Time development of Čerenkov radiation

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Most developments of Čerenkov radiation are in terms of the Fourier components of the fields and power emitted by a single electron. When many electrons in a compact bunch are emitted from an accelerator, the bunch radiates coherently and at a lower frequency than for a single electron. The theory for the time structure of the fields arising from a charge bunch is developed, and it is shown that the source of the radiation is di/dt. Present detector technology should be able to resolve these fields.

TIME DEVELOPMENT OF ČERENKOV RADIATION INTRODUCTION

Čerenkov radiation, produced by a charge or group of charges, moving faster than the speed of electromagnetic radiation in a medium, has been investigated, starting with the experiments of Čerenkov[1] in 1934 and the explanation by Frank and Tamm[2] in 1937. Since power radiated by a single charged particle is proportional to the frequency, most of the research effort has been devoted to the relatively intense optical radiation which is favored over the microwave region by a factor of about 10^4. The optical results[3,4] are given in terms of the Fourier components of the fields and the radiated power.

In our previous work[5,6] it was noted that microwave radiation can be significant because all the electrons in an accelerator bunch (about 10^9) radiate coherently; an effect which more than offsets the single particle increase in radiated power with frequency. For an electron beam generated by a traveling wave Linac and passing through air, it was shown that the various harmonics of the basis frequency up to about the tenth are emitted. (In the case of an L or S band Linac, these correspond to 10 and 30 GHz, respectively.)

The time structure of Čerenkov radiation fields in the optical and even in the higher frequency microwave regions is difficult to observe because the detectors register power. One of the few treatments of the time dependence, by Tamm[7] in 1939, showed that the optical radiation by an electron is singular on the Čerenkov front. Here we consider the time structure of fields generated when electron bunches radiate coherently; in a development which complements the frequency domain analysis of our earlier work.[5,6] The fields should be observable for beams from induction accelerators which produce bunches much longer than those produced by S- or L-band linear accelerators.

MAGNETIC RADIATION FIELD

The purpose of this paper is to present a development of the time dependence of the electric field generated by the Čerenkov mechanism. The method is first to determine the potentials from the moving charge distribution, and subsequently to obtain the fields (in cgs units) from the potentials by

$$\mathbf{B}=\nabla\times\mathbf{A}\ , \tag{1}$$

$$\mathbf{E}=-\nabla\Phi-\frac{1}{c_0}\frac{\partial\mathbf{A}}{\partial t}\ . \tag{2}$$

We assume a charge density function ρ_v and a current density $j_v=\rho_v v/c_0$ with the velocity v in the plus z direction. The charge and current are assumed to be concentrated along the z axis such that

$$\rho_v(\mathbf{r},t)=\rho(z,t)\delta(x)\delta(y) \tag{3}$$

and the charge is assumed to move with no change in shape so that the z and t dependence of the charge is

$$\rho(z,t)=\rho_0(z-vt)\ . \tag{4}$$

Note that ρ_v and j_v represent the usual charge and current densities, while ρ and ρ_0 throughout this paper are charge per unit length. The velocity of light is c and c_0 in the medium and free space, respectively.

The potentials are found by taking the usual retarded solutions to the wave equations; which become under the assumption of a line distribution of charge (3),

$$\Phi(\mathbf{r},t)=\epsilon^{-1}\int R^{-1}\rho(\mathbf{r}',t')dz'\ , \tag{5}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mathbf{v}}{c_0}\int R^{-1}\rho(\mathbf{r}',t')dz'\ , \tag{6}$$

where $\mathbf{R}=\mathbf{r}-\mathbf{r}'$ and t' is the retarded time

$$t'=t-|\mathbf{r}-\mathbf{r}'|/c\ . \tag{7}$$

Now (4), the assumption of rigid motion of the charge distribution, can be incorporated into the potentials, and a new variable $u(z')=z'-vt'$ can be introduced so that the potentials (5) and (6) become

$$\Phi(\mathbf{r},t)=\epsilon^{-1}\int R^{-1}\rho_0(u)dz'\ , \tag{8}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mathbf{v}}{c_0}\int R^{-1}\rho_0(u)dz'\ . \tag{9}$$

