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Renormalization of the quasiperiodic transition to chaos for arbitrary winding numbers

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Previous renormalization analyses have demonstrated universal properties for the quasiperiodic transition to chaos. These theories have the unpleasant feature that universal properties depend on the winding number. We modify the renormalization transformation so that it has stable attractors. This allows us to study nonlocal properties by solving the equations numerically *without linearizing*. The resulting universal strange attractor contains the unstable fixed points of previous theories and has exponents that are independent of winding number.

One of the most common transitions to chaos occurs when quasiperiodic motion with two irrationally related frequencies becomes chaotic. In phase space, a two-torus attractor turns into a strange attractor. Inspired by the universal behavior associated with the period-doubling transition to chaos,[1] in recent years considerable effort has been invested toward understanding the universal properties underlying the breakdown of a torus.[2,3] In spite of the fact that there are numerical indications of universal properties valid at all winding numbers,[4] previous renormalization theories have only been able to deal with special parameter values, finding universal properties that depend on the winding number. Experimental verification of these theories requires control of the winding number which, in general, is not feasible. Our central purpose in this paper is to modify the approach of the previous theories making possible a numerical method that can describe the breakdown of a torus for all parameter values. We demonstrate that the renormalization transformation has a universal ergodic attractor, with universal exponents that are the same for almost all winding numbers. The unstable fixed points of the previous theories lie on this ergodic attractor.

The dynamics of the torus can be simplified by using a Poincaré section to convert a continuous dynamical system into a discrete mapping. To visualize this procedure, imagine slicing the torus, producing a mapping of a closed curve (equivalent to a circle) onto itself. An example of such a circle map is

$$x_{i+1}=x_i-\frac{k}{2\pi}\sin(2\pi x_i)+\omega=f(x_i) \quad . \tag{1}$$

x is taken modulo 1, so that x_i represents the position on the circle at the ith iteration. k is a nonlinearity parameter and ω is a parameter that determines the rate at which points travel around the circle, called the *winding number*, defined as $\rho=\lim_{m\to\infty}[f^m(x_0)-x_0]/m$. ρ is independent of x_0. For $k<1$, this map has a unique attractor. For rational winding numbers this attractor is a finite sequence of periodic points, but for irrational values it fills the circle. For $k>1$ this map is no longer one to one, and may have strange attractors. The critical value $k=1$ is of special interest as the parameter value where a transition to chaos can take place. For $k<1$ (the noncritical case) the map has a differentiable inverse, but at $k=1$ (the critical case) the derivative of the inverse does not exist for $x=0$. Previous work[2,3] has shown that the critical and subcritical cases belong to different universality classes.

Even though Eq. (1) appears simple, its behavior as parameters are varied is extremely complicated, as can be seen by examining Fig. 1(a). Furthermore, it is only one of an infinite number of possible circle maps. The purpose of the renormalization treatment given here is to find regularities in this bifurcation diagram, and to extract properties that are common to an entire class of analytic circle maps.

A one-parameter family of critical circle maps may be visualized as a landscape suspended over the bifurcation diagram of Fig. 1(a). Regularities in this landscape are evident as regularities in the bifurcation diagram. For example, by magnifying the illustrated portion of the bifurcation diagram shown in Fig. 1(a), the entire diagram reproduces itself, as shown in Fig. 1(b). This process can be continued *ad infinitum.* The self-reproducing parts of the diagram correspond to subregions in which a (somewhat distorted) landscape is reproduced for some iterate of the map. As will be explained below, a remarkable property is that it is possible to find such a set of nested reproducing diagrams centered about *any* irrational value of the winding number. The renormalization transformation is a "microscope" for finding such regularities and determining their universal features. This microscope operates by (1) examining a higher iterate of the map, (2) zooming in on the relevant subregion of the map, and (3) blowing up this subregion.

Previous renormalization treatments[2,3,5] begin by noting that any winding number ρ can be expanded in a continued fraction expansion, of the form

$$\rho=\cfrac{1}{n_1+\cfrac{1}{n_2+\cdots}} \quad .$$

Truncating this expansion provides a sequence of rational numbers that for a given denominator are the best rational approximants to ρ. The denominators of this sequence are given by the recursion relation $q_{i+1}=n_{i+1}q_i+q_{i-1}$. This implies that this sequence of iterates can be expressed in terms of the previous two iterates, i.e.,

$$f^{q_{i+1}}=f^{(n_{i+1}-1)q_i}f^{q_i+q_{i-1}} \quad .$$

For a circle map with winding number ρ, the sequence of iterates given by q_i are the recurrences of the orbit.

