Transmission of acoustic waves in a random layered medium

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The transmission of acoustic waves through a sequence of alternating layers with random thicknesses but otherwise fixed characteristics is studied by means of the transfer-matrix formalism of one-dimensional disordered chains. The law $\lim_{N\to\infty}\ln(|T_N|/N)\equiv-\lambda(\omega)$ of the exponential decay of the transmission coefficient T_N as a function of the number $(2N)$ of layers is determined in a weak- (strong-) disorder regime for an arbitrary (uniform) distribution of layer thicknesses. The localization constant $\lambda(\omega)$ has a particularly simple form at extreme low and high frequencies ω. Namely $\lambda(\omega\to 0)=\text{const}\times\omega^2$ with a slope given in terms of physical characteristics of the layers and $\lambda(\omega\to\infty)=\text{const}$ defined by a transmission coefficient of a single interface. The predictions are tested by Monte Carlo simulations of a simple model with characteristics of certain rocks. For all frequencies beyond the weak-strong disorder turnover region discrepancies between theoretical and numerical results are merely a few percent.

I. INTRODUCTION

The subject of this paper is the transmission of acoustic waves through a sequence of alternating layers with random thicknesses but otherwise fixed characteristics. The analysis is carried out in the framework of the general theory of one-dimensional disordered chains.[1]

The propagation of acoustic waves in a random-layered medium has been previously examined by several authors. Gilbert originally analyzed the problem by means of Monte Carlo methods focusing his attention on the reflection coefficient of acoustic waves.[2] Unfortunately the results of such analysis are not amenable to a standard statistical interpretation for the reflection coefficient does not possess a statistically meaningful ensemble average, i.e., its variance scales singularly. This point has been particularly emphasized by Anderson *et al.* in a different context.[3] Levine and Willemsen followed up Gilbert's analysis with an attempt to cast it into the framework of the general theory of one-dimensional random chains.[4] Finally, Hodges applied heuristic consideration of Anderson *et al.*[3] to the propagation of acoustic waves through one-dimensional chains in general, but not to layered media.[5]

The present analysis of the random-layered media proceeds from general mathematical results due to Furstenberg.[6] These results have been surmised by Borland in the context of one-dimensional liquids,[7] but their relevance for a description of one-dimensional chains in general was first recognized by Matsuda and Ishii.[8,9] Furstenberg's master formula determines the basic characteristic of the disordered chain, the localization constant, in terms of its transfer matrix.

The localization constants of some standard models have been calculated by Borland,[7] Matsuda and Ishii,[8] and Hirota[10] by means of successive approximations intrinsically tailored to the specific form of the transfer matrices in question. The simple method employed in the present work is flexible enough to apply to any random-transfer matrix with effectively small fluctuations.

In what follows we will first define the transfer matrix and state Furstenberg's master formula for the localization constant. Next, the localization constant will be evaluated in the weak- (low frequency) and strong- (high frequency) disorder regimes for a given probability distribution of layer thicknesses [see Eqs. (3.16), (3.19), and (4.10)]. Finally the results of computer simulations will be presented in the conclusion.

II. TRANSFER MATRIX AND LOCALIZATION CONSTANT

Consider an alternating sequence of layers $A_0=(x<x_1)$, $B_1=(x_1<x<x_1')$, $A_1=(x_1'<x<x_2),\ldots,$ $A_N(x_{N-1}'<x_N)$ consisting of different materials with densities ρ_X, $X=A,B$ as indicated in Fig. 1. It will be assumed that the thicknesses $\Delta_i^A=x_{i+1}-x_i'$ and $\Delta_i^B=x_i'-x_i$ of the layers of different pairs $(B_i,A_i), i=1,2,\ldots,N$ are drawn independently from the common probability distribution $d\mu(\Delta_i^A,\Delta_i^B)$.

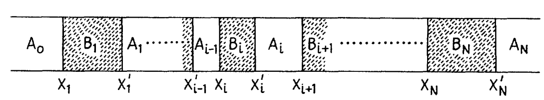

FIG. 1. A one-dimensional medium of alternating layers A_i and B_i, $i=1,2,\ldots,N$ with random thicknesses $\Delta_i^A=x_{i+1}-x_i'$ and $\Delta_i^B=x_i'-x_i$.

