Theory of a dithered-ring-laser gyroscope: A Floquet-theory treatment

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This paper discusses the properties of a periodically dithered-ring-laser gyroscope in the approximation of a single phase-locking equation. We discuss both modulation of the input rotation, the phase of the backscattering, and its amplitude. The first two are found to be mathematically equivalent, and the last case is found to offer no advantages as compared with the undithered case. These conclusions are supported by a heuristic argument. The detailed mathematical treatment is based on Floquet theory, which allows us to obtain results by integrating over one dither period only. The locking condition can be determined from the Floquet exponent. For large input parameters the integration of the differential equation for the Floquet problem becomes numerically overwhelming, and the equivalent formulation in terms of an infinite matrix is utilized. This is evaluated using a method based on matrix continued fractions. In this way no restrictions on the parameters are necessary. The method is applied to the single-frequency dithering, and it is confirmed that the locking at zero rotation rate can be completely eliminated. The calculations also confirm the existence of higher-order lock-in zones, which are large just in those conditions which are optimal near zero rotation rate. Thus we conclude that with sinusoidal dither of one frequency it is not possible to avoid nonlinearities in the gyroscope response. In forthcoming publications we intend to discuss possible schemes to overcome this difficulty.

I. INTRODUCTION

It is generally accepted that the simple two-mode—ring-laser gyroscope can be modeled by an equation describing the phase driven by an externally imposed rotation of the device. The backscattering introduces a nonlinear term owing to the phase difference between the two counterpropagating waves. This treatment assumes that no appreciable amplitude excursions appears even at the lock-in threshold. The amplitudes are assumed constant and usually of equal magnitude (see Aronowitz[1]). In this simple form the phase equation can be solved exactly and the lock-in threshold is reached when the rotation rate exactly equals the backscattering coefficient. Thus the only way to improve the performance of a device for low rotation rates is to reduce the coupling between the modes. This meets eventually with technical difficulties and in practice only a mode difference of a few hundred Hertz can be reached. This corresponds approximately to a rotation rate of 0.1° per second.

In order to improve the sensitivity of the device its motion has been overlaid by a periodically varying dither motion. Its amplitude makes the ring laser pass through the locking region during each period and the locking error decreases. It can be made plausible[2] that in this way one may eliminate the lock-in band at zero rotation rate totally. The complication is, however, that there appears a series of higher-order lock-in bands at multiples of the dither frequency.[3] These cannot be made to disappear together with the lowest-order lock-in band.

In the present series of papers we want to consider the detailed behavior of a dithered-laser system in an approximation based on the single phase-locking equation. Its validity will not be questioned at the present time. To determine the locking behavior by direct integration of the dithered phase equation is found to be time consuming and very sensitive to numerical inaccuracies. Our experience is that 10^4—10^5 dither periods are needed to reach convergence. This is not a very practical approach.

Our present treatment utilizes the fact that the problem can be transformed to a second-order differential equation with periodic coefficients. Then the powerful Floquet theory[4] can be applied; this provides some exact mathematical results, which can be utilized to greatly shorten the numerical computations. The equations need to be integrated over one dither period only. In addition, the problem can be formulated in analogy with an eigenvalue problem, and the full behavior is determined by an eigenvalue condition in terms of an infinite Hill determinant.[5] We have developed a method[6] to treat this numerically, which provides fast and accurate convergence of the numerical problem.

Even if the mathematical theory of differential equations with periodic coefficients is well treated in the literature,[4,5] we present the basic tools needed in Sec. II. This is useful because there are physical aspects to the treatment which are relevant for the present problem only. Furthermore, we need the results for later reference in our numerical applications. The numerical method developed in Sec. III rests heavily on the theory exposed in Sec. II.

In order to round off the sharp thresholds at lock-in the experimenters have introduced random noise. In a highly precise device the operation is limited by quantum noise. The state of ring-laser technology now appears to be such that quantum noise sets a limit to the operation. In a series of papers Cresser *et al.*[7-9] have treated the ring laser with noise. They treat the system analytically in special cases[7] and discuss the detailed numerical behavior of the beat spectrum in various limits.[8,9] When dither is added to their problem the ensuing system is highly complicated. Recently Schleich *et al.*[10] have managed to treat the dithered laser with noise both approximately and numerically with high precision.

In order to gain some understanding of the phenomenon of higher-order lock-in bands we present in Sec. II B a heuristic treatment based on the theory of phase modulation. The theory gives the lock-in regions in terms of Bessel functions. The same approach has been developed into a systematic approximation method by Schleich *et al.*[10]; here we only use it as a point of comparison for the calculated results.

In the present paper we apply our calculational method to the simplest case of sine-wave dithering. This is mainly given as an illustration; nothing new or unexpected emerges. The possibility of eliminating the zero-rotation-rate lock-in band is confirmed, but this occurs at the expense of the appearance of higher-order lock-in bands. The calculational convenience of Floquet theory is adequately demonstrated, and the accuracy of the frequency modulation scheme can be estimated.

The degree of freedom offered by the dither waveform has, however, not yet been utilized. We contend that by applying an arbitrarily assigned waveform the behavior of the gyroscope can be optimized over its whole range of operation. This can be supported by the heuristic argument developed in Sec. II B and proven by detailed calculations utilizing the Floquet theory developed in this paper. These results will be the subject of a subsequent paper. Its main results were reported in Ref. 6.

