New formalism for two-photon quantum optics. II. Mathematical foundation and compact notation

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This paper provides the mathematical foundation for the two-mode formalism introduced in the preceding paper. A vector notation is introduced; it allows two-mode properties to be written as compactly as the comparable properties for a single mode. The fundamental unitary operators of the formalism are described and their properties are examined; particular attention is paid to the two-mode squeeze operator. Special quantum states associated with the formalism are considered, with emphasis on the two-mode squeezed states.

I. INTRODUCTION

The present series of papers introduces a new formalism for two-photon quantum optics. The goal is to develop a formalism suited to analyzing two-photon devices, such as parametric amplifiers and phase-conjugate mirrors, in which photons are created or destroyed in the output modes two at a time. In the preceding paper[1,2] (henceforth referred to as I) we introduced the basic building blocks of our two-mode formalism: (i) new operators, the quadrature phases and the quadrature-phase amplitudes, and (ii) new quantum states, the two-mode squeezed states. The emphasis in I was on developing a sound physical interpretation of these fundamental entities. A conversational style invited the reader to become familiar with the elementary, but most important properties of the quadrature-phase amplitudes and the two-mode squeezed states. In the present paper the emphasis shifts—from physical interpretation to mathematical details. We introduce a compact vector notation which simplifies the mathematical description and at the same time highlights the important physics underlying our two-mode formalism. With the help of this notation we examine in detail the components of the formalism. The reward for the persistent reader is to proceed to a future paper (paper III), where the notation and results of this paper are used to construct the working tools of the new formalism—a set of "two-photon" quasiprobability distributions.

The present paper is largely independent of I, but a complete understanding does require familiarity with some of the material in I. (Equations in I are referred to here by affixing I to the equation number.) Since we make no attempt in this paper to motivate the definitions of the quadrature-phase amplitudes and the two-mode squeezed states, the reader might find it helpful to be familiar with the physical interpretation developed in I. The reader should also be comfortable with our potentially confusing habit of writing equations which contain operators defined in different pictures (see Sec. II of I); in particular, he should be familiar with the relations among the Schrödinger picture (SP), the modulation picture (MP), and the interaction picture (IP) [Eqs. (I.4.3) and (I.4.4)] and with the convention introduced in Sec. IV C of I by which we specify for each physical quantity the picture in which the operator corresponding to that quantity is always written.

Given this minimal familiarity with the material in I, we can cast aside the interpretative superstructure used in I and extract only the essentials needed in this paper. We deal with two electromagnetic field modes whose frequencies are $\Omega\pm\epsilon$, where Ω is a carrier frequency and $\epsilon<\Omega$ is a modulation frequency. The SP creation and annihilation operators for the two modes are denoted by $a_\pm^\dagger$ and $a_\pm$; they satisfy the standard commutation relations

$$[a_+,a_-]=[a_+,a_-^\dagger]=0\ , \tag{1.1a}$$

$$[a_+,a_+^\dagger]=[a_-,a_-^\dagger]=1\ . \tag{1.1b}$$

The free Hamiltonian for the two modes is

$$H_0=H_R+H_M\ , \tag{1.2}$$

where

$$H_R\equiv\Omega(a_+^\dagger a_+ + a_-^\dagger a_-)\ , \tag{1.3a}$$

$$H_M\equiv\epsilon(a_+^\dagger a_+ - a_-^\dagger a_-)\ , \tag{1.3b}$$

$$[H_R,H_M]=0\ . \tag{1.4}$$

The MP *quadrature-phase amplitudes* are defined by

$$\alpha_1\equiv 2^{-1/2}(\lambda_+a_+ + \lambda_-a_-^\dagger)\ , \tag{1.5a}$$

$$\alpha_2\equiv 2^{-1/2}(-i\lambda_+a_+ + i\lambda_-a_-^\dagger)\ , \tag{1.5b}$$

$$\lambda_\pm\equiv[(\Omega\pm\epsilon)/\Omega]^{1/2} \tag{1.6}$$

[Eqs. (I.4.25)]; they obey the following commutation relations:

$$[\alpha_1,\alpha_1^\dagger]=[\alpha_2,\alpha_2^\dagger]=\epsilon/\Omega\ , \tag{1.7a}$$

$$[\alpha_1,\alpha_2]=0\ , \tag{1.7b}$$

$$[\alpha_1,\alpha_2^\dagger]=[\alpha_1^\dagger,\alpha_2]=i\ . \tag{1.7c}$$

The important new unitary operator in our formalism is the *two-mode squeeze operator*

$$S(r,\varphi)\equiv\exp[r(a_+a_-e^{-2i\varphi}-a_+^\dagger a_-^\dagger e^{2i\varphi})] \tag{1.8}$$

[Eq. (I.4.9)], where r is a real number called the *squeeze factor* and φ is a (real) phase angle. The two-mode squeeze operator satisfies

$$S^{-1}(r,\varphi)=S^\dagger(r,\varphi)=S(-r,\varphi)=S(r,\varphi+\pi/2)\ , \tag{1.9}$$

and it generates the *squeezed annihilation operators,* which in the MP are defined by

$$\alpha_\pm(r,\varphi)\equiv S(r,\varphi)a_\pm S^\dagger(r,\varphi)=a_\pm\cosh r+a_\mp^\dagger e^{2i\varphi}\sinh r \tag{1.10}$$

[Eq. (I.4.14)]. The squeezed annihilation operators are unitarily equivalent to the annihilation operators, so they have the same commutator algebra [Eqs. (1.1)].

This paper is built on Eqs. (1.1)—(1.10). Section II introduces the compact vector notation which is used throughout this and subsequent papers. The components of our formalism are a set of fundamental unitary operators and a set of special quantum states. Section III examines in detail the fundamental unitary operators, and Sec. IV does the same for the special quantum states, with emphasis on the two-mode squeezed states. A concluding section meditates on the formalism developed here and hints at the results to come in subsequent papers. Some of the important results are developed in appendices: Appendix A lists properties of various transformation matrices associated with the vector notation; Appendix B derives useful factored forms for the degenerate and two-mode squeeze operators and an expression for the product of two different squeeze operators; Appendix C considers the inner product of two squeezed states. Throughout this paper we use units with $\hbar=c=1$.

II. VECTOR NOTATION

The most important feature of the two-mode squeeze operator $S(r,\varphi)$ [Eq. (1.8)] is that under a unitary transformation generated by $S(r,\varphi)$, $a_\pm$ is transformed into a linear combination of $a_\pm$ and $a_\mp^\dagger$. This association of a_+ with $a_-^\dagger$ (and a_- with $a_+^\dagger$) is evident in the definitions of the squeezed annihilation operators [Eq. (1.10)] and the quadrature-phase amplitudes [Eqs. (1.5)]. We have found it natural and useful to introduce an operator column vector[2]

$$\underset{\sim}{\mathrm{a}}\equiv\begin{bmatrix}a_+\\ a_-^\dagger\end{bmatrix} \tag{2.1}$$

which recognizes explicitly this association. This vector notation has been used by Collett and Gardiner[3] in an analysis of parametric amplification. Mollow[4] and Yuen and Shapiro[5] have also used a two-component vector notation, but they use a column vector formed from a_+ and a_-. The adjoint of the vector (2.1) is the row vector

$$\underset{\sim}{\mathrm{a}}^\dagger\equiv(a_+^\dagger\ \ a_-)\ . \tag{2.2}$$

Products of the vectors (2.1) and (2.2) are calculated using the usual rules for matrix multiplication, i.e.,

$$\underset{\sim}{\mathrm{a}}^\dagger\underset{\sim}{\mathrm{a}}=a_+^\dagger a_+ +a_-a_-^\dagger\ , \tag{2.3a}$$

$$\underset{\sim}{\mathrm{a}}\underset{\sim}{\mathrm{a}}^\dagger=\begin{bmatrix}a_+a_+^\dagger & a_+a_-\\ a_-^\dagger a_+^\dagger & a_-^\dagger a_-\end{bmatrix}\ . \tag{2.3b}$$

Also useful is an operator column vector for the quadrature-phase amplitudes,

$$\underset{\sim}{\mathscr{A}}\equiv\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\underline{A}\,\underline{\lambda}\,\underset{\sim}{\mathrm{a}}\ , \tag{2.4}$$

where

$$\underline{A}\equiv2^{-1/2}\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix}=(\underline{A}^\dagger)^{-1}\ , \tag{2.5}$$

$$\underline{\lambda}\equiv\begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{bmatrix}=\underline{\lambda}^\dagger \tag{2.6}$$

[Eqs. (1.5) and (1.6)]. A list of useful properties of $\underline{A}$ and $\underline{\lambda}$ appears in Appendix A; many of the properties are most conveniently written in terms of the unit matrix $\underline{1}$ and the Pauli matrices

$$\underline{\sigma}_1\equiv\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \underline{\sigma}_2\equiv\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \underline{\sigma}_3\equiv\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\ . \tag{2.7}$$

The matrix $\underline{\lambda}^2=\underline{1}+(\epsilon/\Omega)\underline{\sigma}_3$ plays an important role because it appears in the vector expression for the free Hamiltonian (1.2),

$$\begin{aligned}H_0+\Omega-\epsilon&=\Omega\underset{\sim}{\mathrm{a}}^\dagger\underline{\lambda}^2\underset{\sim}{\mathrm{a}}=(\Omega+\epsilon)a_+^\dagger a_+ +(\Omega-\epsilon)a_-a_-^\dagger\\ &=\Omega\underset{\sim}{\mathscr{A}}^\dagger\underset{\sim}{\mathscr{A}}=\Omega(\alpha_1^\dagger\alpha_1+\alpha_2^\dagger\alpha_2)\end{aligned} \tag{2.8}$$

[cf. Eq. (I.6.10b)]. Other matrices that turn up repeatedly in the following are

$$\underline{\Delta}\equiv\underline{A}\,\underline{\lambda}^2\underline{A}^\dagger=\underline{1}-(\epsilon/\Omega)\underline{\sigma}_2=\begin{bmatrix}1 & i\epsilon/\Omega\\ -i\epsilon/\Omega & 1\end{bmatrix}=\underline{\Delta}^\dagger \tag{2.9}$$

[Eqs. (A17) and (A6)] and

$$\begin{aligned}\underline{\Pi}\equiv\underline{A}\,\underline{\lambda}\,\underline{\sigma}_3\underline{\lambda}\,\underline{A}^\dagger&=-\underline{\sigma}_2\underline{\Delta}=(\epsilon/\Omega)\underline{1}-\underline{\sigma}_2\\ &=\begin{bmatrix}\epsilon/\Omega & i\\ -i & \epsilon/\Omega\end{bmatrix}=\underline{\Pi}^\dagger\end{aligned} \tag{2.10}$$

[Eqs. (A20) and (A6)]. The matrix $\underline{\Pi}$ is the matrix of commutators for the quadrature-phase amplitudes:

$$\Pi_{mn}=[\alpha_m,\alpha_n^\dagger] \tag{2.11}$$

[cf. Eqs. (1.7)].