The normal propagation of an acoustic wave of frequency ω through such a medium may be described by the complex amplitudes

$$\Phi_{i-1}^{A}(x,t)=e^{i\omega t}(a_{i-1}^{A}e^{ik_A x}+b_{i-1}^{A}e^{-ik_A x}), \quad x'_{i-1}<x<x_i \tag{2.1a}$$

$$\Phi_{i}^{B}(x,t)=e^{i\omega t}(a_{i}^{B}e^{ik_B x}+b_{i}^{B}e^{-ik_B x}), \quad x_i<x<x'_i \tag{2.1b}$$

with the wave numbers $k_X=\omega/c_X$, $X=A,B$. The constant coefficients (a_i^X,b_i^X) are related by virtue of the boundary conditions which ensure the continuity of the wave $\Phi(x,t)$ and its pressure $\rho c^2\partial\Phi/\partial x$ through the interfaces at $x=x_i(x'_i)$, $i=1,2,\ldots,N$ (ρ_X, $X=A,B$ stand for densities). In particular the waves in adjacent layers are matched by the matrix equations

$$\begin{bmatrix} a_{i-1}^A \\ b_{i-1}^A \end{bmatrix}=\underline{h}^*(\phi_{Ai})[z\underline{k}(\gamma)]\underline{h}(\phi_{Bi})\begin{bmatrix} a_i^B \\ b_i^B \end{bmatrix}, \tag{2.2a}$$

$$\begin{bmatrix} a_i^B \\ b_i^B \end{bmatrix}=\underline{h}^*(\phi'_{Bi})[z\underline{k}(\gamma)]^{-1}\underline{h}(\phi'_{Ai})\begin{bmatrix} a_{i+1}^A \\ b_{i+1}^A \end{bmatrix}, \tag{2.2b}$$

defined by the abbreviations

$$\phi_{Xi}=k_X x_i, \quad \phi'_{Xi}=k_X x'_i, \quad X=A,B \tag{2.3a}$$

$$\cosh\gamma=\frac{1}{2}\left[\sqrt{Z}+\frac{1}{\sqrt{Z}}\right], \quad Z=(c_B\rho_B/c_A\rho_A), \tag{2.3b}$$

$$\underline{h}(\phi)=\exp(i\underline{\sigma}_3\phi)=\begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}, \tag{2.4a}$$

$$\underline{k}(\gamma)=\exp(-\underline{\sigma}_1\gamma)=\begin{bmatrix} \cosh\gamma & -\sinh\gamma \\ -\sinh\gamma & \cosh\gamma \end{bmatrix}. \tag{2.4b}$$

Here the $\underline{\sigma}_1$ and $\underline{\sigma}_3$ denote ordinary Pauli matrices. Evidently one can relate the incoming and outgoing waves in the semi-infinite layer A_N to those in A_0 by a straightforward iteration of Eqs. (2.2):

$$\begin{bmatrix} a_N^A \\ b_N^A \end{bmatrix}=\underline{h}^*(\phi'_{AN})\underline{M}_N\underline{h}(\phi_{A1})\begin{bmatrix} a_0^A \\ b_0^A \end{bmatrix}, \tag{2.5}$$

where M_N defines the transfer matrix of all layers compounded of its elementary components m_i:

$$\underline{M}_N=\underline{m}(\alpha_N^B\,|\,\gamma)\underline{h}(\alpha_{N-1}^A)\underline{m}(\alpha_{N-1}^B\,|\,\gamma)\times\underline{h}(\alpha_{N-2}^A)\cdots\underline{h}(\alpha_1^A)\underline{m}(\alpha_1^B\,|\,\gamma), \tag{2.6}$$

$$\underline{m}_i\equiv\underline{m}(\alpha_i^B\,|\,\gamma)=\underline{k}^{-1}(\gamma)\underline{h}(\alpha_i^B)\underline{k}(\gamma), \tag{2.7}$$

with

$$\alpha_{Ai}=k_A(x_{i+1}-x'_i)\equiv k_A\Delta_i^A, \tag{2.8a}$$

$$\alpha_{Bi}=k_B(x'_i-x_i)\equiv k_B\Delta_i^B. \tag{2.8b}$$

Both the transfer matrices $\underline{m}_i$'s and $\underline{M}_N$ share an important group-theoretical property, being elements of a noncompact group SU(1,1):

$$\underline{m}_i^\dagger\underline{\sigma}_3\underline{m}_i=\underline{\sigma}_3, \quad \underline{m}_i\in \mathrm{SU}(1,1), \quad i=1,2,\ldots,N. \tag{2.9}$$