Also, since the charge is confined to the z' axis, the new variable $u(z')$ can be written more explicitly,

$$u(z')=z'-vt+\frac{v}{c}[x^2+y^2+(z-z')^2]^{1/2} . \tag{10}$$

The magnetic field **B** may be calculated from (1) and since **A** had only a z component, **B** has only the x and y components, $B_x=(\partial/\partial y)A_z$ and $B_y=-(\partial/\partial x)A_z$. Carrying out the differentiation for the x component gives

$$B_x=\frac{v}{c_0}\int \frac{\partial}{\partial y}R^{-1}\rho_0(u)dz' + \frac{v}{c_0}\int R^{-1}\frac{\partial}{\partial y}\rho_0(u)dz' . \tag{11}$$

For radiation, the first integral, falling off as R^{-2} at large distances, will be neglected and only the second term will be considered further. From (10), it is seen that u is a function of x and y so that the second integral can be written

$$B_x=\frac{v^2}{cc_0}\int \frac{y}{R^2}\rho_0'(u)dz' , \tag{12}$$

where $\rho_0'(u)$ is the derivative of ρ_0 with respect to its argument u. The corresponding expression for B_y has y replaced by $(-x)$. These two components can be combined to give the total magnetic radiation field **B**. In the cylindrical coordinates, (s,θ,z) where s is the radius vector $s=(x^2+y^2)^{1/2}$, **B** is tangential (i.e., in the θ direction) with a magnitude given by

$$B=\frac{v^2}{cc_0}\int \frac{s}{R^2}\rho_0'(u)dz' . \tag{13}$$

TIME DEVELOPMENT

In order to evaluate (13) for B, it is necessary to consider the dependence of u on z' as given in (10). In the u-z' plane, the first two terms are a straight line with unit slope and an intercept which changes with time, while the third term is a hyperbola opening in the $+u$ direction with asymptotic slopes of $\pm v/c$. The sum of these two curves is $u(z')$. In the Čerenkov case with $v>c$, the result is a curve whose ends both point upward as shown in Fig. 1. As time increases, the entire curve will translate downward to smaller u values as a result of the negative second term in (10).

Only changing currents (those with a nonzero ρ_0') will contribute to the magnetic radiation field (13). To proceed and demonstrate the method, a ramp-front current pulse is chosen as a simple example. Assuming that the front end of a current pulse increases linearly up to a constant value, the derivative $\rho_0'(u)$ will be a constant valued square pulse of magnitude ρ_m' as is also shown in Fig. 1. The corresponding negative $\rho_0'(u)$ pulse occurring at the tail of a current pulse is not shown and its effect is considered separately.

For large negative times, the $u(z')$ curve (t_1) is completely above the pulselike nonzero portion of the $\rho_0'(u)$ curve so that the contribution in (13) to B from $\rho_0'(u)$ is zero and therefore, B is zero. As time increases, the $u(z')$ curve moves downward until the B pulse begins when

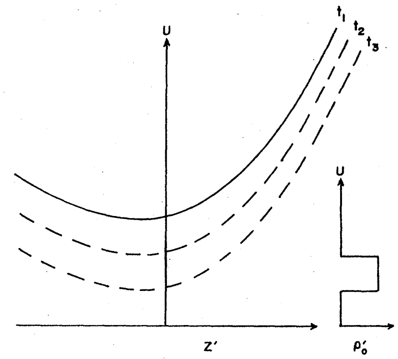

FIG. 1. Function $u=z'-vt'$, defined in the text is plotted for increasing times t_1,t_2,t_3 at the point of observation. The corresponding current derivative profile, on the right, is a function of u only and remains fixed in time. Field signal pulse starts at t_2, and reaches a maximum at t_3.

$u(z')$ is tangent [curve (t_2)] to the upper portion of the $\rho_0'(u)$ pulse. The value of the integral in (13) increases as $u(z)$ continues its constant downward motion with increasing time until $u(z')$ becomes tangent [curve (t_3)] with the lower part of the $\rho_0'(u)$ pulse. At this time the nonzero part of the integral has the largest extent—from z_1 to z_2. At later times, the integral breaks into two regions of the z' axis and if $\rho_0'(u)$ is constant, the value of the integral decreases with increasing time because the extent of the integral in the two regions continues to decrease as a result of the upward turn of $u(z')$.