The naturalness of this vector notation is revealed most clearly by examining the operator matrix

$$(\Delta\underset{\sim}{\mathscr{A}}\,\Delta\underset{\sim}{\mathscr{A}}^\dagger)_{\rm sym}=\begin{bmatrix}(\Delta\alpha_1\,\Delta\alpha_1^\dagger)_{\rm sym} & (\Delta\alpha_1\,\Delta\alpha_2^\dagger)_{\rm sym}\\ (\Delta\alpha_2\,\Delta\alpha_1^\dagger)_{\rm sym} & (\Delta\alpha_2\,\Delta\alpha_2^\dagger)_{\rm sym}\end{bmatrix}\ , \tag{2.12}$$

where $\Delta\underset{\sim}{\mathscr{A}}\equiv\underset{\sim}{\mathscr{A}}-\langle\underset{\sim}{\mathscr{A}}\rangle$, $\Delta\alpha_m\equiv\alpha_m-\langle\alpha_m\rangle$, and "sym" denotes a symmetrized product (see Sec. II of I). The expectation value of the matrix (2.12) is the (Hermitian) reduced spectral-density matrix

$$\underline{\Sigma}\equiv\langle\Delta\underset{\sim}{\mathscr{A}}\,\Delta\underset{\sim}{\mathscr{A}}^\dagger\rangle_{\rm sym} \tag{2.13}$$

[Eq. (I.5.2)]; the components of $\underline{\Sigma}$, $\Sigma_{mn} = \langle \Delta\alpha_m \Delta\alpha_n^\dagger \rangle_{\rm sym} = \Sigma_{nm}^*$, are the second-order noise moments that characterize time-stationary quadrature-phase (TSQP) noise (see Sec. V of I). Thus the vector notation is tailored to describing TSQP noise—the kind of noise produced by two-photon devices—because it generates naturally the second-order noise moments that characterize TSQP noise. In contrast, the noise moments $\langle \Delta\alpha_m \Delta\alpha_n \rangle$, which vanish for TSQP noise [Eq. (I.5.1)], are not generated naturally by the vector notation.

Corresponding to the matrix (2.12) is a matrix involving the creation and annihilation operators,

$$(\Delta\underline{a}\,\Delta\underline{a}^\dagger)_{\rm sym} = \begin{bmatrix} |\Delta a_+|^2 & \Delta a_+ \Delta a_- \\ \Delta a_-^\dagger \Delta a_+^\dagger & |\Delta a_-|^2 \end{bmatrix}, \tag{2.14}$$

where $|\Delta a_\pm|^2 \equiv (\Delta a_\pm \Delta a_\pm^\dagger)_{\rm sym}$ [cf. Eq. (I.2.8)]. Its expectation value is the Hermitian matrix

$$\overline{\underline{\Sigma}} \equiv \langle \Delta\underline{a}\,\Delta\underline{a}^\dagger \rangle_{\rm sym}, \tag{2.15}$$

which gives the second-order noise moments that characterize TSQP noise in terms of the creation and annihilation operators instead of in terms of the quadrature-phase amplitudes. The relation between the two kinds of noise moments can be written in the compact matrix form

$$\underline{\Sigma} = \underline{A}\,\underline{\lambda}\,\overline{\underline{\Sigma}}\,\underline{\lambda}\,\underline{A}^\dagger, \tag{2.16}$$

which is equivalent to Eqs. (I.5.8).

Natural decompositions of $\underline{\Sigma}$ and $\overline{\underline{\Sigma}}$ are afforded by the unit matrix $\underline{1}$ and the Pauli matrices $\underline{\sigma}_1$, $\underline{\sigma}_2$, and $\underline{\sigma}_3$:

$$\underline{\Sigma} = \Sigma_0 \underline{1} + \Sigma_j \underline{\sigma}_j, \tag{2.17}$$

$$\overline{\underline{\Sigma}} = \overline{\Sigma}_0 \underline{1} + \overline{\Sigma}_j \underline{\sigma}_j, \tag{2.18}$$

where repeated indices are summed over $j = 1,2,3$. The coefficients Σ_0, Σ_j and $\overline{\Sigma}_0, \overline{\Sigma}_j$, which are guaranteed to be real by the Hermiticity of $\underline{\Sigma}$ and $\overline{\underline{\Sigma}}$, are related to the noise moments as follows:

$$\Sigma_0 = \tfrac{1}{2}(\Sigma_{11} + \Sigma_{22}), \quad \Sigma_1 = \mathrm{Re}(\Sigma_{12}),$$
$$\Sigma_2 = -\mathrm{Im}(\Sigma_{12}), \quad \Sigma_3 = \tfrac{1}{2}(\Sigma_{11} - \Sigma_{22}), \tag{2.19}$$

$$\overline{\Sigma}_0 = \tfrac{1}{2}(\langle |\Delta a_+|^2 \rangle + \langle |\Delta a_-|^2 \rangle),$$
$$\overline{\Sigma}_1 = \mathrm{Re}(\langle \Delta a_+ \Delta a_- \rangle),$$
$$\overline{\Sigma}_2 = -\mathrm{Im}(\langle \Delta a_+ \Delta a_- \rangle), \tag{2.20}$$
$$\overline{\Sigma}_3 = \tfrac{1}{2}(\langle |\Delta a_+|^2 \rangle - \langle |\Delta a_-|^2 \rangle);$$

they are related to each other by

$$\Sigma_0 = \overline{\Sigma}_0 + (\epsilon/\Omega)\overline{\Sigma}_3, \quad \Sigma_1 = -(1-\epsilon^2/\Omega^2)^{1/2}\overline{\Sigma}_2,$$
$$\Sigma_2 = -\overline{\Sigma}_3 - (\epsilon/\Omega)\overline{\Sigma}_0, \quad \Sigma_3 = (1-\epsilon^2/\Omega^2)^{1/2}\overline{\Sigma}_1 \tag{2.21}$$

[Eq. (2.16); cf. Eqs. (I.5.8)]. The differential distribution of noise in phase is specified by Σ_1 and Σ_3 or, equivalently, by $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$. TSQP noise that is distributed randomly in phase is called time-stationary (TS) noise. For TS noise $\mathrm{Re}(\underline{\Sigma})$ is a multiple of the unit matrix ($\Sigma_1 = \Sigma_3 = 0$), and $\overline{\underline{\Sigma}}$ is diagonal [$\overline{\Sigma}_1 = \overline{\Sigma}_2 = 0$; cf. Eqs. (I.5.9)].

The final important operator column vector is a vector for the squeezed annihilation operators (1.10),

$$\underline{\alpha}_{r,\varphi} \equiv \begin{bmatrix} \alpha_+(r,\varphi) \\ \alpha_-^\dagger(r,\varphi) \end{bmatrix} = S(r,\varphi)\underline{a}S^\dagger(r,\varphi) = \underline{C}_{r,\varphi}\underline{a}, \tag{2.22}$$

where

$$\underline{C}_{r,\varphi} \equiv \begin{bmatrix} \cosh r & e^{2i\varphi}\sinh r \\ e^{-2i\varphi}\sinh r & \cosh r \end{bmatrix} = \underline{C}_{r,\varphi}^\dagger. \tag{2.23}$$

Notice that $\underline{a} = \underline{\alpha}_{0,\varphi}$. In the expression $S(r,\varphi)\underline{a}S^\dagger(r,\varphi)$ the operators $S(r,\varphi)$ and $S^\dagger(r,\varphi)$ act separately on each component of $\underline{a}$. Hence the adjoint of $\underline{\alpha}_{r,\varphi}$ is given by

$$\underline{\alpha}_{r,\varphi}^\dagger = S(r,\varphi)\underline{a}^\dagger S^\dagger(r,\varphi) = \underline{a}^\dagger \underline{C}_{r,\varphi}^\dagger. \tag{2.24}$$

The inverse of Eq. (2.22) takes the form

$$\underline{a} = S^\dagger(r,\varphi)\underline{\alpha}_{r,\varphi}S(r,\varphi) = \underline{C}_{r,\varphi}^{-1}\underline{\alpha}_{r,\varphi}. \tag{2.25}$$

The matrix $\underline{C}_{r,\varphi}$ describes the matrix transformation of $\underline{a}$ that is induced by a unitary transformation of $\underline{a}$ generated by $S(r,\varphi)$. Useful properties of $\underline{C}_{r,\varphi}$ are listed in Appendix A. Any unitary transformation U which generates a matrix transformation of $\underline{a}$ (linear transformation of a_+ and $a_-^\dagger$) is a canonical transformation, described by a matrix $\underline{M}$:

$$U\underline{a}U^\dagger = \underline{M}\underline{a}. \tag{2.26}$$

Since a canonical transformation preserves commutators, $\underline{M}$ must satisfy

$$\underline{M}\underline{\sigma}_3\underline{M}^\dagger = \underline{\sigma}_3, \tag{2.27}$$

which is equivalent to

$$\underline{M}^\dagger\underline{\sigma}_3\underline{M} = \underline{\sigma}_3. \tag{2.28}$$

If $\underline{M}$ has unity determinant, then it is an element of the group SU(1,1).[6] The most general element of SU(1,1) [generated by $U = S(r,\varphi)R^\dagger(\theta)$; see Eq. (3.12)] is $\underline{M} = e^{-i\theta\underline{\sigma}_3}\underline{C}_{r,\varphi}$. The matrices $\underline{C}_{r,\varphi}$, generated by $U = S(r,\varphi)$, are the Hermitian elements of SU(1,1). They must satisfy Eq. (2.28):

$$\underline{C}_{r,\varphi}^\dagger\underline{\sigma}_3\underline{C}_{r,\varphi} = \underline{\sigma}_3. \tag{2.29}$$

Equation (2.29) is the key property of $\underline{C}_{r,\varphi}$. It says that $\underline{C}_{r,\varphi}$ preserves the scalar product with respect to the "metric" $\underline{\sigma}_3$; it is an expression of the fact that a unitary transformation generated by $S(r,\varphi)$ preserves the difference in the number of quanta in the two modes.[7] In terms of the vector notation this fact is most easily written as preservation of the scalar product $\underline{a}^\dagger\underline{\sigma}_3\underline{a} = a_+^\dagger a_+ - a_- a_-^\dagger$, i.e.,

$$S(r,\varphi)\underline{a}^\dagger\underline{\sigma}_3\underline{a}S^\dagger(r,\varphi) = \underline{\alpha}_{r,\varphi}^\dagger\underline{\sigma}_3\underline{\alpha}_{r,\varphi}$$
$$= \underline{a}^\dagger\underline{C}_{r,\varphi}^\dagger\underline{\sigma}_3\underline{C}_{r,\varphi}\underline{a} = \underline{a}^\dagger\underline{\sigma}_3\underline{a}. \tag{2.30}$$

In addition to the above operator vectors [Eqs. (2.1), (2.4), and (2.22)], it is useful to have available the c-number vectors defined in Table I. The components of each c-number vector are complex numbers. With each operator vector we associate two c-number vectors, an "active-role" vector and a "passive-role" vector. The

TABLE I. Two-mode vector notation.