It is easy to see that this is nothing more than the statement of unitarity of the scattering matrix (cf. Ref. 3). Indeed a scattering through a single layer B_i may be described by a unitary $\underline{s}_i$ matrix which relates incoming (a_{i-1}^A,b_i^A) and outgoing (a_i^A,b_{i-1}^A) waves and is conveniently parametrized in terms of complex transmission (t_i) and reflection (r_i) coefficients

$$\begin{bmatrix} a_i^A \\ b_{i-1}^A \end{bmatrix}=\underline{s}_i\begin{bmatrix} a_{i-1}^A \\ b_i^A \end{bmatrix}, \tag{2.10}$$

$$\underline{s}_i=\begin{bmatrix} t_i & r_i \\ r'_i & t'_i \end{bmatrix}, \quad |t_i|^2+|r_i|^2=1, \tag{2.11}$$

where $r'_i=-r_i^*t'_i/t_i^*$ by unitarity, $\underline{s}_i\underline{s}_i^\dagger=\mathbb{1}$, and $t'=t$ by time reversal invariance. It is sufficient to recall that the transfer matrix $\underline{m}_i$ relates incoming (a_i^A) and outgoing (b_i^A) waves on the right of the layer B_i to those (a_{i-1}^A,b_{i-1}^A) on the left to infer the desired representation

$$\underline{m}_i=\begin{bmatrix} 1/t_i & -r_i/t_i \\ -r_i^*/t_i^* & 1/t_i^* \end{bmatrix}, \quad |t_i|^2+|r_i|^2=1. \tag{2.12}$$

Of course it satisfies Eq. (2.9) trivially as claimed above.

An obvious extension of Eq. (2.12) is

$$\underline{M}_N=\begin{bmatrix} 1/T_N & -R_N/T_N \\ -R_N^*/T_N^* & 1/T_N^* \end{bmatrix}, \quad |T_N|^2+|R_N|^2=1, \tag{2.13}$$

where T_N and R_N denote, respectively, the transmission and reflection coefficients of N B-type layers separated by $N-1$ A-type layers (cf. Fig. 1).

Referring back to Eq. (2.6) one immediately concludes that the transmission coefficient of the random-layered media in question is determined by a product of N random matrices $\underline{h}(\alpha_i^A)\underline{m}(\alpha_i^B\,|\,\gamma)\in \mathrm{SU}(1,1)$ whose parameters $\alpha_i=(\alpha_{Ai},\alpha_{Bi})$ have a common probability distribution, $d\mu(\alpha_i)$. Large-N limits of such products have been systematically studied in the mathematical literature.[6] The results relevant to the subsequent discussion will now be stated.

Let $g_i\equiv g(\alpha_i)\in \mathrm{SU}(1,1)$, $i=1,2,\ldots,N$ be random matrices with a common probability distribution $d\mu(\alpha_i)$ of parameters $\alpha_i=\{\alpha_{Xi},X=A,B,\ldots\}$, acting in the complex z plane as

$$\begin{bmatrix} z(g) \\ z^*(g) \end{bmatrix}\equiv\begin{bmatrix} g_{11} & g_{12} \\ g_{12}^* & g_{11}^* \end{bmatrix}\begin{bmatrix} z \\ z^* \end{bmatrix}=\begin{bmatrix} g_{11}z+g_{12}z^* \\ (g_{11}z+g_{12}z^*)^* \end{bmatrix}. \tag{2.14}$$

Then the limit

$$\frac{1}{N}\ln\left|\frac{z(g_1,g_2,\ldots,g_N)}{z}\right|\to\lambda\geq 0 \quad \text{as } N\to\infty \tag{2.15}$$

is reached with probability unity, and is determined by the system of equations

$$\lambda=\int \ln\left|\frac{z[g(\alpha)]}{z}\right|\nu(\theta)d\theta\,d\mu(\alpha)\,, \tag{2.16a}$$

$$\nu(\theta)=\int \nu(\theta(\alpha))[d\theta(\alpha)/d\theta]d\mu(\alpha)\,, \tag{2.16b}$$

$$\theta=\arg z,\quad \theta(\alpha)=\arg z(g(\alpha))\,. \tag{2.17}$$

The reader should consult Ziman's book for an excellent heuristic review of these rigorous results.[1]

Equation (2.16b) is the Dyson-Schmidt self-consistency condition. The auxiliary function $\nu(\theta)$ above is called a stationary measure and defines an induced probability distribution of the random complex number z. Observe that there is an arbitrariness in the definition of $\nu(\theta)$. Indeed it depends on the choice of the random transfer matrix which as far as λ is concerned is defined up to an arbitrary similarity transformation

$$\underline{g}\to\underline{R}\underline{g}\underline{R}^{-1},\quad \underline{R}\in \mathrm{SU}(1,1)\,. \tag{2.18}$$

This freedom will be explored in the following sections to facilitate calculations of λ.