A. Periodic variation of parameters in the phase-locking equation

Over a wide range of parameters it is believed that the locking in a two-mode ring laser can be described by the equation

$$\dot{\varphi}=A+B\sin\varphi\ , \tag{1}$$

where A is proportional to the rotation rate of the device, and B is determined by the mode coupling due to backscattering. When a more exact semiclassical theory is utilized, there appears a power shift of the parameter A and a variation of the coefficient B with the amplitudes of the laser fields. In Eq. (1) there appears phase locking when $|A|\le|B|$ and φ becomes fixed at the value

$$\varphi=\arcsin(A/B)\ . \tag{2}$$

This will considerably deteriorate the use of the device as a rotation sensor.

In order to decrease the tendency to lock the laser gyroscopes are often subjected to mechanical dithering. The rotation rate A is periodically modulated at a frequency Ω, with an amplitude C which has to exceed the backscatter coefficient B in magnitude to allow the gyroscope to swing out of the locking region for each period. The equation to be solved is then

$$\dot{\varphi}=A+B\sin\varphi+C\cos(\Omega t)\ . \tag{3}$$

An interesting observation emerges if we use the substitution

$$y=\varphi-\frac{C}{\Omega}\sin(\Omega t) \tag{4}$$

in Eq. (3) to find the equivalent equation

$$\dot{y}=A+B\sin\left[y+\frac{C}{\Omega}\sin(\Omega t)\right]\ . \tag{5}$$

It can be interpreted as a periodic phase modulation of the backscattered wave. This case can thus be treated as exactly equivalent with the dithering case in Eq. (3).

We can for completeness also consider a third kind of modulation, namely a periodic variation of the amplitude of the backscatter coefficient

$$\dot{\varphi}=A+B[1+C\cos(\Omega t)]\sin\varphi\ . \tag{6}$$

Here we usually may assume $|C|<1$. This case cannot be reduced directly to the previous one. We thus have three cases of modulating the phase-locking equation (1).

Case I. Modulation of the rotating rate mechanically or by any other means giving an optical-path difference to the counter-propagating waves.

Case II. Phase modulation of the backscattered wave causing the mode locking. At the present level of treatment this is equivalent with case I.

Case III. Amplitude modulation of the backscattering.

Case III is found to offer no advantages but for completeness its properties are briefly summarized in the Appendix.

B. A heuristic treatment

In order to obtain some preliminary understanding of the behavior of the theory we give a simple, heuristic discussion valid for cases I and II. From earlier treatments[11] we know that the dithering in Eq. (3) introduces additional lock-in zones near rotation rates being multiples of the dither frequency Ω. We hence expand A around $m\Omega$, where m is an integer,

$$A=m\Omega+a\ . \tag{7}$$

Near $m\Omega$ we use the substitution

$$y=\varphi-m\Omega t-\frac{C}{\Omega}\sin(\Omega t) \tag{8}$$

instead of (4) and obtain

$$\begin{aligned}\dot{y}&=a+B\sin[y+m\Omega t+(C/\Omega)\sin(\Omega t)]\\&=a-i\tfrac{1}{2}B\left[e^{i(y+m\Omega t)}\sum_{n=-\infty}^{\infty}J_n(C/\Omega)e^{in\Omega t}-\text{c.c.}\right]\ ,\end{aligned} \tag{9}$$

where we have used a well-known expansion in terms of Bessel functions J_n.

The physical meaning of Eq. (9) is obvious. The periodic rotation modulates the phase of the oscillating fields in the laser, and according to the theory of phase modulation there appear sidebands around the central frequency of the laser, their strength being given by the Bessel functions.

The observation in a laser gyroscope is integrated over many periods of the dither frequency and hence we expect the time-independent term of (9) to be most influential; the others average to zero over many periods. Thus for fixed m we obtain the approximate equation

$$\dot{y}=a+BJ_{-m}(C/\Omega)\sin y \ . \quad (10)$$

This shows locking when the rotation rate A differs from $m\Omega$ by an amount smaller than $|BJ_{-m}(C/\Omega)|$ according to the theory for the simple equation (1). The locking regions are of magnitude

$$B_m=B\,|J_{-m}(C/\Omega)| \ . \quad (11)$$

In particular B_0 goes to zero if (C/Ω) equals any zero of the Bessel function J_0. At these points no other J_m is exactly zero, but because $J_m(z)\sim z^{-1/2}$ for large values of z, we can reduce the higher-order lock-in regions by going to larger-order zeros of $J_0(C/\Omega)$, but there are severe technical limitations on this approach.

The present approach is similar in spirit to the work by Schleich *et al.*[10] They carry out the time averaging in some detail by using a Fourier series expansion of the distribution function for a dithered ring laser with noise. In the lowest-order, time-averaged approximations they obtain our results [Eqs. (10) and (11)] near the lock-in frequencies (7). They also obtain perturbation approximations to the behavior between the locked regions. Their main interest is to describe the influence of noise, which we will not discuss here.

In this paper we only want to use these considerations as a heuristic device to justify the approximate step size (11) and to use them later to suggest further directions for our work.