Operator vector	Associated c-number vectors: Active role	Associated c-number vectors: Passive role
$\underset{\sim}{\mathbf{a}} \equiv \begin{pmatrix} a_+ \\ a_-^\dagger \end{pmatrix}$	$\underset{\sim}{\mu} \equiv \begin{pmatrix} \mu_+ \\ \mu_-^* \end{pmatrix}$	$\underset{\sim}{\nu} \equiv \begin{pmatrix} \nu_+ \\ \nu_-^* \end{pmatrix}$
$\underset{\sim}{\alpha}_{r,\varphi} \equiv \underset{\sim}{\alpha} \equiv \begin{pmatrix} \alpha_+ \\ \alpha_-^\dagger \end{pmatrix} = \underset{\sim}{C}_{r,\varphi} \underset{\sim}{\mathbf{a}}$	$\underset{\sim}{\mu}_\alpha \equiv \begin{pmatrix} \mu_{\alpha_+} \\ \mu_{\alpha_-}^* \end{pmatrix} \equiv \underset{\sim}{C}_{r,\varphi} \underset{\sim}{\mu}$	$\underset{\sim}{\nu}_\alpha \equiv \begin{pmatrix} \nu_{\alpha_+} \\ \nu_{\alpha_-}^* \end{pmatrix} \equiv \underset{\sim}{C}_{r,\varphi} \underset{\sim}{\nu}$
$\underset{\sim}{\mathscr{A}} \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \underset{\sim}{A} \underset{\sim}{\lambda} \underset{\sim}{\mathbf{a}}$	$\underset{\sim}{\xi} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \equiv \underset{\sim}{A} \underset{\sim}{\lambda} \underset{\sim}{\mu}$	$\underset{\sim}{\eta} \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \equiv \underset{\sim}{A} \underset{\sim}{\lambda}^{-1} \underset{\sim}{\sigma}_3 \underset{\sim}{\nu}$

active-role vectors are used in contexts where the c-numbers act as surrogates for the corresponding operators, e.g., as eigenvalues or expectation values of the operators or as variables of a quasiprobability distribution. The passive-role vectors are used when the c-numbers appear as variables of a characteristic function. (Characteristic functions and quasiprobability distributions will be discussed in paper III.) Notice that there is no real difference between the active-role and passive-role vectors in the first two rows of Table I; nonetheless, we maintain the distinction because of the difference encountered in the third row. The second row of Table I introduces a further notational convenience: for the squeezed annihilation operators and their vectors we drop explicit reference to a particular r and φ unless this leads to confusion. When we need additional c-number vectors for either role, we denote them by attaching primes to all vectors in the appropriate column of Table I.

The crucial properties of the vectors in Table I are the following invariants:

$$a_+^\dagger \nu_+ - a_- \nu_-^* = \underset{\sim}{\mathbf{a}}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu} = \underset{\sim}{\alpha}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu}_\alpha = \underset{\sim}{\mathscr{A}}^\dagger \underset{\sim}{\eta} = \alpha_1^\dagger \eta_1 + \alpha_2^\dagger \eta_2 , \tag{2.31a}$$

$$\mu_+^* \nu_+ - \mu_- \nu_-^* = \underset{\sim}{\mu}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu} = \underset{\sim}{\mu}_\alpha^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu}_\alpha = \underset{\sim}{\xi}^\dagger \underset{\sim}{\eta} = \xi_1^* \eta_1 + \xi_2^* \eta_2 . \tag{2.31b}$$

In Eqs. (2.31) the second equality follows from Eq. (2.29)—that $\underset{\sim}{C}_{r,\varphi}$ preserves the scalar product with respect to $\underset{\sim}{\sigma}_3$; it is the analog of Eq. (2.30). The desire to have the third equality in Eqs. (2.31) is responsible for the peculiar definition of $\underset{\sim}{\eta}$ in Table I. In addition to the invariants (2.31) it is useful to note the relations

$$\underset{\sim}{\xi}^\dagger \underset{\sim}{\xi}' = \underset{\sim}{\mu}^\dagger \underset{\sim}{\lambda}^2 \underset{\sim}{\mu}' , \tag{2.32}$$

$$\underset{\sim}{\nu}^\dagger \underset{\sim}{\nu}' = \underset{\sim}{\eta}^\dagger \underset{\sim}{\Delta} \underset{\sim}{\eta}' , \tag{2.33}$$

$$\underset{\sim}{\nu}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu}' = \underset{\sim}{\eta}^\dagger \underset{\sim}{\Pi} \underset{\sim}{\eta}' , \tag{2.34}$$

which reveal the significance of the matrices $\underset{\sim}{\lambda}^2$, $\underset{\sim}{\Delta}$, and $\underset{\sim}{\Pi}$ [Eqs. (2.8)–(2.10)].

The vector notation introduced in this section allows us to manipulate easily the components of our formalism. Good examples are provided by the relations

$$S(r,\varphi) \underset{\sim}{\mathbf{a}}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu} S^\dagger(r,\varphi) = \underset{\sim}{\alpha}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu} = \nu_+ \alpha_+^\dagger - \nu_-^* \alpha_- , \tag{2.35a}$$

$$S^\dagger(r,\varphi) \underset{\sim}{\mathbf{a}}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu} S(r,\varphi) = \underset{\sim}{\mathbf{a}}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu}_\alpha = \nu_{\alpha_+} a_+^\dagger - \nu_{\alpha_-}^* a_- , \tag{2.35b}$$

the first of which follow directly from Eq. (2.24) and the second of which requires the invariant (2.31a) and Eq. (2.25). Another example is the commutator

$$[\underset{\sim}{\nu}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\mathbf{a}}, \underset{\sim}{\mathbf{a}}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu}'] = \underset{\sim}{\nu}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu}' \tag{2.36}$$

[Eqs. (1.1)], which the invariants (2.31) and Eq. (2.34) allow us to write immediately in the equivalent forms

$$[\underset{\sim}{\nu}_\alpha^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\alpha}, \underset{\sim}{\alpha}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu}_\alpha'] = \underset{\sim}{\nu}_\alpha^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\nu}_\alpha' , \tag{2.37}$$

$$[\underset{\sim}{\eta}^\dagger \underset{\sim}{\mathscr{A}}, \underset{\sim}{\mathscr{A}}^\dagger \underset{\sim}{\eta}'] = \underset{\sim}{\eta}^\dagger \underset{\sim}{\Pi} \underset{\sim}{\eta}' . \tag{2.38}$$

The space on which quasiprobability distributions are defined is a complex phase space, and the space on which characteristic functions are defined is the corresponding complex Fourier space. An active-role vector and the corresponding passive-role vector form a pair of vectors under a complex Fourier transform. It is useful to note here the relations among integration measures for the c-number vectors in each column of Table I. Begin by defining, for a complex integration variable ξ, an integration measure $d^2\xi \equiv d(\mathrm{Re}\xi)\, d(\mathrm{Im}\xi)$.[8] For a pair of complex numbers ξ_1 and ξ_2, which form the c-number vector $\underset{\sim}{\xi}$, define an integration measure

$$d^4\xi \equiv d^2\xi_1\, d^2\xi_2 = d(\mathrm{Re}\xi_1)\, d(\mathrm{Im}\xi_1)\, d(\mathrm{Re}\xi_2)\, d(\mathrm{Im}\xi_2) . \tag{2.39}$$

The relations among integration measures are then given by

$$d^4\mu = d^4\mu_\alpha = (1-\epsilon^2/\Omega^2)^{-1} d^4\xi , \tag{2.40a}$$

$$d^4\nu = d^4\nu_\alpha = (1-\epsilon^2/\Omega^2) d^4\eta . \tag{2.40b}$$

There are analogous relations among the δ functions of the c-number vectors in each column of Table I. For a complex number ξ, let $\delta^2(\xi) \equiv \delta(\mathrm{Re}\xi)\delta(\mathrm{Im}\xi)$, and for a c-number vector $\underset{\sim}{\xi}$, let

$$\delta^4(\underset{\sim}{\xi}) \equiv \delta^2(\xi_1)\delta^2(\xi_2) = \delta(\mathrm{Re}\xi_1)\delta(\mathrm{Im}\xi_1)\delta(\mathrm{Re}\xi_2)\delta(\mathrm{Im}\xi_2) . \tag{2.41}$$

Then one finds that

$$\delta^4(\underset{\sim}{\mu}) = \delta^4(\underset{\sim}{\mu}_\alpha) = (1-\epsilon^2/\Omega^2)\delta^4(\underset{\sim}{\xi}) , \tag{2.42a}$$

$$\delta^4(\underset{\sim}{\nu})=\delta^4(\underset{\sim}{\nu}_\alpha)=(1-\epsilon^2/\Omega^2)^{-1}\delta^4(\underset{\sim}{\eta}) . \tag{2.42b}$$

Equations (2.39)–(2.42) find application in Sec. III C 2, where we define the complex Fourier transform, and they will be used extensively in paper III.

We take the remainder of this section to consider the degenerate limit of our two-mode formalism, because most of the current work on two-photon devices deals with the degenerate limit. By the degenerate limit we mean the limit in which the two modes at frequencies $\Omega\pm\epsilon$ coalesce into a single mode at frequency Ω ($\epsilon=0, a_+=a_-$). The formal method for taking this limit is described in Sec. VIII A of I. Here we list the quantities—analogs of the corresponding two-mode quantities—that are used in the degenerate limit. The SP creation and annihilation operators for the single mode are denoted by $a^\dagger$ and a. The (Hermitian) IP *degenerate quadrature-phase amplitudes* are defined by

$$x_1\equiv 2^{-1/2}(a+a^\dagger) , \tag{2.43a}$$

$$x_2\equiv 2^{-1/2}(-ia+ia^\dagger) , \tag{2.43b}$$

$$a=2^{-1/2}(x_1+ix_2) \tag{2.44}$$

[Eqs. (I.8.16); cf. Eqs. (1.5)]; their commutator is

$$[x_1,x_2]=i \tag{2.45}$$

[cf. Eqs. (1.7)]. Analogous to the two-mode squeeze operator (1.8) is the *degenerate squeeze operator*[9,10]

$$S_1(r,\varphi)\equiv\exp\{\tfrac{1}{2}r[a^2e^{-2i\varphi}-(a^\dagger)^2e^{2i\varphi}]\} \tag{2.46}$$

[Eq. (I.8.19d)], which satisfies

$$S_1^{-1}(r,\varphi)=S_1^\dagger(r,\varphi)=S_1(-r,\varphi)=S_1(r,\varphi+\pi/2) \tag{2.47}$$

[cf. Eq. (1.9)]. The IP *squeezed annihilation operator* is defined by

$$\alpha(r,\varphi)\equiv S_1(r,\varphi)aS_1^\dagger(r,\varphi)=a\cosh r+a^\dagger e^{2i\varphi}\sinh r \tag{2.48}$$

[Eq. (I.8.20); cf. Eq. (1.10)].

