Sudden increase of the fractal dimension in a hydrodynamic system

P. Tabeling*
Laboratoire de Génie Electrique de Paris,† Ecole Supérieure d'Electricité,
Université de Paris VI et Université de Paris XI,
Plateau du Moulon, F-91190 Gif-sur-Yvette, France
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An experiment, performed on a circular liquid metal flow subjected to an external magnetic field, shows that the fractal dimension of the attractors describing turbulence can discontinuously increase with the control parameter.

It is now well established that in many hydrodynamic systems, the transition to weak turbulence can be interpreted in terms of dynamical systems. Recently, it has been shown experimentally that weak turbulence can be described by strange attractors in some low-dimensional phase space.[1] The objective of this Brief Report is to extend this result to spiraling Taylor vortex flows and to show that the fractal dimension of such attractors can increase discontinuously with the control parameter.

The experiment consists in driving a flow of mercury between concentric steady cylinders, by means of electromagnetic forces, produced by a radial current I and an external magnetic field B_0. The experimental system has been described elsewhere.[2] The size of the gap between the two cylinders is 1.17 mm; the height of the mercury cell is 20.20 mm and the radius of the inner cylinder is 40 mm. All the experiments have been performed at fixed values of the external magnetic field B_0; the first stability of the laminar flow is in the form of spiraling vortices whose characteristics have been described in a previous paper.[2] In the present experiment, the system included about 12 spiraling vortices. Figure 1 represents the time dependence of a local quantity related to the flow, obtained for $B_0 \simeq 1.15$ T, in the two-spirals regime. The signal is pseudoperiodic with two incommensurate frequencies $f_1 \simeq 3.40$ Hz and $f_1' \simeq 3.52$ Hz. Those oscillators correspond, in the physical space, to two systems of spiraling vortices characterized by the same angular wave number $m=4$, but with helicities of opposite signs.[2] This regime is stable in a finite range of values of the control parameter I. Figure 2 shows the subsequent states of flow, obtained for different values of I lying between 20.25 and 20.36 A. When I is greater than 20.28 A, the regime is chaotic. The dynamical origin of the onset of chaos has been characterized by means of Fourier spectras; the transition involves the competition between two uncompleted subharmonic cascades, one corresponding to the division by 2 of the difference $\delta f = f_1' - f_1$, and the other corresponding to divisions by 3 of δf. Such a competition between two uncompleted cascades is similar to that observed in Rayleigh-Benard experiments.[3]

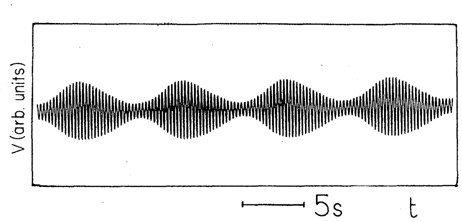

FIG. 1. Direct time recording of a local quantity related to the flow, for $B_0 = 1.15$ T and $I = 20.26$ A. (The physical quantity is the fluctuating voltage measured at the output of a small electrode in contact with the fluid.)

In our experiment, the chaotic nature of the regime can be easily visualized on the direct time recordings by examining the envelopes of the signals. In the physical space, the flow is strictly periodic in the angular direction but it is aperiodic in the direction parallel to the common axis of the cylinder (axial direction). The angular wave number of the flow, as deduced from phase measurements, is $m=4$.

As I is increased above 20.29 A, the system remains chaotic [Fig. 2(b)]. A new dynamical event appears on the direct time recordings for $I=20.36$ A [see Fig. 2(c)]. In this case, the signal cannot be described in terms of two basic frequencies associated with a chaotic envelope; the direct time recording of Fig. 2(c) exhibits a new frequency $f_2 \simeq 0.85$ Hz. Physically this frequency corresponds to the onset of a new mode of instability characterized by an azimuthal wave number $m=1$. This new mode breaks the precedent angular periodicity of the system. The resulting flow is now aperiodic in both the axial and angular directions.[4]

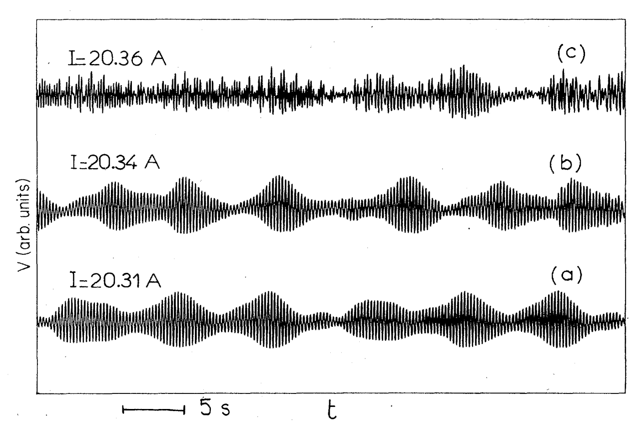

FIG. 2. Direct time recordings showing the various states of flow obtained for $B_0 = 1.15$ T and distinct values of the control parameter I.

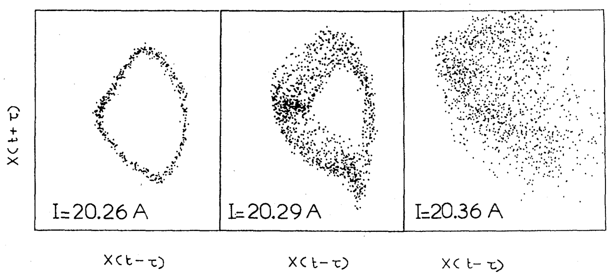

FIG. 3. Poincaré sections defined by intersections of orbits in a three-dimensional phase space with plane $X(t)=0$. The phase portrait is constructed from vectors $X(t_k-\tau)$, $X(t_k)$, $X(t_k+\tau)$, where X is a local quantity related to the flow, t_k are successive values of time t and $\tau=1.66$ s.