The meaning of the integrand (2.16a) becomes more transparent if one recalls the local isomorphism between SU(1,1) and the group SL(2,R) of 2×2 real unimodular matrices $\underline{r}$:

$$\underline{g}=\underline{U}^{\dagger}\underline{r}\underline{U},\quad \underline{g}\in\mathrm{SU}(1,1),\quad \underline{r}\in\mathrm{SL}(2,R)\,, \tag{2.19}$$

where

$$\underline{U}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix},\quad \underline{U}\underline{U}^{\dagger}=\mathbb{1}\,. \tag{2.20}$$

Evidently r generates motion in the plane ($x_1=\mathrm{Re}z$, $x_2=\mathrm{Im}z$), i.e., $x_1(r)=r_{11}x_1+r_{12}x_2$, $x_2(r)=r_{21}x_1+r_{22}x_2$. In terms of the norm $|x|=(x_1^2+x_2^2)^{1/2}$ the desired relation is

$$\lambda(\theta,g)\equiv\ln\left|\frac{z(g)}{z}\right|=\ln\frac{|x(r)|}{|x|}\,,$$
$$\tan\theta=x_1/x_2\,. \tag{2.21}$$

Hence it becomes obvious that the quantity $\lambda(\theta,g)$ is the dilation of the vector $\mathbf{x}=(x_1,x_2)^T$ induced by the random matrix $r(\alpha)$ when the probability distribution of $\mathbf{x}(\theta)$ and $r(\alpha)$ are $\nu(\theta)d\theta$ and $d\mu(\alpha)$, respectively.

One readily infers from the preceding discussion that for sufficiently large number of layers [cf. Eqs. (2.6), (2.13), and (2.15)],

$$\frac{1}{N}\ln|T_N|\to-\lambda\,. \tag{2.22}$$

Here λ is given by Eq. (2.16) defined in terms of a suitably chosen transfer matrix as suggested by Eq. (2.18) (Ref. 11)

$$\begin{aligned}\underline{g}(\alpha_A,\alpha_B)&=\underline{R}^{-1}\underline{h}(\alpha_A)\underline{m}(\alpha_B\,|\,\gamma)\underline{R}\\ &=\underline{R}^{-1}\underline{h}(\alpha_A)\underline{k}^{-1}(\gamma)\underline{h}(\alpha_B)\underline{k}(\gamma)\underline{R}\,.\end{aligned} \tag{2.23}$$

Henceforth the average dilation λ will be referred to as a localization constant to conform with the accepted terminology in the literature.[1]

III. LOCALIZATION CONSTANT IN THE WEAK-DISORDER REGIME

Here a successive approximation scheme will be set up for the system of Eqs. (2.16) and (2.23). This will be achieved by assuming that the fluctuations of the disorder variable α_X about its average value $\bar{\alpha}_X$ is effectively small and therefore its variance σ_X may be used as an expansion parameter:

$$\sigma_X^2=\int(\alpha_X-\bar{\alpha}_X)^2d\mu\ll1,\quad X=A,B \tag{3.1}$$

$$\bar{\alpha}_X\doteq\int\alpha_X d\mu\equiv k_X\bar{\Delta}_X,\quad X=A,B\,. \tag{3.2}$$

Obviously the average thicknesses $\bar{\Delta}_X$ characterize the regular counterpart of the random-layered media. Note also that the above definition (3.1) of the weak disorder encompasses the regime of sufficiently low frequencies.

The calculation of the localization constant (2.16a) and the stationary measure (2.16b) will proceed from the small parameter expansions

$$\lambda=\lambda^{(0)}+\lambda^{(1)}+o(\sigma_X^2)\,, \tag{3.3}$$

$$\nu(\theta)=\nu^{(0)}(\theta)+\nu^{(1)}(\theta)+o(\sigma_X^2)\,. \tag{3.4}$$

A significant simplification will be achieved by exploring the freedom of choice of the auxiliary matrix $\underline{R}$ in Eq. (2.23). Namely, it will be assumed that $\underline{R}$ renders diagonal the transfer matrix of the regular layered media

$$\bar{\underline{g}}\equiv\underline{g}(\bar{\alpha})=\begin{bmatrix}\Lambda_+ & 0\\ 0 & \Lambda_-\end{bmatrix}. \tag{3.5}$$

The eigenvalues $\Lambda_\pm$ and matrix $\underline{R}$ may be easily expressed in terms of the matrix elements a and b of the regular transfer matrix:

$$\underline{h}(\bar{\alpha}_A)\underline{k}^{-1}(\gamma)\underline{h}(\bar{\alpha}_B)\underline{k}(\gamma)\equiv\begin{bmatrix}a & b\\ b^* & a^*\end{bmatrix}, \tag{3.6}$$

$$\underline{R}=\begin{bmatrix}\dfrac{|b|}{N_+} & \dfrac{ib}{N_-}\\[2mm] \dfrac{-ib^*}{N_-} & \dfrac{|b|}{N_+}\end{bmatrix}, \tag{3.7a}$$

$$\Lambda_\pm=a_r\pm i(1-a_r^2)^{1/2}\equiv e^{\pm i\chi},\quad |a_r|\le1\,, \tag{3.7b}$$

where

$$N_\pm^2=2(1-a_r^2)^{1/2}[a_i\mp(1-a_r^2)^{1/2}]\,, \tag{3.8a}$$

$$a_r=\mathrm{Re}a,\quad a_i=\mathrm{Im}a\,. \tag{3.8b}$$

Observe that the condition $|a_r|\le1$, i.e., $|\Lambda_\pm(\omega)|=1$ defines the spectral band of normal modes of the regular layered medium. The following calculations are restricted to the frequencies in the spectral band for in the region of the spectral gap $|\Lambda_\pm(\omega)|>1$ the regular layered medium already exhibits a finite localization constant, i.e., an exponential decay of the transmission amplitude and a weak disorder merely adds small corrections to it.