The degenerate quadrature-phase amplitudes (2.43) look deceptively like a dimensionless coordinate and momentum. To avoid confusion, we remind the reader that Eqs. (2.43) and (2.44) are written in mixed pictures: a and $a^\dagger$ are (constant) SP operators, whereas x_1 and x_2 are (constant) IP operators. In the SP the degenerate quadrature-phase amplitudes are explicitly time-dependent operators, which we denote by

$$x_1(t)\equiv e^{-i\Omega t a^\dagger a}x_1e^{i\Omega t a^\dagger a}$$
$$=2^{-1/2}(ae^{i\Omega t}+a^\dagger e^{-i\Omega t}) , \tag{2.49a}$$

$$x_2(t)\equiv e^{-i\Omega t a^\dagger a}x_2e^{i\Omega t a^\dagger a}$$
$$=2^{-1/2}(-iae^{i\Omega t}+ia^\dagger e^{-i\Omega t}) \tag{2.49b}$$

[cf. Eqs. (I.4.22)]. In contrast, the dimensionless coordinate and momentum are constant operators in the SP, defined by

$$x\equiv 2^{-1/2}(a+a^\dagger) , \tag{2.50a}$$

$$p\equiv 2^{-1/2}(-ia+ia^\dagger) , \tag{2.50b}$$

$$a=2^{-1/2}(x+ip) . \tag{2.51}$$

Thus, although $x_1=x$ and $x_2=p$, a picture-consistent equation relating the degenerate quadrature-phase amplitudes to the coordinate and momentum takes the form

$$x_1(t)=x\cos(\Omega t)-p\sin(\Omega t) , \tag{2.52a}$$

$$x_2(t)=x\sin(\Omega t)+p\cos(\Omega t) . \tag{2.52b}$$

Table II summarizes the operators and associated c-number quantities which are used in the degenerate limit. In Table II we introduce a vector notation for a single mode analogous to the two-mode vector notation summarized in Table I. It is hoped that use of nearly the same symbols for the two cases will not lead to confusion, because we never deal with the two cases simultaneously. Similar single-mode vector notations have been used by Yuen,[11] Milburn,[12] and Collett and Gardiner.[3] The single mode that exists in the degenerate limit has two degrees of freedom—two fewer than in the original two modes. In the first two rows of Table II this reduction shows up in that the components of the vectors are not independent quantities; in the third row it shows up in that the components of the vectors are, reading across the table from left to right, Hermitian operators, real numbers, and pure imaginary numbers. In the degenerate limit it is sometimes convenient to use a different passive-role vector

$$\underset{\sim}{\zeta}\equiv\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}\equiv i\underset{\sim}{\eta}=i\underline{A}\,\underline{\sigma}_3\underset{\sim}{\nu}=2^{1/2}\begin{bmatrix}-\mathrm{Im}(\nu)\\ \mathrm{Re}(\nu)\end{bmatrix}=\underset{\sim}{\zeta}^* , \tag{2.53}$$

whose components are real.

In the degenerate limit the reduced spectral-density matrix (2.13) becomes an ordinary (real, symmetric) covariance matrix

$$\underline{\Sigma}\equiv\langle\Delta\underset{\sim}{x}\,\Delta\underset{\sim}{x}^\dagger\rangle_{\rm sym}$$
$$=\begin{bmatrix}\langle(\Delta x_1)^2\rangle & \langle\Delta x_1\,\Delta x_2\rangle_{\rm sym}\\ \langle\Delta x_2\,\Delta x_1\rangle_{\rm sym} & \langle(\Delta x_2)^2\rangle\end{bmatrix} . \tag{2.54}$$

The corresponding (Hermitian) matrix that gives the second-order noise moments in terms of the creation and annihilation operators is

$$\overline{\underline{\Sigma}}\equiv\langle\Delta\underset{\sim}{a}\,\Delta\underset{\sim}{a}^\dagger\rangle_{\rm sym}=\begin{bmatrix}\langle\,|\Delta a|^2\rangle & \langle(\Delta a)^2\rangle\\ \langle(\Delta a^\dagger)^2\rangle & \langle\,|\Delta a|^2\rangle\end{bmatrix} \tag{2.55}$$

[cf. Eq. (2.15)]. These two matrices are related by

$$\underline{\Sigma}=\underline{A}\,\overline{\underline{\Sigma}}\,\underline{A}^\dagger \tag{2.56}$$

[cf. Eq. (2.16)]. Just as in the two-mode case, one can decompose the matrices $\underline{\Sigma}$ and $\overline{\underline{\Sigma}}$ in terms of the unit matrix and the Pauli matrices. Equations (2.17)–(2.20) retain their forms in the degenerate limit, but note that $\Sigma_2=0$ [Eq. (2.19)] and $\overline{\Sigma}_3=0$ [Eq. (2.20)]. Equations (2.21) reduce to the simple equations

$$\Sigma_0=\overline{\Sigma}_0,\ \ \Sigma_1=-\overline{\Sigma}_2,\ \ \Sigma_3=\overline{\Sigma}_1 . \tag{2.57}$$

For TS noise the matrices $\underline{\Sigma}$ and $\overline{\underline{\Sigma}}$ are identical and equal to a multiple of the unit matrix ($\underline{\Sigma}=\overline{\underline{\Sigma}}=\Sigma_0\underline{1}$).

The invariants in the degenerate limit are very much

TABLE II. Single-mode vector notation.

	Associated c-number vectors	
Operator vector	Active role	Passive role
$\underline{\mathbf{a}} \equiv \begin{bmatrix} a \\ a^\dagger \end{bmatrix}$	$\underline{\boldsymbol{\mu}} \equiv \begin{bmatrix} \mu \\ \mu^* \end{bmatrix}$	$\underline{\boldsymbol{\nu}} \equiv \begin{bmatrix} \nu \\ \nu^* \end{bmatrix}$
$\underline{\boldsymbol{\alpha}}_{r,\varphi} \equiv \underline{\boldsymbol{\alpha}} \equiv \begin{bmatrix} \alpha \\ \alpha^\dagger \end{bmatrix} = \underline{C}_{r,\varphi}\underline{\mathbf{a}}$	$\underline{\boldsymbol{\mu}}_\alpha \equiv \begin{bmatrix} \mu_\alpha \\ \mu_\alpha^* \end{bmatrix} \equiv \underline{C}_{r,\varphi}\underline{\boldsymbol{\mu}}$	$\underline{\boldsymbol{\nu}}_\alpha \equiv \begin{bmatrix} \nu_\alpha \\ \nu_\alpha^* \end{bmatrix} \equiv \underline{C}_{r,\varphi}\underline{\boldsymbol{\nu}}$
$\underline{\mathbf{x}} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{A}\underline{\mathbf{a}}$	$\underline{\boldsymbol{\xi}} \equiv \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \equiv \underline{A}\underline{\boldsymbol{\mu}} = \underline{\boldsymbol{\xi}}^*$	$\underline{\boldsymbol{\eta}} \equiv \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \equiv \underline{A}\underline{\sigma}_3\underline{\boldsymbol{\nu}} = -\underline{\boldsymbol{\eta}}^*$

like the invariants (2.31):

$$a^\dagger\nu - a\nu^* = \underline{\mathbf{a}}^\dagger\underline{\sigma}_3\underline{\boldsymbol{\nu}} = \underline{\boldsymbol{\alpha}}^\dagger\underline{\sigma}_3\underline{\boldsymbol{\nu}}_\alpha = \underline{\mathbf{x}}^\dagger\underline{\boldsymbol{\eta}} = -i\underline{\mathbf{x}}^\dagger\underline{\boldsymbol{\zeta}} = -i(x_1\zeta_1 + x_2\zeta_2)\ , \tag{2.58a}$$

$$\mu^*\nu - \mu\nu^* = \underline{\boldsymbol{\mu}}^\dagger\underline{\sigma}_3\underline{\boldsymbol{\nu}} = \underline{\boldsymbol{\mu}}_\alpha^\dagger\underline{\sigma}_3\underline{\boldsymbol{\nu}}_\alpha = \underline{\boldsymbol{\xi}}^\dagger\underline{\boldsymbol{\eta}} = -i\underline{\boldsymbol{\xi}}^\dagger\underline{\boldsymbol{\zeta}} = -i(\xi_1\zeta_1 + \xi_2\zeta_2)\ . \tag{2.58b}$$

In the degenerate limit the relations (2.32)–(2.34) become

$$\underline{\boldsymbol{\xi}}^\dagger\underline{\boldsymbol{\xi}}' = \underline{\boldsymbol{\mu}}^\dagger\underline{\boldsymbol{\mu}}'\ , \tag{2.59}$$

$$\underline{\boldsymbol{\nu}}^\dagger\underline{\boldsymbol{\nu}}' = \underline{\boldsymbol{\eta}}^\dagger\underline{\boldsymbol{\eta}}' = \underline{\boldsymbol{\zeta}}^\dagger\underline{\boldsymbol{\zeta}}'\ , \tag{2.60}$$

$$\underline{\boldsymbol{\nu}}^\dagger\underline{\sigma}_3\underline{\boldsymbol{\nu}}' = -\underline{\boldsymbol{\eta}}^\dagger\underline{\sigma}_2\underline{\boldsymbol{\eta}}' = -\underline{\boldsymbol{\zeta}}^\dagger\underline{\sigma}_2\underline{\boldsymbol{\zeta}}'\ , \tag{2.61}$$

because when $\epsilon=0$ ($\underline{\lambda}=\underline{1}$), $\underline{\lambda}^2=\underline{\Delta}=\underline{1}$ and $\underline{\Pi}=-\underline{\sigma}_2$. Equations (2.35)–(2.38) retain their forms in the degenerate limit, with the two-mode vectors replaced by the corresponding single-mode vectors.

The integration measures corresponding to the c-number vectors in Table II deserve special comment; they do not have the form of Eqs. (2.40) with $\epsilon=0$ because of the reduction in the number of degrees of freedom at degeneracy. Defining $d^2\mu \equiv d(\mathrm{Re}\mu)\,d(\mathrm{Im}\mu)$, one finds that

$$d^2\mu = d^2\mu_\alpha = \tfrac{1}{2}d\xi_1\,d\xi_2\ , \tag{2.62a}$$

$$d^2\nu = d^2\nu_\alpha = \tfrac{1}{2}d\zeta_1\,d\zeta_2\ . \tag{2.62b}$$

Notice that $d^2\mu/\pi = d\xi_1\,d\xi_2/2\pi$ is the usual phase-space volume element. Corresponding to Eqs. (2.62) are the following relations among δ functions:

$$\delta^2(\mu) = \delta^2(\mu_\alpha) = 2\delta(\xi_1)\delta(\xi_2)\ , \tag{2.63a}$$

$$\delta^2(\nu) = \delta^2(\nu_\alpha) = 2\delta(\zeta_1)\delta(\zeta_2)\ , \tag{2.63b}$$

[$\delta^2(\mu) \equiv \delta(\mathrm{Re}\mu)\,\delta(\mathrm{Im}\mu)$].