We now estimate the effects of such a phenomenon on the characteristics of the chaotic attractors which represent the various turbulent states of the flow. Figure 3 shows two-dimensional (2D) Poincaré sections given by the intersections of the trajectory representing the system in a three-dimensional (3D) phase space with a given plane. When $I=20.26$ A, the regime is pseudoperiodic with two frequencies and the attractor is a two-dimensional torus. The scatter arises from instrumental noise. When $I=20.29$ A, the points of the Poincaré section seem to lie on a curve of dimension greater than one; the form of the precedent torus remains clearly visible, and it is possible to describe the new attractor as composed of a great number of toroidal sheets in a three-dimensional phase space. When $I=20.36$ A, the structure of the precedent attractor is destroyed and one can ask if the trajectory evolves in a higher-dimensional phase space.

We have determined a lower bound of the dimensions of the attractors by using a method proposed by Grassberger and Procaccia which consists first in calculating the integral correlation $C(r)$ in n-dimensional balls of radius r, and after in determining the correlation exponent D_ν. Figure 4 shows the variations of $C(r)$ with r for various values of n, and for $I=20.29$ A. Such curves have been computed by using about 400 orbits, and averaging over ten points in the phase space. For other values of I, the $C(r)$ curves are very similar to those of Fig. 4.

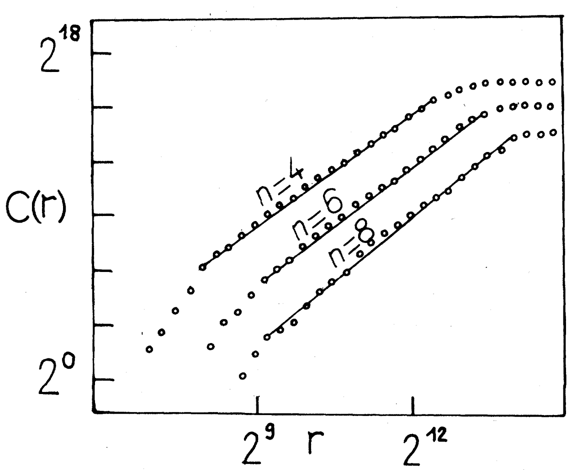

FIG. 4. Variations of the correlation integral $C(r)$ calculated in n-dimensional balls of radius r for increasing values of n, in the case where $I=20.29$ A. The straight lines have been determined by using the least-squares method.

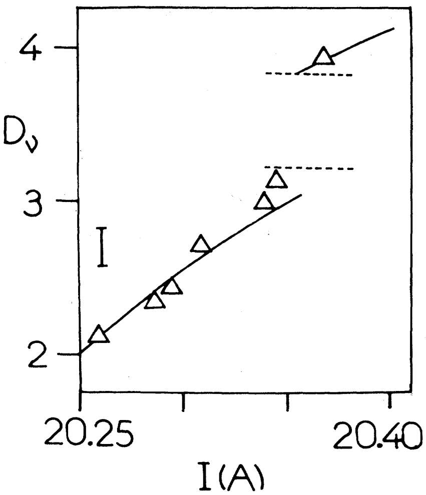

FIG. 5. Variations of D_ν with the control parameter, for $B_0=1.15$ T. The vertical bar in the left of the figure shows an estimate of the numerical error on D_ν.

At low values of r, the experimental noise is predominant and the derivatives of $C(r)$ increase with n. At larger values of r, and in a reasonably wide range, the curves of Fig. 4 can be approximated by straight lines. We have checked that the slopes of such lines converge when n is increased. The correlation exponent D_ν is equal to the limiting value of such slopes as n is increased. We have represented in Fig. 5 dimension D_ν as a function of the control parameter I; the experimental conditions correspond to situations described in Figs. 2 and 3. The curve of Fig. 5 shows a first continuous increase of D_ν with the control parameter; such a region of the figure is associated to attractors whose structure is in form of a large number of toroidal sheets in a 3D phase space. For $I=20.36$ A, a sudden discontinuous increase of D_ν occurs.[5] The jump is associated, in the 3D phase space, to the sudden breaking of the precedent structure of the attractors. In the physical space, it also corresponds, as mentioned earlier, to the onset of a new mode of instability which destroys the spatial angular periodicity of the flow.

The present results indicate that, in a limited range of values of the control parameter, weak turbulence can be described by strange low-dimensional attractors.

This result is in agreement with recent studies performed on other hydrodynamical systems.[1] However, the onset of new modes of instabilities or the changes in the spatial symmetries of the flow cause jumps in the dimension of the chaotic attractor which represent turbulence.

Therefore, the approach of developed turbulence by increasing some generic Reynolds number should be viewed, in phase space, as a discontinuous process.

*Present address: James Franck Institute, University of Chicago, Chicago, IL 60637.
†Laboratoire (No. 127) associé au Centre National de la Recherche Scientifique.

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[2]P. Tabeling and C. Trakas, J. Phys. (Paris) Lett. **45**, L159 (1984).
[3]A. Libchaber and S. Fauve, in *Melting, Localization and Chaos*, edited by R. K. Kalia and P. Vaskishta (Elsevier, New York, 1982), p. 195.
[4]Or eventually pseudoperiodic with two wave numbers in the angular direction, and aperiodic in the axial direction. In any event, the flow ceases to be singly periodic in the angular direction.
[5]When I is increased above 20.36 A, D_ν keeps increasing and the results on the value of D_ν become less accurate.