C. Transformation of the equations

To carry out our treatment we transform the variable φ in three successive stages, see Ref. 6, $\varphi\to\theta\to y\to Z$, defined by

$$\theta=e^{i\varphi};\ \ \dot{y}=-\tfrac{1}{2}\theta By \ ; \quad (12a)$$

$$Z=\exp\left[-\frac{i}{2}[At+(C/\Omega)\sin(\Omega t)]\right]y \ . \quad (12b)$$

The same transformations have also been utilized by the authors of Ref. 8; in Ref. 10, footnote 22, they mention that it leads to a Floquet problem. We obtain

$$\ddot{Z}+R(t)Z=0 \quad (13a)$$

with

$$R(t)=\tfrac{1}{4}\{[A+C\cos(\Omega t)]^2-B^2-2iC\Omega\sin(\Omega t)\} \ . \quad (13b)$$

Equation (13a) is a differential equation with periodic coefficients which we can solve with standard methods (vide infra).

In a laser gyroscope the ideal observed quantity is the time-averaged rate of change of the phases over many periods of the dither frequency. The technical arrangements cause complications, and the fact that a reasonably rapid response is required forces one to take the average approximately over a finite time only. In Ref. 6 we showed that the time-averaged rate of change of φ is determined directly by the phase of Z according to

$$\bar{\dot{\varphi}}=\lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{+T}\dot{\varphi}\,dt\right]$$
$$=-\lim_{T\to\infty}\left[\frac{1}{T}\{\arg[Z(T)]-\arg[Z(-T)]\}\right] \ . \quad (14)$$

It is, of course, also possible to evaluate directly the spectrum of the beat note in the laser gyroscope; this has been done by Cresser *et al.*[7–9]

D. Floquet theory

The mathematical theory of differential equation with periodic coefficients is well developed[4]; it usually goes under the name of Floquet theory. The function $R(t)$ in (13b) is clearly periodic

$$R(t+\tau)=R(t) \quad (15a)$$

with the period

$$\tau=\frac{2\pi}{\Omega} \ . \quad (15b)$$

According to the theory the general solution of (12) is then of the form

$$\eta_i(t)=e^{\nu_i t}\omega_i(t) \ , \quad (16)$$

where ω_i has the period τ and ν_i is the Floquet exponent; i takes two values.

The general solution of Eq. (13a) can be obtained from two particular solutions $Z_1(t)$ and $Z_2(t)$ as linear superposition. Since the differential equation has periodic coefficients, with period τ, the functions $Z_1(t+\tau)$, $Z_2(t+\tau)$ must satisfy (13a) equally well. Owing to the presence of the Floquet factors $e^{\nu_1 t}$, $e^{\nu_2 t}$, the solutions themselves are not, generally speaking, periodic in t, but, due to the linearity of the system, the solutions $Z_1(t+\tau)$, $Z_2(t+\tau)$ can always be expressed as linear combination of $Z_1(t)$, $Z_2(t)$:

$$Z_1(t+\tau)=a_{11}Z_1(t)+a_{12}Z_2(t) \ ,$$
$$Z_2(t+\tau)=a_{21}Z_1(t)+a_{22}Z_2(t) \ . \quad (17)$$

Only if we have chosen the original solutions $Z_i(t)$ so that they directly give the form (16) can we obtain a separation of the time evolution of Z_1 and Z_2. This condition is similar to an eigenvalue problem and serves to determine the exponents ν_i.

We express the Floquet functions (16) in terms of our two arbitrary function Z_i as

$$\eta(t)=\alpha Z_1(t)+\beta Z_2(t) \quad (18)$$

and requiring that

$$\eta(t+\tau)=e^{\nu\tau}\eta(t) \ , \quad (19)$$

we find the consistency equations

$$\alpha(a_{11}-e^{\nu\tau})+\beta a_{21}=0 \,,$$
$$\alpha a_{12}+\beta(a_{22}-e^{\nu\tau})=0 \,. \tag{20}$$

If we, for simplicity, introduce the notation

$$\rho=e^{\nu\tau} \; [\nu=(\ln\rho)/\tau] \,, \tag{21}$$

the exponents ν_i are determined by the two roots of the equation

$$\rho^2-(a_{11}+a_{22})p-a_{12}a_{21}+a_{11}a_{22}=0 \,. \tag{22}$$

It is convenient to use as the two fundamental solutions

$$Z_1(0)=1; \;\; \dot{Z}_1(0)=0 \,, \tag{23a}$$

$$Z_2(0)=0; \;\; \dot{Z}_2(0)=1 \,, \tag{23b}$$

which give directly

$$Z_1(\tau)=a_{11}; \;\; Z_2(\tau)=a_{21} \,,$$
$$\dot{Z}_1(\tau)=a_{12}; \;\; \dot{Z}_2(\tau)=a_{22} \,. \tag{24}$$