III. FUNDAMENTAL UNITARY OPERATORS

A. Modulation-picture free evolution operator

The basic picture in our formalism is the modulation picture (see discussion in Sec. IV C of I), so the fundamental free evolution operator is the *modulation-picture free evolution operator*

$$U_M(t) \equiv e^{-iH_Mt} = \exp[-i\epsilon t(a_+^\dagger a_+ - a_-^\dagger a_-)] = e^{-i\epsilon t}\exp(-i\epsilon t\underline{\mathbf{a}}^\dagger\underline{\sigma}_3\underline{\mathbf{a}}) \tag{3.1}$$

[Eq. (I.4.37)], which satisfies

$$U_M^{-1}(t) = U_M^\dagger(t) = U_M(-t)\ . \tag{3.2}$$

The MP free evolution operator is used to evolve states in the MP when the two modes are evolving freely. It unitarily transforms $a_\pm$ as

$$U_M^\dagger(t)a_\pm U_M(t) = a_\pm e^{\mp i\epsilon t}\ , \tag{3.3}$$

which in vector notation becomes

$$U_M^\dagger(t)\underline{\mathbf{a}}U_M(t) = \underline{\mathbf{a}}e^{-i\epsilon t}\ . \tag{3.4}$$

Multiplying Eq. (3.4) first by $\underline{C}_{r,\varphi}$ and then by $\underline{A}\underline{\lambda}$, one finds that

$$U_M^\dagger(t)\underline{\boldsymbol{\alpha}}U_M(t) = \underline{\boldsymbol{\alpha}}e^{-i\epsilon t}\ , \tag{3.5}$$

$$U_M^\dagger(t)\underline{\mathscr{A}}U_M(t) = \underline{\mathscr{A}}e^{-i\epsilon t} \tag{3.6}$$

[cf. Eq. (I.4.27)]. An important property of $U_M(t)$ is that it commutes with $S(r,\varphi)$:

$$U_M(t)S(r,\varphi)U_M^\dagger(t) = S(r,\varphi)\ . \tag{3.7}$$

In the degenerate limit $U_M(t)$ becomes the identity operator, i.e.,

$$U_M(t) \underset{p}{\rightarrow} 1 \tag{3.8}$$

[Eq. (I.8.19a)], which means that the MP and the IP coincide.

B. Rotation operator

An important feature of our formalism is the phase freedom in the definition of the quadrature-phase amplitudes (see discussion in Sec. IV C of I). The operator that describes this phase freedom is the *rotation operator*

$$R(\theta) \equiv \exp[-i\theta(a_+^\dagger a_+ + a_-^\dagger a_-)] = e^{i\theta}\exp(-i\theta\underline{\mathbf{a}}^\dagger\underline{\mathbf{a}}) \tag{3.9}$$

[Eq. (I.4.33)], which satisfies

$$R^{-1}(\theta) = R^\dagger(\theta) = R(-\theta)\ . \tag{3.10}$$

A unitary transformation generated by $R(\theta)$ produces a common phase change of the annihilation operators,

$$R^{\dagger}(\theta)a_{\pm}R(\theta)=a_{\pm}e^{-i\theta} \quad (3.11)$$

[Eq. (I.4.35)], which in vector notation becomes

$$R^{\dagger}(\theta)\underset{\sim}{\mathbf{a}}R(\theta)=e^{-i\theta\underset{\sim}{\sigma}_3}\underset{\sim}{\mathbf{a}} , \quad (3.12)$$

$$e^{-i\theta\underset{\sim}{\sigma}_3}=\underline{1}\cos\theta-i\underset{\sim}{\sigma}_3\sin\theta=\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta}\end{bmatrix} , \quad (3.13)$$

and it produces a rotation of the quadrature-phase amplitudes,

$$R^{\dagger}(\theta)\underset{\sim}{\mathscr{A}}R(\theta)=e^{i\theta\underset{\sim}{\sigma}_2}\underset{\sim}{\mathscr{A}} , \quad (3.14)$$

$$e^{i\theta\underset{\sim}{\sigma}_2}=\underline{1}\cos\theta+i\underset{\sim}{\sigma}_2\sin\theta=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} \quad (3.15)$$

[Eq. (A6); cf. Eqs. (I.4.36)].

Important properties of the rotation operator include the following: (i) $R(\theta)$ "rotates" the squeeze operator, i.e.,

$$R^{\dagger}(\theta)S(r,\varphi)R(\theta)=S(r,\varphi+\theta) \quad (3.16)$$

[Eq. (3.11)], a property which is the operator analog of Eq. (A27); (ii) $R(\Omega t)=e^{-iH_R t}$ is the unitary transformation that connects the SP and the MP [Eq. (I.4.4)]; (iii) $R(\theta)$ commutes with $U_M(t)$ [cf. Eq. (1.4)]; and (iv) the SP free evolution operator is $e^{-iH_0 t}=R(\Omega t)U_M(t)$. Equation (3.16) implies immediately that

$$R^{\dagger}(\theta)\underset{\sim}{\alpha}_{r,\varphi}R(\theta)=e^{-i\theta\underset{\sim}{\sigma}_3}\underset{\sim}{\alpha}_{r,\varphi+\theta} . \quad (3.17)$$

In the degenerate limit $R(\theta)$ becomes a single-mode rotation operator, i.e.,

$$R(\theta)\underset{P}{\rightarrow}\exp(-i\theta a^{\dagger}a) \quad (3.18)$$

[Eq. (I.8.19b)].

C. Two-mode displacement operator

At the heart of one-photon optics lies the displacement operator,[8] which generates coherent states from the vacuum. It continues to occupy an important place in two-photon optics. Here we begin by reviewing some well-known properties of the displacement operator for a single mode. We then proceed to the displacement operator for two modes and write its properties in terms of the vector notation.

1. Single-mode displacement operator

The *single-mode displacement operator*[8] is defined by

$$D(a,\mu)\equiv\exp(\mu a^{\dagger}-\mu^{*}a) \quad (3.19)$$

[Eq. (I.3.7)]. It satisfies the following string of equalities:

$$D^{-1}(a,\mu)=D^{\dagger}(a,\mu)=D(a,-\mu)=D(-a,\mu) . \quad (3.20)$$

The key property of the displacement operator is that it displaces the annihilation operator,[8] i.e.,

$$D^{\dagger}(a,\mu)aD(a,\mu)=a+\mu . \quad (3.21)$$

One can write Eq. (3.21) in an equivalent form involving the degenerate quadrature-phase amplitudes (2.43):

$$D^{\dagger}(a,\mu)\underset{\sim}{\mathbf{x}}D(a,\mu)=\underset{\sim}{\mathbf{x}}+\underset{\sim}{\xi} \quad (3.22)$$

(see Table II).

We use a two-slot notation for the displacement operator: $D(a,\mu)$ can be regarded as an operator-valued function of an operator a (first slot) and a complex number μ (second slot). The most important reason for this two-slot notation is that one can replace a with another operator that has the same commutator with its adjoint and the resulting "displacement operator" has the same properties as the original. In practice, it is sometimes useful to replace a with the squeezed annihilation operator $\alpha(r,\varphi)$ [Eq. (2.48)]. The resulting operator

$$D(\alpha,\mu_\alpha)=e^{\mu_\alpha\alpha^{\dagger}-\mu_\alpha^{*}\alpha} , \quad (3.23)$$

which we conventionally write with μ_α as the label for the complex variable, displaces α:

$$D^{\dagger}(\alpha,\mu_\alpha)\alpha D(\alpha,\mu_\alpha)=\alpha+\mu_\alpha . \quad (3.24)$$

Notice that $D(\alpha,\mu)$ is unitarily equivalent to $D(a,\mu)$:

$$S_1(r,\varphi)D(a,\mu)S_1^{\dagger}(r,\varphi)=D(\alpha,\mu) . \quad (3.25)$$

Further, the invariant (2.58a) implies

$$D(a,\mu)=D(\alpha,\mu_\alpha) . \quad (3.26)$$

Equations (3.25) and (3.26) can be used to obtain the result

$$S_1^{\dagger}(r,\varphi)D(a,\mu)S_1(r,\varphi)=D(a,\mu_\alpha) . \quad (3.27)$$

One can also define the operator

$$\mathscr{D}(\underset{\sim}{\mathbf{x}},\underset{\sim}{\eta})\equiv\exp(\underset{\sim}{\mathbf{x}}^{\dagger}\underset{\sim}{\eta})=\exp(-i\underset{\sim}{\mathbf{x}}^{\dagger}\underset{\sim}{\xi})=e^{-i(\xi_1x_1+\xi_2x_2)} , \quad (3.28)$$

which is the displacement operator written in terms of x_1 and x_2, i.e.,

$$D(a,\nu)=D(\alpha,\nu_\alpha)=\mathscr{D}(\underset{\sim}{\mathbf{x}},\underset{\sim}{\eta}) \quad (3.29)$$

[Eq. (2.58a)].

A second reason for the two-slot notation is that one can replace the operator a with a complex number μ to obtain a complex-valued function of two complex variables,

$$D(\mu,\nu)\equiv e^{\mu^{*}\nu-\nu^{*}\mu} , \quad (3.30)$$

which satisfies

$$D^{-1}(\mu,\nu)=D^{*}(\mu,\nu)=D(\nu,\mu)=D(\mu,-\nu)=D(-\mu,\nu) , \quad (3.31a)$$

$$D(\mu,\nu)D(\mu,\nu')=D(\mu,\nu+\nu') . \quad (3.31b)$$

The importance of $D(\mu,\nu)$ lies in its role as the expansion factor for complex Fourier transforms.[13] A function $f(\mu)$ is related to its complex Fourier transform $F(\nu)$ by

$$f(\mu)=\int\frac{d^2\nu}{\pi}F(\nu)D(\nu,\mu) \quad (3.32)$$

$[d^2\nu\equiv d(\mathrm{Re}\nu)\,d(\mathrm{Im}\nu)]$. Employing the property

$$\int \frac{d^2\mu}{\pi} D(\mu,\nu) = \int \frac{d^2\mu}{\pi} D(\nu,\mu) = \pi\delta^2(\nu) \tag{3.33}$$

$[\delta^2(\nu) \equiv \delta(\mathrm{Re}\nu)\,\delta(\mathrm{Im}\nu)]$, one can invert Eq. (3.32) to give

$$F(\nu) = \int \frac{d^2\mu}{\pi} f(\mu) D(\mu,\nu) \, . \tag{3.34}$$

Equations (3.32) and (3.34) are a neat, symmetric way to write the relations between a function and its complex Fourier transform. The invariant (2.58b) implies

$$D(\mu,\nu) = D(\mu_\alpha,\nu_\alpha) = e^{\underset{\sim}{\xi}^\dagger \underset{\sim}{\eta}} \equiv \mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta}) \tag{3.35}$$

[cf. Eq. (3.29)].

We end this review of the single-mode displacement operator by listing a few other important properties:

$$D^\dagger(a,\mu) D(a,\nu) D(a,\mu) = D(a+\mu,\nu) = D(\mu,\nu) D(a,\nu) \, , \tag{3.36}$$

$$D^\dagger(a,\mu') D(a,\mu) = D(\tfrac{1}{2}\mu',\mu) D(a,\mu-\mu') \, , \tag{3.37}$$

$$D(a,\mu) = e^{-|\mu|^2/2} e^{\mu a^\dagger} e^{-\mu^* a} = e^{|\mu|^2/2} e^{-\mu^* a} e^{\mu a^\dagger} \, . \tag{3.38}$$

2. Two-mode displacement operator

We turn now to the displacement operator for two modes, our objective being to generalize (trivially) the properties of the single-mode displacement operator and to write the resulting two-mode properties in terms of the vector notation. Use of the vector notation gives the two-mode properties an appearance as compact and elegant as the single-mode properties.