We are now in a position to proceed to the determination of individual terms of the series (3.3). A simple comparison of the same order terms in Eq. (2.16) immediately leads to

$$\lambda^{(0)}=\int d\theta\,\nu^{(0)}(\theta)\ln\left|\frac{z[g(\bar{\alpha})]}{z}\right| , \quad (3.9a)$$

$$\nu^{(0)}(\theta)d\theta=\nu^{(0)}(\theta(\bar{\alpha}))d\theta(\bar{\alpha}) , \quad (3.9b)$$

$$\lambda^{(1)}=\int d\theta\left[\nu^{(0)}(\theta)\sum_X\frac{1}{2}\sigma_X^2\frac{\partial^2}{\partial\alpha_X^2}+\nu^{(1)}(\theta)\right]\times\ln\left|\frac{z(g(\alpha))}{z}\right|_{\alpha=\bar{\alpha}} , \quad (3.10a)$$

$$\nu^{(1)}(\theta)d\theta=\nu^{(1)}(\theta(\bar{\alpha}))d\theta(\bar{\alpha})+\tfrac{1}{2}\sum_X\sigma_X^2\frac{\partial^2}{\partial\alpha_X^2}[\nu^{(0)}(\theta(\alpha))d\theta(\alpha)]_{\alpha=\bar{\alpha}} . \quad (3.10b)$$

Evidently Eq. (3.9a) determines the localization length $\lambda^{(0)}$ of the regular layered media. As expected it vanishes because $|z(g(\bar{\alpha}))|=|z|$ by Eqs. (2.14) and (3.5). As concerns $\lambda^{(1)}$ it is entirely determined by $\nu^{(0)}(\theta)$ since the coefficient of $\nu^{(1)}(\theta)$ in the integrand (3.10a) vanishes identically. On the other hand the solution to Eq. (3.9b) is obviously $\nu^{(0)}(\theta)=\text{const}$ since by virtue of definitions (2.17) and (3.7b) $\theta(\bar{\alpha})=\theta+2\chi$. Hence one concludes that

$$\lambda=\lambda^{(1)}=\int_0^{2\pi}\frac{d\theta}{2\pi}\sum_X\frac{1}{2}\sigma_X^2\frac{\partial^2}{\partial\alpha_X^2}\ln|g_{11}(\alpha)+g_{12}(\alpha)e^{-i\theta}|_{\alpha=\bar{\alpha}}+o(\sigma^2) . \quad (3.11)$$

Note that the measure has been properly normalized $\nu^{(0)}=1/2\pi$.

The evaluation of the above integral is trivial and yields

$$\lambda=\tfrac{1}{2}\sum_X\sigma_X^2[\ln|g_{11}(\alpha)|]''_{\alpha_X=\bar{\alpha}_X}+o(\sigma^2) . \quad (3.12)$$

Thus the problem is reduced to the calculation of the matrix element $g_{11}(\alpha)$. It is a matter of straightforward algebraic manipulations to show that

$$g_{11}(\alpha)=A_r(\alpha)+i\frac{a_iA_i(\alpha)-\mathrm{Re}[b^*B(\alpha)]}{(1-a_r^2)^{1/2}} , \quad (3.13)$$

where the functions $A(\alpha)$ and $B(\alpha)$ are direct extensions of a and b, respectively, to arbitrary parameters α, i.e., $A(\bar{\alpha})=a$ and $B(\bar{\alpha})=b$. Noting that $A'_{\alpha=\bar{\alpha}}=a'_{\bar{\alpha}}$, $A''_{\alpha=\bar{\alpha}}=a''_{\bar{\alpha}}$, etc., one easily derives

$$[g_{11}(\alpha)]'_{\alpha=\bar{\alpha}}=\Lambda'_+ , \quad (3.14a)$$

$$[g_{22}(\alpha)]''_{\alpha=\bar{\alpha}}=-\Lambda_+ . \quad (3.14b)$$

Returning to Eq. (3.12) we conclude that

$$\lambda=-\sum_X\tfrac{1}{2}[1+\mathrm{Re}(\partial\ln\Lambda_+/\partial\bar{\alpha}_X)^2]\sigma_X^2+o(\sigma^2) \quad (3.15)$$

or more explicitly

$$\lambda=\frac{1}{2}\frac{\sinh^2(2\gamma)}{1-a_r^2}(\sigma_A^2\sin^2\bar{\alpha}_B+\sigma_B^2\sin^2\bar{\alpha}_A)+o(\sigma^2) \quad (3.16)$$

with

$$a_r=\cos\bar{\alpha}_A\cos\bar{\alpha}_B-\cosh^2\gamma\,\sin\bar{\alpha}_A\sin\bar{\alpha}_B . \quad (3.17)$$