This calculation may be carried further to determine the time structure (shape) of the resulting B pulse. Although the expression (13) for B can be integrated directly in the case where the slope $\rho_0'(u)$ is constant, it is instructive to carry out the calculation by developing u in a power series. Denoting z_0' as the value of z' at which $u(z')$ has zero slope, the values of z_0', $u(z_0')$ and the second derivative are

$$z_0'=-s\left[\frac{v^2}{c^2}-1\right]_{+2}^{-1/2}+z , \tag{14}$$

$$u(z_0')=s\left[\frac{v^2}{c^2}-1\right]_{+2}^{+1/2}+z-vt , \tag{15}$$

$$\left.\frac{\partial^2 u}{\partial z'^2}\right|_{z'=z_0'}=\frac{1}{s}\frac{c^2}{v^2}\left[\frac{v^2}{c^2}-1\right]^{3/2}\equiv 2A , \tag{16}$$

so that u can be expressed as a power series about the minimum

$$u=u(z_0')+A(z'-z_0')^2 . \tag{17}$$

The limits z_1' and z_2' can be written in terms of the minimum value as $z_2'=z_0'+\Delta z'$ and $z_1'=z_0'-\Delta z'$, where $\Delta z'$ is the value of $z'-z_0'$ such that the difference $u(z')-u(z_0')=a$, the width of the current derivative pulse ρ_0'. Then from (17), $a=A(\Delta z')^2$ or

$$\Delta z'=(a/A)^{1/2} . \tag{18}$$

Using this value, the maximum magnetic radiation field for the rising front of the magnetic field pulse is easily evaluated from (13) under the assumption that s and R are slowly varying to give

$$B_{\max}=\frac{v^2}{cc_0}\frac{s}{R^2}\rho_m' 2\left[\frac{a}{A}\right]^{1/2}, \tag{19}$$

where a is the length of the linear rise of the current pulse and A is given by (16).

Values of B for the rise up to the peak value given above are found by the same process but using appropriately smaller values of $\Delta z'$. The result is that the integral (and therefore B) increases as $t^{1/2}$ after the onset of the pulse. After the maximum magnetic field is reached, the integral splits into two parts. If the expression on the right side of (19) is called $I(a)$, the value of B at later times becomes

$$B=I(a'+a)-I(a'), \tag{20}$$

where a' is the distance by which the minimum in $u(z')$ is below the lower step of the ρ_0' pulse in Fig. 1. The first term in (20) increases slowly with a', but the second term decreases as $(a')^{1/2}$ leading to the sharp falloff of the magnetic field after the maximum as shown in Fig. 2.

A complete current pulse may be considered as a linear rise, followed by a constant current, and then a linear decrease. The latter part gives rise to a negative $\rho'(u)$ and a reversed magnetic field pulse so that the magnetic field for a complete current pulse has the double-peaked structure shown in Fig. 2.

ELECTRIC RADIATION FIELD

In a manner similar to the derivation of (13), the electric radiation field may be found from (2), (8), and (9). The details are omitted, but the result is

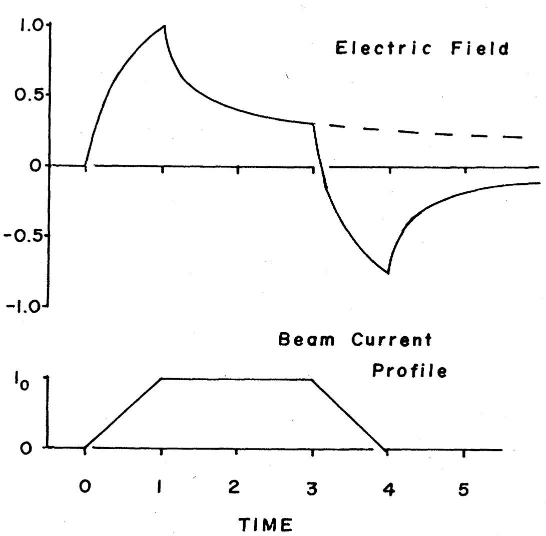

FIG. 2. Electric field pulse, shown in the upper curve, as generated by the beam current profile, shown in the lower curve. In the text, the electric and magnetic radiation fields are shown to have the ratio $E/B=c/c_0$, the same as for a plane wave in the medium. The $\mathbf{E}$ and $\mathbf{B}$ fields and the propagation direction are mutually perpendicular.

$$\mathbf{E}=-\frac{v}{c_0}\int\left[\frac{\mathbf{R}c}{Rc_0}-\frac{\mathbf{v}}{c_0}\right]\frac{1}{R}\rho_0'(u)dz'. \tag{21}$$