We begin by writing the *two-mode displacement operator*[8] [Eq. (I.4.12)] in the form

$$\begin{aligned} \mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) &\equiv \exp(\underset{\sim}{\mathbf{a}}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\mu} - \underset{\sim}{\mu}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\mathbf{a}}) \\ &= D(a_+,\mu_+) D(a_-,\mu_-) \end{aligned} \tag{3.39}$$

[cf. Eq. (3.19)], which satisfies

$$\mathbf{D}^{-1}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) = \mathbf{D}^\dagger(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) = \mathbf{D}(\underset{\sim}{\mathbf{a}},-\underset{\sim}{\mu}) = \mathbf{D}(-\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) \tag{3.40}$$

[cf. Eq. (3.20)]. The two-mode displacement operator displaces $\underset{\sim}{\mathbf{a}}$, i.e.,

$$\mathbf{D}^\dagger(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) \underset{\sim}{\mathbf{a}} \mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) = \underset{\sim}{\mathbf{a}} + \underset{\sim}{\mu} \tag{3.41}$$

[cf. Eq. (3.21)], and it generates two-mode coherent states from the vacuum (see Sec. IV A 2). Multiplying Eq. (3.41) by $\underset{\sim}{A}\underset{\sim}{\lambda}$ yields

$$\mathbf{D}^\dagger(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) \underset{\sim}{\mathscr{A}} \mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) = \underset{\sim}{\mathscr{A}} + \underset{\sim}{\xi} \tag{3.42}$$

[see Table I; cf. Eq. (3.22)].

We use a two-slot notation for the two-mode displacement operator: $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})$ can be regarded as an operator-valued function of an operator vector $\underset{\sim}{\mathbf{a}}$ and a c-number vector $\underset{\sim}{\mu}$. Just as in the single-mode case, the main reason for this two-slot notation is that we can also consider the operator

$$\begin{aligned} \mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha) &= e^{\underset{\sim}{\alpha}^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\mu}_\alpha - \underset{\sim}{\mu}_\alpha^\dagger \underset{\sim}{\sigma}_3 \underset{\sim}{\alpha}} \\ &= D(\alpha_+,\mu_{\alpha_+}) D(\alpha_-,\mu_{\alpha_-}) \end{aligned} \tag{3.43}$$

[cf. Eq. (3.23)]. An important connection between $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})$ and $\mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha)$ is that properties of $\mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha)$ can be obtained directly from those of $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})$ because the squeezed annihilation operators have the same commutation algebra as the annihilation operators. For example, one can say immediately that $\mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha)$ displaces $\underset{\sim}{\alpha}$:

$$\mathbf{D}^\dagger(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha) \underset{\sim}{\alpha} \mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha) = \underset{\sim}{\alpha} + \underset{\sim}{\mu}_\alpha \, . \tag{3.44}$$

An equivalent way of stating this connection is that $\mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu})$ is unitarily equivalent to $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})$, i.e.,

$$S(r,\varphi) \mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) S^\dagger(r,\varphi) = \mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}) \tag{3.45}$$

[cf. Eqs. (2.35a) and (3.25)]. Thus Eq. (3.44) could be obtained by unitarily transforming Eq. (3.41) with $S(r,\varphi)$ and replacing $\underset{\sim}{\mu}$ with $\underset{\sim}{\mu}_\alpha$. A different and crucial connection between $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})$ and $\mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha)$ is that they are the same operator, a consequence of the invariant (2.31a):

$$\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) = \mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha) \tag{3.46}$$

[cf. Eq. (3.26)]. Equation (3.46) means that Eq. (3.44) results from multiplying Eq. (3.41) by $\underset{\sim}{C}_{r,\varphi}$. A further important relation is a consequence of Eqs. (3.46) and (2.25):

$$S^\dagger(r,\varphi) \mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}) S(r,\varphi) = \mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}_\alpha) \tag{3.47}$$

[cf. Eqs. (2.35b) and (3.27)].

We find it useful to write the two-mode displacement operator in terms of the quadrature-phase amplitudes α_1 and α_2. Therefore, we define the operator

$$\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta}) \equiv \exp(\underset{\sim}{\mathscr{A}}^\dagger \underset{\sim}{\eta} - \underset{\sim}{\eta}^\dagger \underset{\sim}{\mathscr{A}}) = e^{\eta_1 \alpha_1^\dagger + \eta_2 \alpha_2^\dagger - \eta_1^* \alpha_1 - \eta_2^* \alpha_2} \tag{3.48}$$

[cf. Eq. (3.28)], which satisfies

$$\mathscr{D}^{-1}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta}) = \mathscr{D}^\dagger(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta}) = \mathscr{D}(\underset{\sim}{\mathscr{A}},-\underset{\sim}{\eta}) = \mathscr{D}(-\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta}) \tag{3.49}$$

and which can be regarded as an operator-valued function of an operator vector $\underset{\sim}{\mathscr{A}}$ and a c-number vector $\underset{\sim}{\eta}$. Since it is not the same function as $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})$, we distinguish it by using a script letter. Nonetheless, the invariant (2.31a) guarantees that $\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})$ and $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})$ are the same operator:

$$\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu}) = \mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\nu}_\alpha) = \mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta}) \tag{3.50}$$

[cf. Eq. (3.29)].

The introduction of $\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})$ provides a good opportunity to elucidate the distinction between the active-role and passive-role vectors introduced in Table I. As noted in Sec. II, an active-role vector is used as a surrogate for the corresponding operator vector, e.g., as an eigenvalue or an expectation value of the operator vector or as the vector variable of a quasiprobability distribution. Thus the active-role vectors $\underset{\sim}{\mu}$ and $\underset{\sim}{\mu}_\alpha$ are used in the second slot of the two-mode displacement operator when it is used in its active role, i.e., as a unitary operator that transforms states and operators. A passive-role vector is used as the vector variable of a characteristic function. Thus the passive-role vectors $\underset{\sim}{\nu}$ and $\underset{\sim}{\nu}_\alpha$ are used in the second slot of the two-mode displacement operator when it is used in its passive role, i.e., when one takes its expectation value to obtain a characteristic function.

In the first two rows of Table I there is no real difference between the active-role and passive-role vectors, since $\underset{\sim}{\nu}_\alpha$ is related to $\underset{\sim}{\nu}$ in the same way that $\underset{\sim}{\mu}_\alpha$ is related to $\underset{\sim}{\mu}$; the difference is merely a matter of choosing different labels for vectors in the two roles. To find a real difference, one must proceed to the third row. The definition of the active-role vector $\underset{\sim}{\xi}=\underset{\sim}{A}\,\underset{\sim}{\lambda}\underset{\sim}{\mu}$ is determined by the fact that $\underset{\sim}{\xi}$ and $\underset{\sim}{\mu}$ can stand for the expectation values of $\underset{\sim}{\mathscr{A}}$ and $\underset{\sim}{\mathbf{a}}$, respectively; thus $\underset{\sim}{\xi}$ must be related to $\underset{\sim}{\mu}$ in the same way that $\underset{\sim}{\mathscr{A}}$ is related to $\underset{\sim}{\mathbf{a}}$. This natural definition of $\underset{\sim}{\xi}$ is to be contrasted with the definition of the passive-role vector $\underset{\sim}{\eta}=\underset{\sim}{A}\,\underset{\sim}{\lambda}^{-1}\underset{\sim}{\sigma}_3\underset{\sim}{\nu}$, which at first appears very peculiar indeed. The explanation for this peculiar definition lies in the form of the operator $\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})$, which is the displacement operator written in terms of quadrature-phase amplitudes. The simple form of $\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})$ is a consequence of the invariant (2.31a) and, hence, of the definition of $\underset{\sim}{\eta}$. More illuminating is to put things the other way around: the peculiar definition of $\underset{\sim}{\eta}$ is dictated by the desire to have a simple form for the two-mode displacement operator when it is written in terms of the quadrature-phase amplitudes; thus this desire is ultimately responsible for the distinction we make between the active role and the passive role. This discussion also makes clear why $\underset{\sim}{\nu}_\alpha$ is related to $\underset{\sim}{\nu}$ in the same way that $\underset{\sim}{\mu}_\alpha$ is related to $\underset{\sim}{\mu}$. The definition of $\underset{\sim}{\mu}_\alpha=\underset{\sim}{C}_{r,\varphi}\underset{\sim}{\mu}$ is determined by the relation $\underset{\sim}{\alpha}=\underset{\sim}{C}_{r,\varphi}\underset{\sim}{\mathbf{a}}$; the same transformation $\underset{\sim}{\nu}_\alpha=\underset{\sim}{C}_{r,\varphi}\underset{\sim}{\nu}$ is appropriate for the passive-role vector because of the property (2.29) of $\underset{\sim}{C}_{r,\varphi}$.

The operator $\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})$ is defined in terms of the passive-role vector $\underset{\sim}{\eta}$, and it is used exclusively in the passive role. We could write the two-mode displacement operator in terms of $\underset{\sim}{\mathscr{A}}$ and the active-role vector $\underset{\sim}{\xi}$ simply by substituting $\underset{\sim}{\mathbf{a}}=\underset{\sim}{\lambda}^{-1}\underset{\sim}{A}^\dagger\underset{\sim}{\mathscr{A}}$ and $\underset{\sim}{\mu}=\underset{\sim}{\lambda}^{-1}\underset{\sim}{A}^\dagger\underset{\sim}{\xi}$ into $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})$. The result does not have a simple form, nor do we find it useful, so we do without it.

A particularly important form of Eq. (3.41) can be obtained by using the passive-role vector $\underset{\sim}{\nu}$ and then writing Eq. (3.41) as the commutator

$$[\underset{\sim}{\mathbf{a}},\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})]=\underset{\sim}{\nu}\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})\ . \tag{3.51}$$

Multiplying Eq. (3.51) by $\underset{\sim}{A}\,\underset{\sim}{\lambda}$ and substituting $\underset{\sim}{\nu}=\underset{\sim}{\sigma}_3\underset{\sim}{\lambda}\underset{\sim}{A}^\dagger\underset{\sim}{\eta}$, one finds

$$[\underset{\sim}{\mathscr{A}},\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})]=\underset{\sim}{\Pi}\underset{\sim}{\eta}\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta}) \tag{3.52}$$

[Eqs. (2.10) and (3.50)]. Equations (3.51) and (3.52) will play a crucial role in the operator-ordering formalism of paper III. Equation (3.52) expresses the same relation that Eq. (3.42) does; the apparent difference is due to the use of the passive-role vector $\underset{\sim}{\eta}$ in Eq. (3.52), in contrast to the use of the active-role vector $\underset{\sim}{\xi}$ in Eq. (3.42).

