , \quad (A2)$$

where a_{jk} are the elements of an orthogonal matrix $\underline{A}$,

$$\sum_k a_{ik}a_{jk}=\delta_{ij} , \quad (A3)$$

and we assume that

$$\{q_k\}=\{\theta_{11},\ldots,\theta_{1n},\theta_{21},\ldots,\theta_{2n},\theta_{31},\ldots,\theta_{3n}\}\ ,$$
$$\{p_k\}=\{N_{11},\ldots,N_{1n},N_{21},\ldots,N_{2n},N_{31},\ldots,N_{3n}\}\ . \qquad \text{(A4)}$$

From this we can get the general form of the matrix $\underline{A}\equiv(a_{jk})$:

$$\underline{A}=\frac{1}{\sqrt{n}}\begin{bmatrix}\underline{I}_1 & \underline{I}_2 & \underline{I}_3\\ \underline{A}_1 & \underline{A}_2 & \underline{A}_3\end{bmatrix}\ . \qquad \text{(A5)}$$

$\underline{I}_i$ are $3\times n$ matrices with all the elements zero, except those of the ith row which are equal to 1, and A_i are $(3n-3)\times n$ matrices which we do not need to specify here. The factor $n^{-1/2}$ is necessary to guarantee the orthogonality condition (A3). This also implies that the definition of the macroscopic variables N_j and θ_j is now

$$N_j=\frac{1}{\sqrt{n}}\sum_{k=1}^{n}N_{jk},\quad \theta_j=\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\theta_{jk}\ . \qquad \text{(A6)}$$

The difference between this definition and Eqs. (8) is irrelevant to the theory. The inversion of (A2) leads to

$$N_{lk}=N_k+\sum_{i=4}^{3n}a_{ik}P_i,\quad \theta_{lk}=\theta_l+\sum_{i=4}^{3n}a_{ki}Q_i\ . \qquad \text{(A7)}$$

The generating function for this orthogonal transformation is simply written as[17]

$$F=N_1\theta_1+N_2\theta_2+N_3\theta_3+\sum_{i=4}^{3n}P_iQ_i\ . \qquad \text{(A8)}$$

This function does not depend explicitly on time and the new Hamiltonian $H'(P_i,Q_i)$ can easily be obtained replacing (A7) in Eq. (1). Such canonical transformation has the effect of rotating the hyperplanes shown in Fig. 2, making them parallel to the coordinate planes. The macroscopic subdynamics can then be obtained simply by integration on the $6(n-1)$ variables P_i and Q_i corresponding to $i>4$. The corresponding projection operator P' is now an integration operator:

$$u(\mathbf{N},\boldsymbol{\theta})=P'u(P_i,Q_i)$$
$$=S^{-1}\int u(P_i,Q_i)\prod_{k=4}^{3n}dP_k dQ_k\ . \qquad \text{(A9)}$$

Such a projection operator is, in principle, equivalent to the one defined in Eq. (11). Although formally simpler, this new projection operator P' presents practical disadvantages with respect to P in the appropriate choice of the limits of integration.

APPENDIX B

For the sake of completeness we present here the derivation of Eqs. (44) and (45). Replacing θ_j by (8) in the arguments of the Dirac δ functions appearing in (20) we can write

$$P\frac{d\theta_1}{dt}=\omega_1+\frac{w}{S}\sum_{j=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}A'_{jst}(\mathbf{N})B'_{jst}(\theta)\ , \qquad \text{(B1)}$$

where ω_1 is defined by

$$\omega_1=\frac{1}{n}\sum_{p=1}^{n}\omega_{1p}\ . \qquad \text{(B2)}$$

If we assume that the energy associated with the first wave is uniformly distributed over the n integral modes, which means that $N_{1k}\simeq N_1/n$ for all k, we can see that ω_1 is nearly equal to the mean frequency $\bar{\omega}_j$ defined by (9). On the other hand, the action-dependent functions $A'_{jst}(\mathbf{N})$ are of the form

$$A'_{jst}(\mathbf{N})=A'_1A_2A_3\ , \qquad \text{(B3)}$$

where A_2 and A_3 are given by Eq. (26) if we replace the subscript 1 by 2 and 3, and A'_1 is defined by

$$A'_1=\int N_{1j}^{-1/2}\delta\left[N_{1j}+\sum_{\substack{\alpha=1\\ \alpha\neq j}}^{n}N_{1\alpha}-N_1\right]\prod_{k=1}^{n}dN_{1k}\ . \qquad \text{(B4)}$$

More explicitly, we can write

$$A'_1=\int_0^{N_1}dN_{1n}\int_0^{N_1-N_{1n}}dN_{1,n-1}\cdots\int_0^{N_1-\sum_{\alpha=2,\ \alpha\neq j}^{n}N_{1\alpha}}dN_{11}\left[N_1-\sum_{\substack{\alpha=1\\ \alpha\neq j}}^{n}N_{1\alpha}\right]^{-1/2}. \qquad \text{(B5)}$$

After integration we obtain

$$A'_1=\frac{2^{n-1}}{\prod_{p=1}^{n-2}(2p+1)}N_1^{n-3/2}\ . \qquad \text{(B6)}$$

Using these results we can write Eq. (B3) in the form

$$A'_{jst}(\mathbf{N})=\left[\frac{2^{n-1}}{\prod_{p=1}^{n-1}(2p+1)}\right]^3 N_1^{n-3/2}(N_2N_3)^{n-1/2}\ . \qquad \text{(B7)}$$

We now turn to the angle-dependent functions $B'_{jst}(\boldsymbol{\theta})$ appearing in (B1):

$$B'_{jst}(\boldsymbol{\theta})=\int \left[\prod_{\lambda=1}^{3} B_\lambda\right] \cos\theta_{jst}\, d\theta_{1j}\, d\theta_{2s}\, d\theta_{3t} \ , \tag{B8}$$

where B_λ are defined by Eq. (30). Using (40) and replacing (B7) and (B8) in (B1) we obtain

$$P\frac{d\theta_1}{dt}=\omega_1+w\left[\frac{N_2N_3}{N_1}\right]^{1/2} G(\boldsymbol{\theta}) \ . \tag{B9}$$

Making similar calculations for θ_2 and θ_3 we can finally obtain the evolution equation for the phase difference $\theta=\theta_1-\theta_2-\theta_3$, which is precisely Eq. (44), with $\Delta\omega=\omega_1-\omega_2-\omega_3\simeq\overline{\omega}_1-\overline{\omega}_2-\overline{\omega}_3$.

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