Observe that the localization constant increases with the frequency approaching to the edges $|a_r(\omega)|=1$ of the spectral band as should be expected. In the special case of extreme low frequencies $\bar{\alpha}_X\ll 1$ its limiting form can be easily deduced:

$$\lambda=\left[\frac{\sinh^2 2\gamma}{2}\right]\frac{\bar{\alpha}_A^2\sigma_B^2+\bar{\alpha}_B^2\sigma_A^2}{\bar{\alpha}_A^2+\bar{\alpha}_B^2+2\bar{\alpha}_A\bar{\alpha}_B\cosh(2\gamma)}+o(\bar{\alpha},\sigma^2) . \quad (3.18)$$

This is rendered more transparent in terms of the dimensionful physical parameters sound speeds c_X and layer thicknesses Δ_X:

$$\lambda(\omega)=\frac{\sinh^2(2\gamma)}{2}\frac{\delta_A^2/\bar{\Delta}_A^2+\delta_B^2/\bar{\Delta}_B^2}{(c_A/c_B)(\bar{\Delta}_B/\bar{\Delta}_A)+(c_B/c_A)(\bar{\Delta}_A/\bar{\Delta}_B)+2\cosh(2\gamma)}(\bar{\Delta}_A/c_A)(\bar{\Delta}_B/c_B)\omega^2+o(\omega^4) , \quad (3.19)$$

where $\delta_X^2\equiv\overline{\Delta_X^2}-\bar{\Delta}_X^2$ is the variance of the layer thickness.

The preceding analysis has been restricted to leading-order effects. It is not difficult to see from Eq. (3.10b) that higher-order contributions are well defined and amenable to a direct evaluation. Indeed the estimates $\nu^{(1)}(\theta)-\nu^{(1)}[\theta(\bar{\alpha})=\theta+2\chi]\sim\chi\sim\omega$ and $\sigma_X^2\sim\omega^2$ guarantee the validity of the perturbation theory $\nu^{(0)}\gg\nu^{(1)}\sim\omega$ even at low frequencies in contradistinction to the situation encountered in the Andersom model.[12] In addition, only the first few Fourier coefficients of $\nu^{(1)}(\theta)$ are nonvanishing for $d\theta(\bar{\alpha})=d\theta$, and $[d\theta(\alpha)]''_{\alpha=\bar{\alpha}}$ is just a second-order polynomial in $z(z^*)$. Thus if necessary the corrections to the above results (3.16) and (3.19) can be identified in a straightforward manner.

IV. LOCALIZATION CONSTANT IN THE STRONG-DISORDER REGIME

The strong-disorder regime will be characterized by large fluctuations of disorder variables α_X or, to be more precise, by the condition

$$\sigma_X/2\pi \gg 1 \ . \tag{4.1}$$

Obviously the inequality always holds for sufficiently high frequencies. It will be further assumed that the probability distribution of disorder variables is uniform

$$d\mu(\alpha)=\prod_X \frac{d\alpha_X}{2\pi}, \quad 0\le \alpha_X \le 2\pi \ . \tag{4.2}$$

It is legitimate to restrict the range of α's to the interval $[0,2\pi]$ in the light of the condition (4.1) as well as periodicity and positivity properties of the integrands (2.16). Note that the integrands (2.16a) and (2.16b) are positive by virtue of the inequality $|z(g(\alpha))/g| \ge |g_{11}|^2-|g_{12}|^2=1$ and definition of the measure $\nu(\theta)$.