A second reason for the two-slot notation is that one can replace the operator vector in the first slot of $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})$ or $\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})$ with a c-number vector. Hence, one can define the following complex-valued functions of two c-number vectors:

$$\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu})\equiv e^{\underset{\sim}{\mu}^\dagger\underset{\sim}{\sigma}_3\underset{\sim}{\nu}-\underset{\sim}{\nu}^\dagger\underset{\sim}{\sigma}_3\underset{\sim}{\mu}}=D(\mu_+,\nu_+)D(\mu_-,\nu_-)\ , \tag{3.53}$$

$$\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta})\equiv e^{\underset{\sim}{\xi}^\dagger\underset{\sim}{\eta}-\underset{\sim}{\eta}^\dagger\underset{\sim}{\xi}}=D(\xi_1,\eta_1)D(\xi_2,\eta_2) \tag{3.54}$$

[cf. Eq. (3.30)]. These functions satisfy

$$\mathbf{D}^{-1}(\underset{\sim}{\mu},\underset{\sim}{\nu})=\mathbf{D}^*(\underset{\sim}{\mu},\underset{\sim}{\nu})=\mathbf{D}(\underset{\sim}{\nu},\underset{\sim}{\mu})=\mathbf{D}(\underset{\sim}{\mu},-\underset{\sim}{\nu})=\mathbf{D}(-\underset{\sim}{\mu},\underset{\sim}{\nu})\ , \tag{3.55a}$$

$$\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu})\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu}')=\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu}+\underset{\sim}{\nu}')\ , \tag{3.55b}$$

$$\mathscr{D}^{-1}(\underset{\sim}{\xi},\underset{\sim}{\eta})=\mathscr{D}^*(\underset{\sim}{\xi},\underset{\sim}{\eta})=\mathscr{D}(\underset{\sim}{\eta},\underset{\sim}{\xi})=\mathscr{D}(\underset{\sim}{\xi},-\underset{\sim}{\eta})=\mathscr{D}(-\underset{\sim}{\xi},\underset{\sim}{\eta})\ , \tag{3.56a}$$

$$\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta})\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta}')=\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta}+\underset{\sim}{\eta}') \tag{3.56b}$$

[cf. Eqs. (3.31)]. The invariant (2.31b) implies

$$\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu})=\mathbf{D}(\underset{\sim}{\mu}_\alpha,\underset{\sim}{\nu}_\alpha)=\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta}) \tag{3.57}$$

[cf. Eqs. (3.35) and (3.50)]. Either $\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu})$ or $\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta})$ can serve as the expansion factor for complex Fourier transforms. For example, a function $f(\underset{\sim}{\mu})$ is related to its complex Fourier transform $F(\underset{\sim}{\nu})$ by

$$f(\underset{\sim}{\mu})=\int\frac{d^4\nu}{\pi^2}F(\underset{\sim}{\nu})\mathbf{D}(\underset{\sim}{\nu},\underset{\sim}{\mu})\ , \tag{3.58a}$$

$$F(\underset{\sim}{\nu})=\int\frac{d^4\mu}{\pi^2}f(\underset{\sim}{\mu})\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu}) \tag{3.58b}$$

[Eq. (2.39); cf. Eqs. (3.32) and (3.34)]. The orthonormality and completeness relations for $\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu})$ and $\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta})$ are subsumed in the equations

$$\int\frac{d^4\mu}{\pi^2}\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu})=\int\frac{d^4\mu}{\pi^2}\mathbf{D}(\underset{\sim}{\nu},\underset{\sim}{\mu})=\pi^2\delta^4(\underset{\sim}{\nu})\ , \tag{3.59a}$$

$$\int\frac{d^4\xi}{\pi^2}\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta})=\int\frac{d^4\xi}{\pi^2}\mathscr{D}(\underset{\sim}{\eta},\underset{\sim}{\xi})=\pi^2\delta^4(\underset{\sim}{\eta}) \tag{3.59b}$$

[Eq. (2.41); cf. Eq. (3.33)].

Further properties of the two-mode displacement operator include the way it is transformed by the MP free evolution operator,

$$U_M^\dagger(t)\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})U_M(t)=\mathbf{D}(\underset{\sim}{\mathbf{a}}e^{-i\epsilon t},\underset{\sim}{\nu})=\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu}e^{i\epsilon t})\ , \tag{3.60a}$$

$$U_M^\dagger(t)\mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\nu}_\alpha)U_M(t)=\mathbf{D}(\underset{\sim}{\alpha}e^{-i\epsilon t},\underset{\sim}{\nu}_\alpha)=\mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\nu}_\alpha e^{i\epsilon t})\ , \tag{3.60b}$$

$$U_M^\dagger(t)\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})U_M(t)=\mathscr{D}(\underset{\sim}{\mathscr{A}}e^{-i\epsilon t},\underset{\sim}{\eta})=\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta}e^{i\epsilon t}) \tag{3.60c}$$

[Eqs. (3.4)–(3.6)], and the way it is transformed by the rotation operator,

$$R^\dagger(\theta)\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})R(\theta)=\mathbf{D}(e^{-i\theta\underset{\sim}{\sigma}_3}\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})=\mathbf{D}(\underset{\sim}{\mathbf{a}},e^{i\theta\underset{\sim}{\sigma}_3}\underset{\sim}{\nu})\ , \tag{3.61a}$$

$$\begin{aligned}R^\dagger(\theta)\mathbf{D}(\underset{\sim}{\alpha}_{r,\varphi},\underset{\sim}{\nu}_\alpha)R(\theta)&=\mathbf{D}(e^{-i\theta\underset{\sim}{\sigma}_3}\underset{\sim}{\alpha}_{r,\varphi+\theta},\underset{\sim}{\nu}_\alpha)\\&=\mathbf{D}(\underset{\sim}{\alpha}_{r,\varphi+\theta},e^{i\theta\underset{\sim}{\sigma}_3}\underset{\sim}{\nu}_\alpha)\\&=\mathbf{D}(\underset{\sim}{\alpha}_{r,\varphi},e^{i\theta\underset{\sim}{\sigma}_3}\underset{\sim}{C}_{r,\varphi-\theta}\underset{\sim}{\nu})\ ,\end{aligned} \tag{3.61b}$$

$$R^\dagger(\theta)\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})R(\theta)=\mathscr{D}(e^{i\theta\underset{\sim}{\sigma}_2}\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})=\mathscr{D}(\underset{\sim}{\mathscr{A}},e^{-i\theta\underset{\sim}{\sigma}_2}\underset{\sim}{\eta}) \tag{3.61c}$$

[Eqs. (3.12), (3.14), (3.17), and (A27)]. The two-mode versions of Eq. (3.36) are

$$\mathbf{D}^\dagger(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}})\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\nu}})\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}})=\mathbf{D}(\underset{\sim}{\mathbf{a}}+\underset{\sim}{\boldsymbol{\mu}},\underset{\sim}{\boldsymbol{\nu}})=\mathbf{D}(\underset{\sim}{\boldsymbol{\mu}},\underset{\sim}{\boldsymbol{\nu}})\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\nu}})\ , \tag{3.62a}$$

$$\mathbf{D}^\dagger(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}})\mathscr{D}(\mathscr{A},\underset{\sim}{\boldsymbol{\eta}})\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}})=\mathscr{D}(\mathscr{A}+\underset{\sim}{\boldsymbol{\xi}},\underset{\sim}{\boldsymbol{\eta}})$$
$$=\mathscr{D}(\underset{\sim}{\boldsymbol{\xi}},\underset{\sim}{\boldsymbol{\eta}})\mathscr{D}(\mathscr{A},\underset{\sim}{\boldsymbol{\eta}}) \tag{3.62b}$$

[Eqs. (3.41) and (3.42)]. The product of two displacement operators can be written in two useful forms:

$$\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}}')\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}})=\mathbf{D}^*(\tfrac{1}{2}\underset{\sim}{\boldsymbol{\mu}}',\underset{\sim}{\boldsymbol{\mu}})\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}}+\underset{\sim}{\boldsymbol{\mu}}')\ , \tag{3.63a}$$

$$\mathbf{D}^\dagger(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}}')\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}})=\mathbf{D}(\tfrac{1}{2}\underset{\sim}{\boldsymbol{\mu}}',\underset{\sim}{\boldsymbol{\mu}})\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}}-\underset{\sim}{\boldsymbol{\mu}}') \tag{3.63b}$$

[cf. Eq. (3.37)].

In the degenerate limit the two-mode displacement operator reduces to the single-mode displacement operator:

$$\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\boldsymbol{\mu}})\underset{p}{\rightarrow}D(a,\mu),\quad \mu\equiv 2^{-1/2}(\mu_+ +\mu_-) \tag{3.64}$$

[Eq. (I.8.19c)].

D. Two-mode squeeze operator

The last important unitary operator in our formalism is the *two-mode squeeze operator*

$$S(r,\varphi)\equiv\exp[r(a_+a_-e^{-2i\varphi}-a_+^\dagger a_-^\dagger e^{2i\varphi})]$$
$$=\exp\{-ir\underset{\sim}{\mathbf{a}}^\dagger[\sigma_1\sin(2\varphi)+\sigma_2\cos(2\varphi)]\underset{\sim}{\mathbf{a}}\} \tag{3.65}$$

[Eq. (1.8)]. It squeezes the annihilation operators to give the squeezed annihilation operators (1.10), and it generates two-mode squeezed states from coherent states (see Sec. IV B 1).

Almost all the important properties of $S(r,\varphi)$ have been listed elsewhere in Secs. I—III. Little is left to note here, except two properties—factorization of the squeeze operator and the product of two different squeeze operators. The two-mode squeeze operator can be factored into a product of exponentials of $a_+^\dagger a_-^\dagger$, a_+a_-, and $a_+^\dagger a_+ + a_-^\dagger a_-$. The most useful factored form is[14,15]

$$S(r,\varphi)=(\cosh r)^{-1}e^{-a_+^\dagger a_-^\dagger e^{2i\varphi}\tanh r}e^{-(a_+^\dagger a_+ + a_-^\dagger a_-)\ln(\cosh r)}e^{a_+a_-e^{-2i\varphi}\tanh r} \tag{3.66}$$

[Eq. (B9b)]. In Appendix B we derive Eq. (3.66) and other factored forms in which the exponentials appear in different orders. Also in Appendix B we use the factored forms to write the product of two squeeze operators as a rotation operator times a squeeze operator:

$$S^\dagger(r',\varphi')S(r,\varphi)=e^{-i\Theta}R(\Theta)S(R,\Phi)$$
$$=e^{-i\Theta}S(R,\Phi-\Theta)R(\Theta) \tag{3.67}$$

[Eq. (B16)]; here R, Φ, and Θ are defined by the matrix equation

$$\underset{\sim}{C}_{R,\Phi}e^{i\Theta\sigma_3}=\underset{\sim}{C}_{r,\varphi}\underset{\sim}{C}^{-1}_{r',\varphi'}=\underset{\sim}{C}_{r,\varphi}\underset{\sim}{C}_{-r',\varphi'} \tag{3.68}$$

[Eq. (B14)]. Notice that if $\varphi=\varphi'$, then $\Theta=0$, $\Phi=\varphi$, and $R=r-r'$, i.e., $S^\dagger(r',\varphi')S(r,\varphi)=S(r-r',\varphi)$ [cf. Eq. (A26)].