The Wronskian

$$W(t)=\begin{vmatrix} Z_1(t) & Z_2(t) \\ \dot{Z}_1(t) & \dot{Z}_2(t) \end{vmatrix} \tag{25}$$

is seen to be constant in time, and is therefore equal to its initial value [i.e., 1, because of (23)] at any subsequent time. Thus

$$a_{11}a_{22}-a_{12}a_{21}=1 \,. \tag{26}$$

If we introduce the notation

$$P=\tfrac{1}{2}(a_{11}+a_{22}) \tag{27}$$

we find the roots of (22) in the form

$$\rho_\pm=P\pm\sqrt{P^2-1} \,. \tag{28}$$

We have two cases

$$P^2>1, \;\; \rho \text{ real} \tag{29a}$$

$$P^2<1, \;\; \rho \text{ complex} \,. \tag{29b}$$

In (29b) we can define ψ by setting

$$P\equiv\cos\psi \,, \tag{30}$$

which gives $\rho=e^{\pm i\psi}$ and further from (28)

$$i\psi=\pm\nu_i\tau+2\pi n_i i \;\; (i=1,2) \,. \tag{31}$$

Because of the periodicity of $\omega(t)$ we have for the time average in (14)

$$\bar{\dot{\varphi}}=-2\,\mathrm{Im}\nu_i=\frac{2\psi}{\tau}+\frac{4\pi n_i}{\tau}$$
$$=\pm\frac{1}{\pi}\psi\Omega+2n_i\Omega \,. \tag{32}$$

Since ψ is a continuous, monotonically increasing function of A, the ring-laser gyroscope is able to detect changes in rotation rates in the range where $P^2<1$. In other words, the system is not locked in these regions.

If $P>1$ both roots (30) are real and positive and we find, with an appropriate choice of the phase of the multivalued function ln which appears in (21),

$$\bar{\dot{\varphi}}=2n\Omega \,. \tag{33}$$

Thus the phase is locked to an even multiple of the dithering frequency.

For $P<-1$ we have two negative roots which gives a phase factor $e^{\pm i\pi}$ in addition to (32) and hence the observable becomes

$$\bar{\dot{\varphi}}=(2n\pm1)\Omega \,, \tag{34}$$

which gives locking to an odd multiple of the dither frequency. Thus for those values of A for which $P>1$ ($P<-1$), $\bar{\dot{\varphi}}$ is locked at an even (odd) multiple of the dither frequency. Also here, P is a continuous function of the input rotation rates A. Thus, if it happens for some value of A that $P>1$ ($P<-1$), there will be a finite range of values of A for which $P\geq1$ ($P\leq-1$). For all those values, $\bar{\dot{\varphi}}$ will be kept "locked" at an even (odd) multiple of Ω, and in that range the laser gyroscope will be unable to detect any change in the input rotation rate.

E. Use of the Hill determinant

A very convenient method to obtain the Floquet exponents is to use an infinite determinant, the Hill determinant. We write the function (13b) in the form

$$R(t)=\sum_{m=-2}^{+2}\theta_m e^{im\Omega t} \,, \tag{35}$$

with

$$\theta_0=\frac{1}{4}\left[A^2+\frac{C^2}{2}-B^2\right] \,, \tag{36}$$

$$\theta_{\pm1}=\frac{C}{4}(A\mp\Omega) \,, \tag{37}$$

$$\theta_{\pm2}=\frac{1}{16}C^2 \,. \tag{38}$$

We also introduce the new exponent

$$\mu\equiv i\nu \,. \tag{39}$$

If we expand the solution to (13a) as

$$Z(t)=e^{i\mu t}\sum_k \xi_k e^{ik\Omega t} \,, \tag{40}$$

we find for the coefficients the recurrence relation

$$-(\mu+k\Omega)^2\xi_k+\sum_{m=-2}^{+2}\theta_m\xi_{k-n}=0 \,. \tag{41}$$

This is in the form of an infinite eigenvalue problem. We define

$$\lambda_k\equiv\theta_0-(k\Omega+\mu)^2 \tag{42}$$

and introduce the notation

$$C_{k,i}=\frac{\theta_i}{\lambda_k} \,. \tag{43}$$

Then the exponent μ is determined by the zeros of the determinant

$$\Delta=\begin{vmatrix} \cdots & \cdots & & & & & & & \\ 0 & C_{k,2} & C_{k,1} & 1 & C_{k,-1} & C_{k,-2} & 0 & \cdots & \\ & 0 & C_{k+1,2} & C_{k+1,1} & 1 & C_{k+1,-1} & C_{k+1,-2} & 0 & \cdots \\ & & & & \cdots & \cdots & & & \end{vmatrix}, \tag{44}$$

as already given in Ref. 6, Eq. (17).

It can be proved (see, e.g., Whittaker and Watson[5]) that the exponent follows from the relation

$$\cos\left[\frac{2\pi\mu}{\Omega}\right]=1-\Delta(0)\left[1-\cos\left[\frac{2\pi}{\Omega}\theta_0^{1/2}\right]\right]. \tag{45}$$

It consequently suffices to evaluate the infinite determinant (44) for $\mu=0$. The value for μ follows from (45).

From Eq. (45) it follows that when μ is a solution, $-\mu$ is also one because the determinant is unchanged when k is replaced by $-k$. It is obvious that $\cos(2\pi\mu/\Omega)$ is real and consequently from Eqs. (28), (21), and (30),

$$P=\tfrac{1}{2}(\rho_+ + \rho_-)=\tfrac{1}{2}(e^{-i\mu\tau}+e^{i\mu\tau})$$
$$=\cos(\mu\tau)\ . \tag{46}$$

This proves that P is a real number, which has been used in our discussion following Eq. (29). Note that, because Eqs. (45) and (46) contain only cosine functions of μ, they stay real even when μ becomes imaginary; in this case they are only replaced by hyperbolic cosine functions. The argument, however, prevails.