Rand and co-workers[2] have shown that this relation defines a renormalization scheme for two monotonic functions ξ and η, representing f^{q_i} and $f^{q_{i-1}+q_i}$. In principle, their

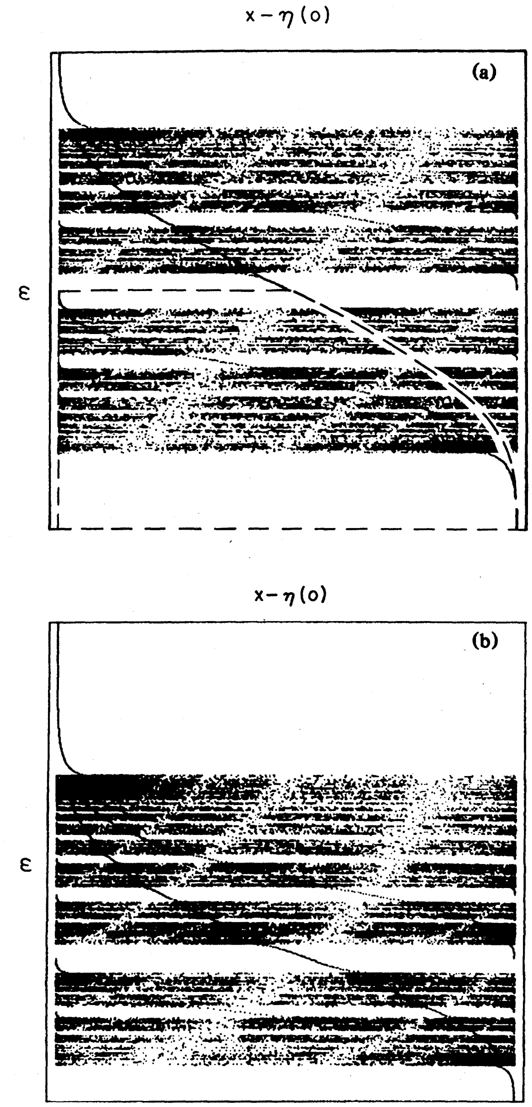

FIG. 1. (a) Bifurcation diagram for the circle map of Eq. (1), made by plotting many iterates at each ω. To keep the scale fixed, the values actually plotted are $x-\omega+1$. (b) A blowup of the dashed region of (a). This sequence can be continued *ad infinitum;* an infinite sequence of such pictures can be obtained about *any* irrational winding number.

scheme is valid for any winding number. In practice, the winding number of the initial functions ξ and η cannot be computed to arbitrary precision, making it impossible to indefinitely iterate the transformation. In the scheme of Rand and co-workers, only the argument x of ξ and η is rescaled, while the parameter ω is kept fixed. The sequence of blowups given in Fig. 1 naturally suggests, however, that the domain of the renormalized functions should be two dimensional, involving both x and ω. This automatically solves the problem of controlling the winding number. Using this new scheme the renormalized function is an entire one-parameter family of circle maps containing all winding numbers $0\leqslant\rho\leqslant 1$. This ensures that a circle map at any winding number of interest is always present. The resulting generalization of the renormalization equations of Ref. 2 is

$$\begin{aligned}\tilde{\xi}_{\tilde{\omega}}(\tilde{x})&=\alpha\xi_{\omega}^{n-1}\eta_{\omega}(\tilde{x}/\alpha)\ ,\\ \tilde{\eta}_{\tilde{\omega}}(\tilde{x})&=\alpha\xi_{\omega}^{n-1}\eta_{\omega}\xi_{\omega}(\tilde{x}/\alpha)\ .\end{aligned}\tag{2}$$

where $\tilde{\omega}=\omega/\delta+\omega_2$ and $\delta=1/(\omega_1-\omega_2)$. ω_2 and ω_1 determine the boundaries in ω of the renormalized domain, and are given by

$$\begin{aligned}\tilde{\xi}_0(0)&=0=\alpha\xi_{\omega_2}^{n-1}\eta_{\omega_2}(0)\ ,\\ \tilde{\xi}_1(0)&=1=\alpha\xi_{\omega_1}^{n-1}\eta_{\omega_1}(0)\ .\end{aligned}\tag{3}$$

As is clear from Fig. 1, α is now a function of ω, and is given by $\alpha(\omega)=[\xi_{\omega}^{n}\eta_{\omega}(0)-\xi_{\omega}^{n-1}\eta_{\omega}(0)]^{-1}$. ξ and η at each ω satisfy several other conditions, given in Ref. 2. The great advantage of formulating the renormalization transformation in this manner is that this transformation has stable attractors, and the transformation can be iterated directly. In contrast, using previous approaches the renormalization transformation is unstable, making it possible to linearize the equations in the neighborhood of a fixed point.

To iterate Eq. (2) we represent ξ and η as values on a discrete two-dimensional lattice, and use a bicubic scheme[6] to interpolate values between lattice points. If we choose, we can follow a periodic winding number by specifying a periodic sequence of n values. In this case, the solution is a stable fixed point or periodic cycle. Since our solution (ξ,η) is a one-dimensional parametrization of the function space along the unstable manifold near the fixed point, our method in this case can be viewed as a computation of the unstable manifold of the fixed point (of the one-dimensional transformation of previous theories). This makes it clear why our transformation is stable: it counteracts the effect of the instability by zooming in along it.