The direction of $\mathbf{E}$ may be determined from the following considerations. If $\mathbf{R}$ is assumed approximately constant and denoted by $\mathbf{R}_m$ in the region which contributes most strongly to the integral, then

$$\mathbf{E}\cdot\frac{\mathbf{R}}{R}=\mathbf{E}\cdot\frac{\mathbf{R}_m}{R_m}=I\left[1-\frac{v}{c}\cos\theta\right], \tag{22}$$

where I represents the integral in (21) without the factor in parenthesis and θ is the angle between $\mathbf{R}_m$ and the z axis. But the value of R_m in the region which contributes to the integral is found by evaluating the general expression (21) at $z'=z_0$. To simplify the expression, let the observer be at $z=0$ and also assume that the ρ_0' pulse is centered near $u=0$. Then $R_m=(s^2+z'^2)^{1/2}$ may be evaluated using (14) to give

$$R_m=s\left[1-\frac{c^2}{v^2}\right]^{-1/2}. \tag{23}$$

If the usual Čerenkov angle is defined as $\cos\theta_C=c/v$, R_m can be written as

$$R_m=\frac{s}{\sin\theta_C}. \tag{24}$$

Consequently $\mathbf{R}_m$ is inclined at an angle θ_C to the velocity, which is along the z axis.

From (22) it is apparent that $\mathbf{E}$ is perpendicular to $\mathbf{R}_m$ when $\theta=\theta_c$ and (24) shows that $\mathbf{R}_m$ is the value of $\mathbf{R}$ at the Čerenkov angle θ_c. Thus, the electric field from the front of the pulse (i.e., $z'=z_m$) is transverse to $\mathbf{R}_m$.

To proceed, it is necessary to evaluate the magnitude of $\mathbf{E}$. Note that the square of the vector quantity in parentheses in (21) is $(c^2+v^2-2c\mathbf{R}\cdot\mathbf{v}/R)c_0^{-2}$. If this factor of the integrand is evaluated at the point $\mathbf{R}=\mathbf{R}_m$, then (24) can be used to give $\mathbf{R}\cdot\mathbf{v}=R_m v\cos\theta_C$ and the magnitude of the parentheses in (21) is $(v/c)\sin\theta_C$. Then, under this approximation, the magnitude of $\mathbf{E}$ is

$$E=\frac{v^2}{c_0^2}\int\frac{\sin\theta_c}{R}\rho_0'(u)dz'. \tag{25}$$

When $\mathbf{R}$ is evaluated at $\mathbf{R}_m$ this integral for E becomes identical to (13), the one for B, but multiplied by c/c_0. Thus, the usual relation for electric and magnetic radiation fields holds for this case, i.e.,

$$E/B=c/c_0. \tag{26}$$

The situation is clarified in Fig. 3. The charge, traveling from left to right, emits a signal from A, which travels to the observer at 0, traversing a distance R_m. The observer is at $z=0$ and at a perpendicular distance s from the path. The field $\mathbf{E}$ is perpendicular to $\mathbf{R}_m$ and lies in the plane of $\mathbf{R}_m$ and $\mathbf{v}$. Since the magnetic field $\mathbf{B}$ was found to be perpendicular to that plane and therefore perpendicular to $\mathbf{E}$, the vector product $\mathbf{E}\times\mathbf{B}$ is along $\mathbf{R}_m$. The signal was emitted from A at an earlier time t' in order to arrive at 0 at the time t, with $c(t-t')=R_m$. By the time the signal reaches 0, the particle is at B, with

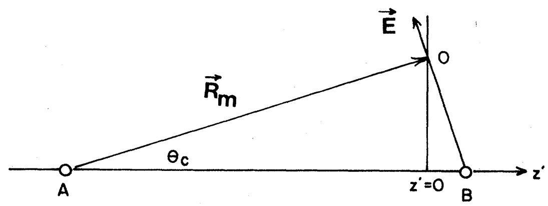

FIG. 3. Geometrical relations for the Čerenkov pulse. The source (ρ_0') at A emits a signal at an early time giving the E field at the observer position O. When the field reaches the observer, the particle is at B.

$D = v(t-t')$. Then $R_m/D = c/v = \cos\theta_c$ as expected. D is the path length from A to B.

From Fig. 3, one should also note that the electric field is transverse to $\mathbf{R}_m$, which points from the earlier (retarded time) position of the particle, and is radial relative to the present position of the particle. The former condition holds for typical dipole radiation, while the latter condition holds for the Lienard-Wiechart field for a particle moving with $v < c$. The Lienard-Wiechart fields fall off as the inverse square of the distance, and do not represent radiation. In contrast, the fields discussed here fall off more slowly than R^{-1} and represent radiation—which is discussed in the next section. The R dependence of B can be seen from (19) where the factors except for $A^{-1/2}$ cause B to fall off as R^{-1}. However, the $A^{-1/2}$ factor contributes a $R^{-1/2}$ factor from (16) so that B actually varies as $R^{-1/2}$ as would be expected from the assumed cylindrical symmetry of the current source.