To see the expedience of the present formulation we set $C=0$, i.e., we have no dithering. Then, $\Delta(0)=1$ and $\mu=\theta_0^{1/2}$ which from (32) gives directly.

$$\bar{\dot{\varphi}}=\sqrt{A^2-C^2} \tag{47}$$

as should be the case. The evaluation of the Hill determinant, i.e., Eq. (44) becomes a convenient alternative tool to the direct-time integration of Eq. (12a).

III. NUMERICAL ANALYSIS

A. The method of numerical integration

The analysis of the previous paragraph has shown that the dithered laser gyroscope may be locked at multiples of the dither frequency; but, of course, the actual existence of these higher-order locking regions can be proved only through numerical calculations. We have performed several time integrations of Eq. (13a) starting from initial conditions (23a), in order to find $Z_1(\tau)\equiv a_{11}$. Symmetry consideration also shows that $a_{22}=a_{11}^*$; we were then able to find the parameter P of Eq. (27) and the value of the average phase rate $\bar{\dot{\varphi}}$ from it. We stress that, using this method, one is able to find the solution by integration over only a single period of the applied dither mechanism instead of the thousand periods that would have been necessary starting from the original equation (3). This procedure thus saves a considerable amount of computer time in numerical work.

The results obtained by this method and by the method of the Hill determinant (vide infra) will be discussed in Sec. III C.

In the numerical work we have chosen dimensionless units by scaling the time to t/τ. The dither period then equals unity and the quantities A, B, C, and $\bar{\dot{\varphi}}$ are all multiplied by the period τ.

B. The Hill determinant

The method of numerical integration of Eq. (13), described in Sec. III A proved to be applicable only for small values of the parameters A and C, say up to 50 (in our dimensionless units). For larger values, which are met in practical applications, numerical integration faces some problems because the value of $Z_1(t)$, chosen equal to 1 at $t=0$, may grow exponentially to very high values within the period of integration. Thus the precision of the calculations may be degraded, or the values of Z_1 may even run out of the range where the computer can handle the numbers. Thus, another method of evaluation of the Floquet parameter has been used, namely the method of Hill determinant, discussed in Sec. III E. The infinite determinant which appears in (44) cannot be evaluated through a standard procedure (i.e., evaluating 3×5, 5×5, ... determinants and looking for convergence) because in practical cases one has to go up to very high orders to achieve convergence. But we can take advantage of the particular structure of the determinant itself (which has only five lines along the diagonal filled with nonzero elements) to set up a procedure for its evaluation. The determinant in Eq. (44) can be partitioned using the 2×2 submatrices

$$\underline{T}_r=\begin{bmatrix} C_{2r-1,2} & C_{2r-1,1} \\ 0 & C_{2r,2} \end{bmatrix}, \tag{48}$$

$$\underline{U}_r=\begin{bmatrix} 1 & C_{2r-1,-1} \\ C_{2r,1} & 1 \end{bmatrix}, \tag{49}$$

$$\underline{V}_r=\begin{bmatrix} C_{2r-1,-2} & 0 \\ C_{2r,-1} & C_{2r,-2} \end{bmatrix}. \tag{50}$$

We also define the additional matrices $\underline{\widetilde{T}}_r$, $\underline{\widetilde{U}}_r$, and $\underline{\widetilde{V}}_r$ from (48)–(50) by replacing $C_{2r-1,i}$ with $C_{2r,i}$, etc. The determinant Δ can now be written as one with U matrices along the diagonal and the row of numbers

$$S=(C_{0,2},C_{0,1},1,C_{0,-1},C_{0,-2}) \tag{51}$$

dividing it into a lower part containing the matrices (48)–(50) and an upper part containing those with the tilde.

We can obtain successive approximation to Δ by setting $\underline{V}_r=0_{2\times2}$, $\underline{\widetilde{T}}_r=0_{2\times2}$, with $r=1,2,3,\ldots$. For instance, if we set $\underline{V}_2=0_{2\times 2}$, $\tilde{\underline{T}}_2=0_{2\times 2}$ the determinant is split into the product of three determinants, a central one $\Delta^{(9)}$ with the structure given in Fig. 1 and two others of infinite order, whose values is 1.[12] Thus, if the elements of $\underline{V}_2$ and $\tilde{\underline{T}}_2$ are small enough, a good approximation of Δ would be $\Delta^{(9)}$. In turn, each determinant of this form may be reduced, eliminating the first two rows and the first two columns to

$$\Delta^{(9)}=(\det\tilde{\underline{U}}_2)D_1\ , \tag{52}$$

where D_1 is the 7×7 determinant given in Fig. 2. This method follows closely the method of Gauss reduction, with the only difference that the eliminants are 2×2 matrices instead of elements. Each time a reduction is performed, one has to multiply the reduced determinant by the determinant of the eliminant, $\det\tilde{\underline{U}}_2$ in the example of (52). Taking the limit $r\rightarrow\infty$, one can show that the infinite determinant is now given by