Now we proceed to the calculation of $\nu(\theta)$ and λ via Eqs. (2.16) and (2.23) and (4.2). First we will check by inspection that the transfer matrix (2.23) with $R^{-1}=k(\gamma)$ leads to the uniform measure $\nu(\theta)=1/2\pi$. It is sufficient to convince oneself that

$$\int \frac{d\theta(\alpha)}{d\theta}\prod_X \frac{d\alpha_X}{2\pi}=1 \Longrightarrow \nu(\theta)=1 \ . \tag{4.3}$$

This is easy to check by recalling that $t(\alpha)=e^{i\theta(\alpha)}$ is related to $t\equiv e^{i\theta}$ via successive conformal transformations $\underline{k}(\gamma)$, $\underline{h}(\alpha_B)$, $\underline{k}^{-1}(\gamma)$, and $\underline{h}(\alpha_A)$, which map a unit circle onto itself. Hence, one readily infers the relations

$$\frac{d\theta(\alpha)}{d\theta}=|g_{11}t+g_{12}|^{-2} \ , \tag{4.4}$$

$$|g_{11}t+g_{12}|=|cu^2t'+s| \ |cv^2t-s| \ , \tag{4.5}$$

defined in terms of the abbreviations

$$u=e^{i\alpha_A}, \quad v=e^{i\alpha_B} \ , \tag{4.6a}$$

$$c=\cosh\gamma, \quad s=\sinh\gamma \ , \tag{4.6b}$$

$$t'=(cv^2t-s)/(c-sv^2t) \ . \tag{4.6c}$$

Returning to the integral (4.3) with the relations (4.4) and (4.5) and invoking the integral of the Poisson kernel

$$\frac{1}{2\pi i}\int_{|u|=1}\frac{(1-|w|^2)}{u\,|u-w|^2}du=1 \tag{4.7}$$

one immediately confirms the statement (4.3).

Now we turn to the localization constant which is given by (2.16a) with $\nu(\theta)=1$:

$$\lambda=\int \ln|g_{11}+g_{12}t|\frac{d\theta}{2\pi}d\mu(\alpha) \ . \tag{4.8}$$

The resulting integral is trivial since its integrand is a harmonic function inside the unit disc $|t|\le 1$. Taking into account the identity (4.5) one finds

$$\lambda=\int \ln|c^2-s^2u^2|\prod_X\frac{d\alpha_X}{2\pi}=\ln c^2 \ . \tag{4.9}$$

This is a very simple result with a clear physical interpretation. Namely, it predicts a decay of the transmission coefficient of N pairs of layers as c^{-2N}, i.e.,

$$\lim_{N\to\infty}\left[\frac{1}{N}\ln|T_N|\right]=-\ln(\cosh^2\gamma) \ . \tag{4.10}$$

This may be viewed as an effect of independent scatterings from $2N$ interfaces since the wave transmitted through the interface of two semi-infinite layers is damped by a factor $(\cosh\gamma)^{-1}$ [see Eqs. (2.4b) and (2.12)]. Perhaps it should be reemphasized that no simplifying assumption has been made concerning the scattering "strength" of individual interfaces (cf. Ref. 13).

It is instructive to rederive the result (4.10) in a heuristic manner. For this purpose we note that the composition law of the transfer matrix (2.6),

$$\underline{M}_{N+1}=\underline{m}(\alpha_{N+1}^B|\gamma)\underline{h}(\alpha_N^B)\underline{M}_N \tag{4.11}$$

implies the relation [cf. Eqs. (2.12) and (2.13)]

$$\ln|T_{N+1}|=\ln|T_N|+\ln|t_{N+1}| \\ -\ln|1-\overset{*}{r}_{N+1}R_N(\overset{*}{T}_N/T_N)\exp(-2i\alpha_N^A)| \ . \tag{4.12}$$

Now it is easy to see that upon averaging according to the probability distribution (4.2) the last term vanishes whereas the second term reproduces the integral (4.9):

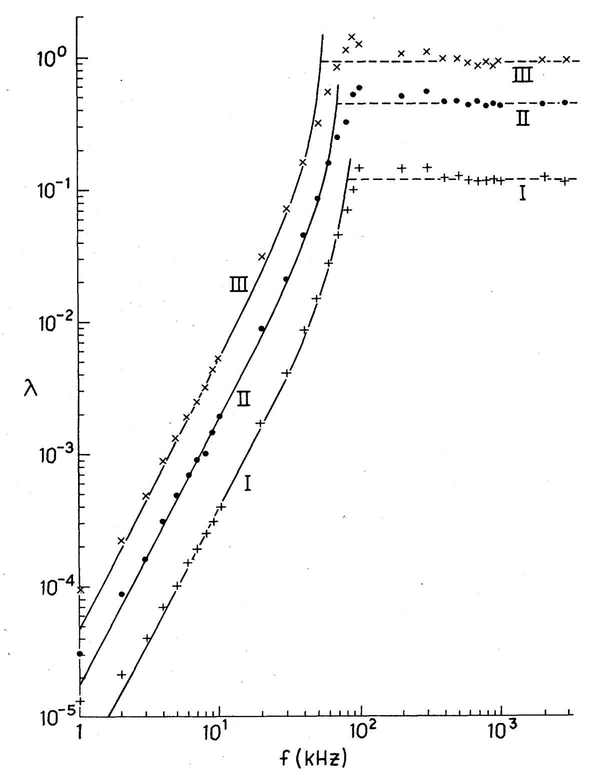

FIG. 2. Continuous (dashed) curves I, II, and III represent the behavior of the localization constant $\lambda(\omega)$ for three distinct values of the parameter $Z=2$ (I), 4 (II), and 8 (III) in the weak- (strong-) disorder regime as described by Eq. (3.16) [Eq. (4.9)]. Points near the theoretical curves indicate the results of Monte Carlo simulations for corresponding values of $\omega=2\pi f$ and Z.