RADIATED POWER

The energy radiated may be found by calculating the Poynting vector and integrating over a surface. If the surface is a cylinder centered on the z axis, the fields at a given time have a pattern independent of angle and a z dependence as shown in the top curve of Fig. 2. The Poynting vector is, of course, along $\mathbf{R}_m$, and the outward component may be integrated over the cylinder to give the total power radiated. As a crude estimate for the integral, replace the field by the peak field (19) and let the spatial width of the rising ramp of the pulse be a. The radiated power is then

$$P(\text{approx}) = \left[\frac{8}{c^2}\right] v I_0^2 \sin^2\theta_c \tag{27}$$

in cgs units. In the mks system, the square bracket is replaced by $2\mu_0/\pi$.

In the earlier calculations,[5,6] the fields and power were expressed in terms of Fourier amplitudes. If the same current pulse is assumed, P_ω has frequency components up to the value of ω such that the wave length of the radiation is equal to the pulse length. If it is assumed that P_ω rises linearly up to this frequency and suddenly drops, the total power radiated becomes (in mks units)

$$\sum P_\omega = \frac{\pi}{2}\mu_0 v I_0^2 \sin^2\theta_c \ . \tag{28}$$

Equations (27) and (28) are both rough estimates and the point is that the similarity of the results is asserted to be confirmation that the calculations in this paper represent the Čerenkov radiation, expressed in terms of time dependence of the fields.

DISCUSSION

In preceding sections, the time structure of the electric and magnetic radiation fields was developed. Only the far fields were retained in the development leading to the B field (13), and the E field (21) and (25), and only the assumption of a rigid charge confined to a line was introduced. It is seen from these equations that the time derivative of the current is the source function.

The simple model chosen to demonstrate the method of developing the time structures was that of a uniform charge distribution with uniformly varying front and rear sections. This model gives the square pulse charge derivative of height ρ_m' shown in Fig. 1 which is easy to use in evaluating the integral (13). Similar remarks hold for the power series expansion of $u(z')$ which is an increasingly better approximation as the time during which the current is changing, decreases. Current variations other than linear may be readily incorporated within the framework given. Also it should be noted that in all cases the variation of R^{-2} in (13) which was assumed constant in the example will tend to sharpen both the leading and trailing edges of a field pulse.

In the evaluation of the time structure of the fields, the peak field arose when the integral (13) had the most widely spaced limits; a situation which occurs because $u(z')$ has a negative slope for sufficiently negative values of $(z'-z)$ as shown at the left side of Fig. 1. In the non-Čerenkov case ($v/c<1$) this situation does not arise since then the slope of the $u(z')$ function always has the same sign. In this case (i.e., $v<c$) the $u(z')$ curve bends downward instead of upward for large negative values of $(z'-z)$ and the only contribution to the integral (13) is from small regions of z'.

It should be noted that the radiation which is calculated is produced by a bunch of electrons of negligible transverse dimension, and with a longitudinal distribution which does not change as the bunch travels along the z axis. The radiated power is calculated to be proportional to ρ_0^2 and therefore to the square of the beam current. The expressions are valid and the evaluation is therefore coherent only insofar as the bunch is not distorted either by the reaction of the radiation or by the instabilities associated with very high current beams.

These results show how the time structure of Čerenkov radiation arises from the time rate of change of the charge distribution in an electron bunch. Present technology is such that this structure is not observable in the Čerenkov radiation from S or L band linacs because of their relatively high fundamental frequency. However, induction accelerators with their longer electron bunch structure should produce Čerenkov signals in air for energies greater than about 25 MeV, which should be observable.

The extension of this method of calculation of the fields for both Čerenkov and sub-Čerenkov charge velocities is easily made for cases for which the charge derivative $\rho'(u)$ is not constant. A detailed report is under preparation.

Finally, we note that although the results of other workers[4,7] often have singularities in the radiated power at the Čerenkov angle, the present results and our previous ones[5,6] show that the radiated power is finite whether calculated in the frequency or time domain. Also it should be noted that causality is satisfied because the fields are zero at times earlier than the edge of the pulse shown in Fig. 2.

ACKNOWLEDGMENTS

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