$$\Delta(0)=\epsilon_+\epsilon_-D_2\ , \tag{53}$$

where D_2 is given in Fig. 3 and where $\tilde{\underline{W}}$ and $\underline{W}$ are 2×2 matrix continued fractions,

$$\tilde{\underline{W}}=\tilde{\underline{U}}_1-\tilde{\underline{V}}_2\frac{1}{\tilde{\underline{U}}_2-\tilde{\underline{V}}_3\dfrac{1}{\tilde{\underline{U}}_3-\tilde{\underline{V}}_4\dfrac{1}{\tilde{\underline{U}}_4-\cdots}\tilde{\underline{T}}_3}\tilde{\underline{T}}_2}\tilde{\underline{T}}_1\ , \tag{54}$$

$$\underline{W}=\underline{U}_1-\underline{T}_2\frac{1}{\underline{U}_2-\underline{T}_3\dfrac{1}{\underline{U}_3-\underline{T}_4\dfrac{1}{\underline{U}_4-\cdots}\underline{V}_3}\underline{V}_2}\underline{V}_1\ , \tag{55}$$

and ϵ_+, ϵ_- are the products of the determinants of the eliminants of the upper and lower part, respectively. Actually, one can show that

$$\epsilon_+=\epsilon_- \tag{56}$$

so that the evaluation of the infinite Hill determinant is reduced to the evaluation of two matrix continued fractions and an infinite product, and, eventually, to the final evaluation of a determinant of 5(rows)$\times$5(columns). This method has proved convergent for any values of the parameters A, B, and C. Moreover, it does not display numerical instabilities as the method of numerical integration of the previous paragraph. Convergence may be reached in some cases after hundreds of steps in the continued fractions, but the overall computational time is less than the time required by a single numerical integration, when the latter is feasible.

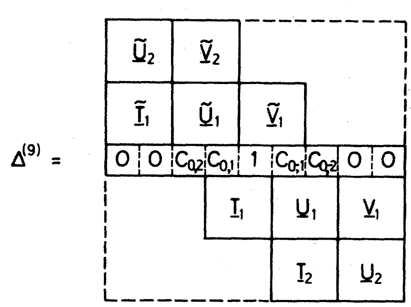

FIG. 1. This shows the structure of the 9×9 subdeterminant defined in Sec. III B. The numbers $C_{0,j}$ are defined in Eq. (51) and the 2×2 $\underline{U}$, $\underline{V}$, and $\underline{T}$ matrices are defined in Eqs. (47)–(49).

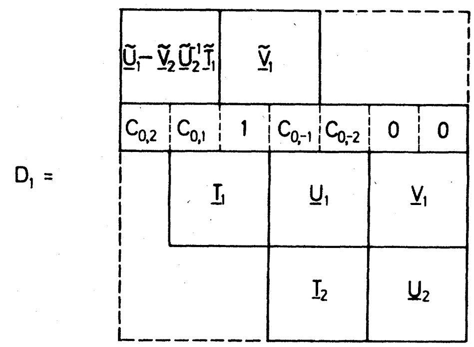

FIG. 2. This is the 7×7 determinant obtained by the extraction of $\det\tilde{\underline{U}}_2$ from $\Delta^{(9)}$ as given in Fig. 1.

C. Numerical results

We show here a few results obtained for the sinusoidally dithered ring-laser gyroscope. Both methods have been used, when it proved possible, and the results have been found to coincide within the precision requirements made for convergence (which, by the way, were set to the very small value of 10^{-10}).

In Figs. 4(a), 4(b), and 4(c) the $\bar{\dot{\varphi}}$ versus A plots are shown for three values of C, namely $C=5$, $C=10$, and $C=15.109\,956$; B has been kept fixed at 1 (the dimensionless units are used throughout this section; in these units, the period of the dither mechanism is equal to 1). In going from Fig. 4(a) to Fig. 4(c) one can see a considerable reduction of the zero-order locking zone. However, as C increases the first nonlinear locking zone, located at $A=\Omega$ ($=2\pi$ is our dimensionless units), also increases; thus a reduction in the lower locking zone at $A=0$ will lead to larger higher-order locking zones.

The last value of C has been chosen as suggested by the heuristic treatment, according to which a total reduction of the locking zone at the origin should be achieved at

$$C=j_{0,1}\Omega\ , \tag{57}$$

where $j_{0,1}$ is the first root of the equation

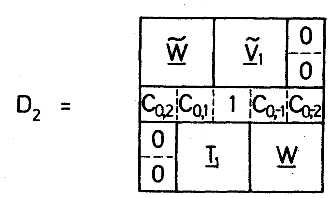

FIG. 3. When the extraction process described in Figs. 1 and 2 has been carried out for the infinite determinant of Eq. (43), the process ends with the 5×5 determinant D_2 centered around the numbers S of Eq. (50). Its structure is shown in this figure.