$$\langle \ln | T_{N+1} | \rangle = \langle \ln | T_N | \rangle - \ln(\cosh^2\gamma) \ . \tag{4.13}$$

Hence the result (4.10) reemerges provided one assumes following Anderson *et al.*[3] that the quantity $\ln | T_N |$ is amenable to a statistical interpretation, i.e., its average $\langle \ln | T_N | \rangle$ is a typical member of the statistical ensemble in question. Similar arguments have been also invoked by Hodges in a general context of transmission of acoustic waves through one-dimensional chains.[5]

V. NUMERICAL TESTS

We have applied the results of previous sections, Eqs. (3.16) and (4.9), to the simple model of certain rocks considered earlier in Ref. 4. The model in question represents alternating layers of sandstone (A) and shale (B) with a uniform distribution of thicknesses characterized by the parameters

$$3 \ \text{mm} \le \Delta_A \le 23 \ \text{mm} \ , \tag{5.1a}$$

$$0.1 \ \text{mm} \le \Delta_B \le 5.1 \ \text{mm} \ , \tag{5.1b}$$

$$c_A = 5 \ \text{km} \ , \tag{5.2a}$$

$$c_B = 2 \ \text{km} \ . \tag{5.2b}$$

The ratio of acoustic impedances Z as defined in Eq. (2.3b) has been considered as a free parameter.

We have computed the logarithm of the transmission coefficient [cf. Eq. (2.22)] for 2500 pairs of layers and averaged over 200 realizations. Results of our calculations along with corresponding theoretical predictions for three representative values of the parameter $Z = 2,4,8$ are displayed in Fig. 2. Evidently for all frequencies beyond the weak-strong disorder turnover region, about $f \approx 100$ kHz, deviations of the Monte Carlo points from the theoretical curves do not exceed a few percent. We believe that the large descrepancies at the low end of the spectrum $f \approx 1$ kHz are entirely due to statistical fluctuations. Indeed, by reducing the number of layers ($N \approx 100$) one discovers such discrepancies even at higher frequencies ($f = 10$ kHz).

A closer examination of Fig. 2 reveals some special features. The departure from theoretical predictions is already evident at $f_m = 50$ kHz when the variance of the phase is still relatively small $\sigma_{A(B)}(f_m) = 0.35$ (0.23) [cf. Eq. (3.1)]. On the other hand, in light of the condition (4.1), the approach to the asymptotic regime at $f_a = 100$ kHz seems very precocious since $\sigma_{A(B)}\ (f_a)/2\pi = 0.11$ $(0.07) \ll 1$.

We conclude that the numerical results are in excellent agreement with the theoretical predictions (3.16) and (4.9).

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[1]For a lucid discussion of this theory, see J. M. Ziman, *Models of Disorder* (Cambridge University Press, Cambridge, England, 1979), Chap. 8.

[2]K. E. Gilbert, J. Acoust. Soc. Am. **68**, 1454 (1980).

[3]P. W. Anderson, D. J. Thoules, E. Abrahms, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980). See also E. Abrahms and M. J. Stephen, J. Phys. C **13**, L377 (1980); **13**, L383 (1980).

[4]H. Levine and J. Willemsen, J. Acoust. Soc. Am. **73**, 32 (1983).

[5]C. H. Hodges, J. Sound Vib. **82**, 411 (1982).

[6]H. Furstenberg, Trans. Am. Math. Soc. **108** (3), 377 (1963). See also Y. N. Tutubalin, Theory Probab. Its Appl. (USSR) **10** (1), 19 (1965), in Russian.

[7]R. E. Borland, Proc. R. Soc. London **A274**, 529 (1963).

[8]H. Matsuda and K. Ishii, Prog. Theor. Phys. (Suppl.) **45**, 56 (1970).

[9]K. Ishii, Prog. Theor. Phys. (Suppl.) **53**, 77 (1973).

[10]T. Hirota, Prog. Theor. Phys. **50** (4), 1240 (1973).

[11]The random transfer matrices of the general form (2.23) do satisfy the conditions of the Furstenberg theorem as shown by Matsuda and Ishii (Ref. 8).

[12]M. Kappus and F. Wegner, Z. Phys. B **45**, 15 (1981).

[13]D. J. Thouless, J. Phys. C **5**, 77 (1972).