While a computation of the unstable manifold might be interesting, the real power of this method is that it can be applied to the more general case in which the winding number is *not* periodic. In this case our renormalization transformation can no longer be viewed as an unstable manifold calculation, since there are no longer any fixed points to compute the unstable manifold of.

As shown by Rand and co-workers,[2] the one-dimensional version of Eq. (2) alters the winding number at each iteration by eliminating n_1 from the continued fraction expansion, sending ρ to $1/\rho-[1/\rho]$, where $[1/\rho]=n_1$ is the integer part of $1/\rho$. This map is called the Gauss map. To iterate our renormalization transformation for arbitrary winding numbers, we simply pick an initial random seed ρ_0, iterate the Gauss map for ρ_0, and generate values of n_i at each step. Numerical difficulties can be caused when n_i is too large. We overcame this by setting $n_i=n_c$ whenever $n_i>n_c$, where n_c is a fixed cutoff. The net effect is that we sample only a restricted set of all possible random numbers. Fortunately, for n_c large the probability that $n_i>n_c$ is small, proportional to $1/n_c$. We obtained numerical estimates of the universal exponents in the limit as $n_c\to\infty$ by iterating our equations for several different values of n_c, and extrapolating the answers. For the random case, ξ and η fluctuate, due to the fact that the renormalization transformation does not have a simple attractor. By taking the geometric mean of the values generated at each iteration of the transformation, we computed the average values of delta and alpha. For the subcritical case, our results are $\bar{\delta}=10.4\pm0.5$ and $\bar{\alpha}=3.2\pm0.2$. The value of $\bar{\delta}$ is within experimental error of the Lyapunov number of the Gauss map, as predicted by Rand and co-workers.[2] For the critical case, we get $\bar{\delta} = 15.5 \pm 0.5$ and $\bar{\alpha} = 1.8 \pm 0.1$. As a test we compared these results with those obtained from a Monte Carlo approach.[7] The critical exponents obtained in each case agreed to the quoted numerical tolerance.

For the critical case there is a problem, however. In order to ensure cubic criticality we have to force $\xi'(0) = \eta'(0) = \eta''(0) = \xi''(0) = 0$. For periodic winding numbers, we find that our computed values of α and δ are correct to only three digits. The limited precision for the periodic case is apparently due to a numerical instability caused by imposing the above conditions. This does not limit the accuracy of our calculation for $\bar{\alpha}$ and $\bar{\delta}$ in the ergodic case, since this error is smaller than the more dominant statistical error due to fluctuations in different regions of the attractor. As demonstrated in Ref. 7, this does, however, wipe out the fine details of the fractal structure of the attractor. Using a Monte Carlo approach[7] applied to several different maps, we have since been able to numerically demonstrate the fractal structure, and to show that the renormalization transformation has a strange attractor.

We wish to emphasize that the universal numbers computed here are statistical quantities, summarizing an average property of a nontrivial global bifurcation sequence. In contrast, the universal exponents of previous theories are local quantities, giving information about a much simpler, repetitive bifurcation sequence. Thus, our universal numbers contain information about the entire bifurcation diagram shown in Fig. 1, in contrast with the usual numbers which give information only about small local regions of the diagram, corresponding to special winding numbers. The dominant source of error in computing the numbers given here is statistical fluctuation, due to the fact that we can iterate our transformation only a finite number of times, and properties vary from point to point. This is analogous to the study of stability properties in dynamical systems: the eigenvalues of a fixed point are easily computed to machine accuracy, while Lyapunov exponents are often difficult to compute to even three digits of precision. However, the eigenvalues give information only about the region in a neighborhood, while the Lyapunov exponents are statistical quantities that can summarize average properties of a nonlocal object such as a strange attractor. Knowing a Lyapunov number of a strange attractor to three decimal places is often much more valuable than knowing the eigenvalue of an unstable fixed point to arbitrary precision.

To summarize, the great advantage of formulating the renormalization transformation on a two- rather than a one-dimensional domain is that the universal features are properties of the *stable attractors* of the transformation. In contrast to previous renormalization treatments, it is not necessary to linearize the equations about special points, making it possible to study sets of positive measure. Special cases such as winding numbers with periodic continued fraction expansions lead to stable periodic orbits, but the more typical case of randomly picked continued fraction expansions leads to a more complicated universal strange attractor. The unstable points of previous theories lie on this attractor, which describes the regularities and universal features of circle map bifurcation diagrams. For periodic continued fractions, these regularities are asymptotically exactly self-similar; but for the more typical random case, self-similarity occurs in only an average sense, summarized by the $\bar{\alpha}$ and $\bar{\delta}$ computed here.

There are two important new points in this paper. First, we have demonstrated that it is possible to stabilize the renormalization equations, so that their nonlocal properties can be studied without resorting to linearizing. To our knowledge, this is the first time that this has been done. Second, Rand and co-workers[2] conjectured that the renormalization transformation has a strange attractor, but they gave no evidence to support this conjecture. We have given the first numerical evidence supporting the idea that the full renormalization equations can have strange attractors. We believe that this approach should have application in other problems besides circle maps.

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