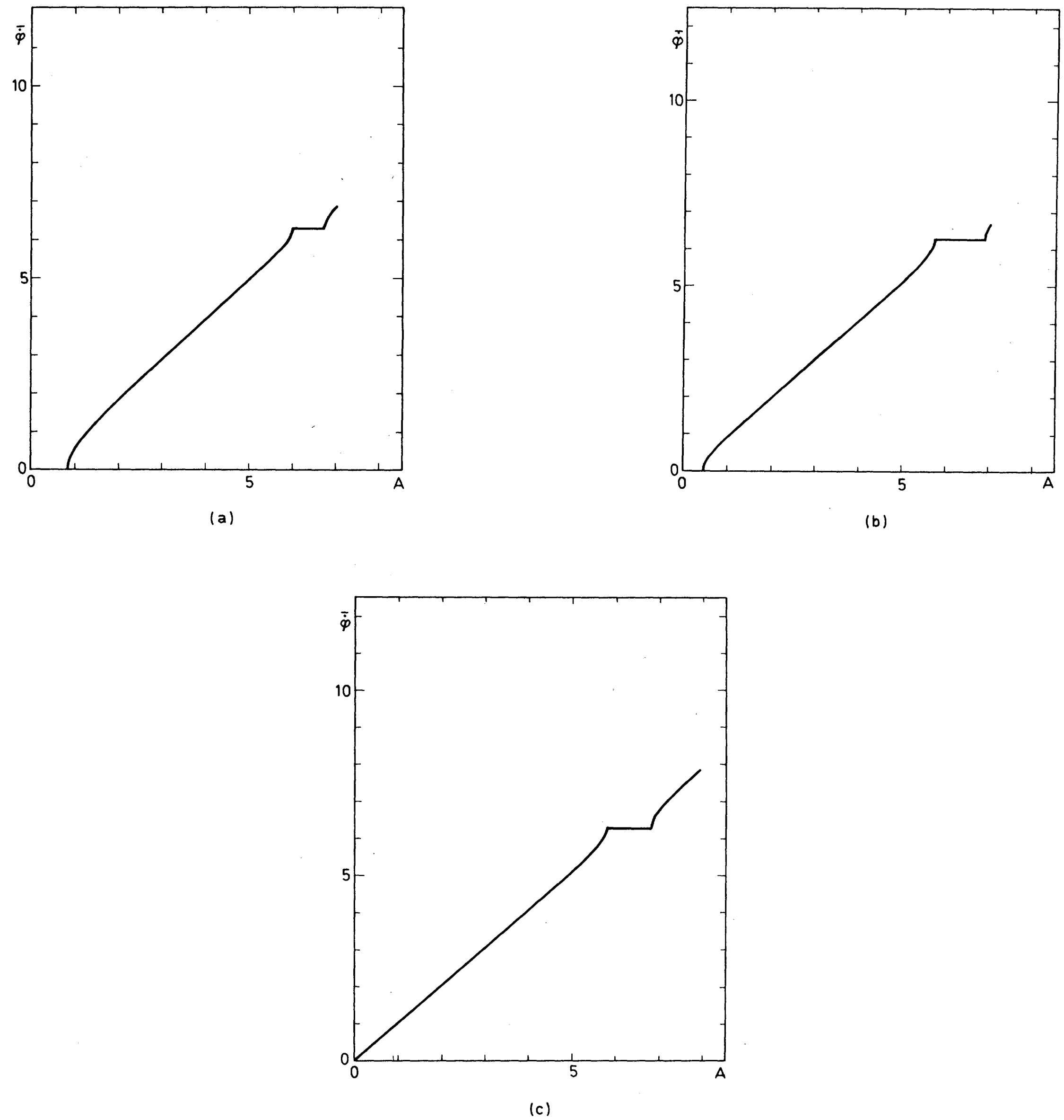

FIG. 4. This sequence of figures shows the gyroscope output $\bar{\dot{\varphi}}$ as a function of input A. We have $B=1$, and for no dither, $C=0$, the output would remain locked for $A \leq 1$ and no higher-order lock-in zones would occur. Here we have (a) $C=5$, (b) $C=10$, and (c) $C=15.109\,965 \simeq 2\pi j_{0,1}$; see Eq. (57).

$$J_0(x)=0 \tag{58}$$

($\Omega=2\pi$ in our units).

With C chosen in this way, we have found at $A=0$ a (very small) residual locking zone, since P was found to be equal to 1.000 001 846 (disappearance of the locking zone would have been revealed by $P=1$). Further calculations have shown that the locking zone may indeed be reduced completely to zero, within the precision of our calculations. A more propitious value for C proved to be

$$C=15.155\,92\ . \tag{59}$$

There are many values of C for which the zero locking region is reduced to zero. The value in (59) is the smallest one. These values are not equally spaced but they display an almost periodic pattern. Figure 5(a) shows such a pattern. Here P (or equivalently $\cos\mu$) is displayed versus C. The zero locking zone at the origin is reached whenever $|P|=1$. At all other points, $P>1$, i.e., the output $\bar{\dot{\varphi}}$ is locked at $A=0$. In Fig. 5(b) we show the P parameter in the next locking zone, i.e., $A=2\pi$, as a function of C. Here too, there are values of C for which $P \geq -1$; at all other points $P<-1$, showing that the output is locked at Ω.

A comparison of Figs. 5(a) and 5(b) shows that, when the zero-order locking zone has been reduced to zero

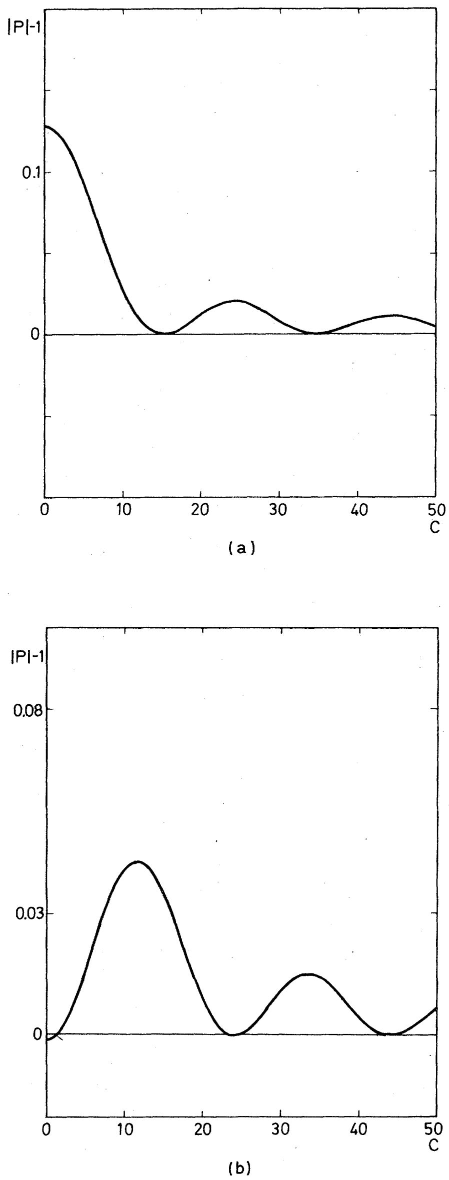

FIG. 5. This figure shows the dependence of the parameter P in Eq. (27) on the dither amplitude C. When $|P|$ is less than unity there is no locking. (a) Here $A=0$ and we consider locking at zero rotation rate. (b) Here we consider the situation at the first of the higher-order lock-in regions, $A=2\pi$. For small values of C there occurs no locking; this situation reappears regularly but with almost exactly the opposite phase to that for which there is no locking at $A=0$ as shown in (a).

($P=1$), the value of $|P|$ in the first-order locking zone gets a maximum, i.e., the locking width there is maximum. Thus, zero-order and first-order zones are out of phase exactly. The same conclusion applies to first-order and second-order locking zones, to second-order and third-order locking zones, and so on. One can see, therefore, that a purely sinusoidal body dithering of the ring-laser gyroscope cannot display a linear response over a wide range of input parameters: if reduction (or total elimination) of locking dead bands is achieved in one zone of the operation range, large deviations from linearity occur at other zones, thus deteriorating the overall performances of the laser gyroscope.

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APPENDIX

Case III is easily treated by the heuristic method of Sec. II B. We write Eq. (6) as

$$\dot{\varphi}=A+B\sin\varphi+\tfrac{1}{2}BC\sin(\varphi+\Omega t)+\tfrac{1}{2}BC\sin(\varphi-\Omega t)\ . \tag{A1}$$

Using again the transformation (7) and

$$y=\varphi-m\Omega t\ , \tag{A2}$$

we find

$$\dot{y}=a+B\sin(y+m\Omega t)+\tfrac{1}{2}BC\sin[y+(m+1)\Omega t] + \tfrac{1}{2}BS\sin[y+(m-1)\Omega t]\ . \tag{A3}$$

For $m=0$ there appears two oscillating terms which average to zero and the zero-rotation-rate dead band of width B is essentially unaffected. For $m=\pm 1$ there appears two new dead bands of width $\frac{1}{2}BC$. No additional dead bands appear if this treatment is valid.

Using the transformations (12a) we find the equation

$$\ddot{y}+\left[-iA+\frac{C\Omega\sin(\Omega t)}{1+C\cos(\Omega t)}\right]\dot{y} -\frac{B^2}{4}[1+C\cos(\Omega t)]^2 y=0\ . \tag{A4}$$

When we introduce the further transformation: $y\rightarrow Z$,

$$y=fZ \tag{A5}$$

with

$$f(t)=\sqrt{1+C\cos(\Omega t)}\,e^{iAt/2}\ , \tag{A6}$$

we obtain again an equation of the type (13a) but with a more complicated expression for $R(t)$. It still remains periodic over τ and our previous treatment remains possible. Only now the Hill determinant becomes more complicated. From Eqs. (A4) and (A6) we can see that the present approach makes sense only when $|C|<1$.

We have carried out the numerical evaluation of this case, too, along the lines presented in the main text. The conclusions of the heuristic model are verified: No appreciable decrease of the zero-lock-in region is found. There appears only two additional lock-in bands and their widths agree with the approximate value $\frac{1}{2}BC$ obtained from Eq. (A3).

This investigation shows that a (not too strong) periodic variation of the backscattering coefficient leaves the ring-laser operation nearly unaffected. It is suggested, even if not entirely warranted. that one connect this with the fact that the phase equation (1) appears to give a good description of the ring laser, even if the amplitude dependence of the backscattering coefficient B is ignored. One could envisage the phase variation causing a periodic modulation of the amplitudes, which might be simulated by the behavior in Eq. (6). Our analysis indicates that, at least near $A=0$, such a variation does not influence the laser operation.

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