In the degenerate limit the two-mode squeeze operator becomes the *degenerate squeeze operator* $S_1(r,\varphi)$ [Eq. (2.46)]:

$$S(r,\varphi)\underset{p}{\rightarrow}S_1(r,\varphi)=e^{(r/2)[a^2e^{-2i\varphi}-(a^\dagger)^2e^{2i\varphi}]} \tag{3.69}$$

[Eq. (I.8.19d)]. The degenerate squeeze operator can be factored in the same way as the two-mode squeeze operator. In particular, its most useful factored form is[15–17]

$$S_1(r,\varphi)=(\cosh r)^{-1/2}e^{-(a^\dagger)^2e^{2i\varphi}(\tanh r)/2}e^{-a^\dagger a\ln(\cosh r)}e^{a^2e^{-2i\varphi}(\tanh r)/2} \tag{3.70}$$

[cf. Eq. (3.66); for the derivation of this and other factored forms see Appendix B]. The product of two degenerate squeeze operators is given by

$$S_1^\dagger(r',\varphi')S_1(r,\varphi)=e^{-i\Theta/2}e^{-i\Theta a^\dagger a}S_1(R,\Phi)$$
$$=e^{-i\Theta/2}S_1(R,\Phi-\Theta)e^{-i\Theta a^\dagger a} \tag{3.71}$$

[Eq. (B16); cf. Eq. (3.67)], where R, Φ, and Θ are again defined by Eq. (3.68).

IV. SPECIAL QUANTUM STATES

Any discussion of special quantum states begins with the two-mode vacuum state $|0\rangle$, the state annihilated by a_+ and a_- ($a_\pm|0\rangle=0$). A useful associated state is the (two-mode) *squeezed vacuum state*

$$|0\rangle_{(r,\varphi)}\equiv S(r,\varphi)|0\rangle\ , \tag{4.1}$$

which is the two-mode squeezed state (see Sec. IV B) with $\langle a_\pm\rangle=0$. A convenient basis is provided by the (two-mode) *number eigenstates*

$$|n_+,n_-\rangle\equiv[(n_+!)(n_-!)]^{-1/2}(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}|0\rangle\ , \tag{4.2}$$

$$a_\pm^\dagger a_\pm|n_+,n_-\rangle=n_\pm|n_+,n_-\rangle\ . \tag{4.3}$$

Another basis, unitarily equivalent to the number-eigenstate basis under the two-mode squeeze operator, consists of the (two-mode) *squeezed number eigenstates*

$$|n_+,n_-\rangle_{(r,\varphi)} \equiv S(r,\varphi)|n_+,n_-\rangle = [(n_+!)(n_-!)]^{-1/2}(\alpha_+^\dagger)^{n_+} \times (\alpha_-^\dagger)^{n_-}|0\rangle_{(r,\varphi)} , \quad (4.4)$$

$$\alpha_\pm^\dagger \alpha_\pm |n_+,n_-\rangle_{(r,\varphi)} = n_\pm |n_+,n_-\rangle_{(r,\varphi)} . \quad (4.5)$$

A. Coherent states

If the displacement operator is the heart of one-photon optics, then the soul is the set of states it generates—the coherent states. Here we review briefly some well-known properties[8,18] of coherent states for a single mode, generalize (trivially) those properties to two-mode coherent states, and write the two-mode properties in our vector notation.

1. Single-mode coherent states

The *single-mode coherent states*[8] are generated from the single-mode vacuum state $|0\rangle$ by the displacement operator:

$$|\mu\rangle_{\text{coh}} \equiv D(a,\mu)|0\rangle \quad (4.6)$$

[Eq. (I.3.9)]. Their most important property is that they are eigenstates of the annihilation operator:

$$a|\mu\rangle_{\text{coh}} = \mu|\mu\rangle_{\text{coh}} \quad (4.7)$$

[Eq. (3.21)]. Equation (3.38) can be used to obtain an expansion of $|\mu\rangle_{\text{coh}}$ in terms of the single-mode number eigenstates $|n\rangle \equiv (n!)^{-1/2}(a^\dagger)^n|0\rangle$:

$$|\mu\rangle_{\text{coh}} = e^{-|\mu|^2/2}e^{\mu a^\dagger}|0\rangle = e^{-|\mu|^2/2}\sum_{n=0}^{\infty}\frac{\mu^n}{(n!)^{1/2}}|n\rangle . \quad (4.8)$$

A coherent state $|\mu\rangle_{\text{coh}}$ has time-stationary noise [$\langle(\Delta a)^2\rangle = 0$; Eq. (I.3.12)]; its important nonzero first- and second-order moments are

$$\langle a\rangle = \mu, \quad \langle|\Delta a|^2\rangle = \tfrac{1}{2} , \quad (4.9a)$$

$$\langle \alpha(r,\varphi)\rangle = \mu_\alpha , \quad (4.9b)$$

$$\langle \underset{\sim}{x}\rangle = \underset{\sim}{\xi}, \quad \langle(\Delta x_1)^2\rangle = \langle(\Delta x_2)^2\rangle = \tfrac{1}{2} \quad (4.9c)$$

(see Table II). The expectation value and variance of the number of quanta are

$$\langle a^\dagger a\rangle = |\mu|^2, \quad \langle[\Delta(a^\dagger a)]^2\rangle = |\mu|^2 . \quad (4.10)$$

The symmetrically ordered characteristic function for a coherent state $|\mu\rangle_{\text{coh}}$ is the expectation value of the displacement operator,

$$_{\text{coh}}\langle\mu|D(a,\nu)|\mu\rangle_{\text{coh}} = e^{-|\nu|^2/2}D(\mu,\nu) = e^{-\underset{\sim}{\nu}^\dagger\underset{\sim}{\nu}/4}D(\mu,\nu) \quad (4.11)$$

[Eq. (3.38)]. A Taylor expansion of Eq. (4.11) with respect to ν and ν^* yields the symmetrically ordered moments[13] of a and $a^\dagger$. Using Eqs. (3.29) and (3.35), one can write the characteristic function (4.11) in terms of the variables η_1 and η_2 (see Table II):

$$_{\text{coh}}\langle\mu|\mathscr{D}(\underset{\sim}{x},\underset{\sim}{\eta})|\mu\rangle_{\text{coh}} = e^{(\eta_1^2+\eta_2^2)/4}\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta}) = e^{-\underset{\sim}{\eta}^\dagger\underset{\sim}{\eta}/4}\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta}) . \quad (4.12)$$

A Taylor expansion of Eq. (4.12) with respect to η_1 and η_2 yields the symmetrically ordered moments of x_1 and x_2. Characteristic functions will be considered in detail in paper III.

The coherent states are not orthonormal,

$$_{\text{coh}}\langle\mu'|\mu\rangle_{\text{coh}} = \langle 0|D^\dagger(a,\mu')D(a,\mu)|0\rangle = D(\tfrac{1}{2}\mu',\mu)e^{-|\mu-\mu'|^2/2} \quad (4.13)$$

[Eqs. (3.37) and (4.11)], but they are complete in the following sense, pointed out by Klauder:[19]

$$1 = \int\frac{d^2\mu}{\pi}|\mu\rangle_{\text{coh}\,\text{coh}}\langle\mu| . \quad (4.14)$$

The completeness relation (4.14) is the starting point for developing expansions in terms of the coherent states. It can also be used to demonstrate that the trace of an operator f is given by

$$\text{tr} f = \int\frac{d^2\mu}{\pi}\,{}_{\text{coh}}\langle\mu|f|\mu\rangle_{\text{coh}} , \quad (4.15)$$

which in turn shows that

$$\text{tr}[D(a,\nu)] = e^{-|\nu|^2/2}\int\frac{d^2\mu}{\pi}D(\mu,\nu) = \pi\delta^2(\nu) \quad (4.16)$$

[Eqs. (4.11) and (3.33)].

2. Two-mode coherent states

A *two-mode coherent state*[8] [Eq. (I.4.11)] is generated from the vacuum state by the two-mode displacement operator:

$$|\underset{\sim}{\mu}\rangle_{\text{coh}} \equiv |\mu_+,\mu_-\rangle_{\text{coh}} \equiv \mathbf{D}(\underset{\sim}{a},\underset{\sim}{\mu})|0\rangle \quad (4.17)$$

[cf. Eq. (4.6)]. It is an eigenstate of both a_+ and a_-, i.e.,

$$a_\pm|\underset{\sim}{\mu}\rangle_{\text{coh}} = \mu_\pm|\underset{\sim}{\mu}\rangle_{\text{coh}} \quad (4.18)$$

[cf. Eq. (4.7)], and its number-eigenstate expansion is given by

$$|\underset{\sim}{\mu}\rangle_{\text{coh}} = e^{-\underset{\sim}{\mu}^\dagger\underset{\sim}{\mu}/2}e^{\mu_+a_+^\dagger}e^{\mu_-a_-^\dagger}|0\rangle = e^{-\underset{\sim}{\mu}^\dagger\underset{\sim}{\mu}/2}\sum_{n_+,n_-}\frac{(\mu_+)^{n_+}(\mu_-)^{n_-}}{[(n_+!)(n_-!)]^{1/2}}|n_+,n_-\rangle \quad (4.19)$$

[cf. Eq. (4.8)]. A coherent state $|\underset{\sim}{\mu}\rangle_{\text{coh}}$ has time-stationary noise [Eqs. (I.5.6) and (I.5.9)]; its nonzero first- and second-order moments are

$$\langle\underset{\sim}{a}\rangle = \underset{\sim}{\mu}, \quad \underset{\approx}{\Sigma}_{\text{coh}} = \langle\Delta\underset{\sim}{a}\,\Delta\underset{\sim}{a}^\dagger\rangle_{\text{sym}} = \tfrac{1}{2}\underset{\approx}{1} , \quad (4.20a)$$

$$\langle\underset{\sim}{\alpha}_{r,\varphi}\rangle = \underset{\sim}{\mu}_\alpha, \quad \langle\Delta\underset{\sim}{\alpha}_{r,\varphi}\Delta\underset{\sim}{\alpha}_{r,\varphi}^\dagger\rangle_{\text{sym}} = \tfrac{1}{2}\underset{\approx}{C}_{2r,\varphi} , \quad (4.20b)$$

$$\langle\underset{\sim}{\mathscr{A}}\rangle = \underset{\sim}{\xi}, \quad \underset{\approx}{\Sigma}_{\text{coh}} = \langle\Delta\underset{\sim}{\mathscr{A}}\,\Delta\underset{\sim}{\mathscr{A}}^\dagger\rangle_{\text{sym}} = \tfrac{1}{2}\underset{\approx}{\Delta} = \tfrac{1}{2}[\underset{\approx}{1} - (\epsilon/\Omega)\underset{\approx}{\sigma}_2] \quad (4.20c)$$

[Eqs. (2.9), (2.13), (2.15), and (2.16); cf. Eqs. (4.9), (I.7.2), and (I.7.3)].

For a coherent state $|\underset{\sim}{\mu}\rangle_{\rm coh}$ the expectation value and variance of the free Hamiltonian (1.2) are

$$\langle H_0\rangle=\Omega\underset{\sim}{\mu}^\dagger\underset{\sim}{\lambda}^2\underset{\sim}{\mu}=\Omega\underset{\sim}{\xi}^\dagger\underset{\sim}{\xi}\ , \tag{4.21a}$$

$$\langle(\Delta H_0)^2\rangle=\Omega^2\underset{\sim}{\mu}^\dagger\underset{\sim}{\lambda}^4\underset{\sim}{\mu}=\Omega^2\underset{\sim}{\xi}^\dagger\underset{\sim}{\Delta}\underset{\sim}{\xi} \tag{4.21b}$$

[Eqs. (2.32) and (2.9)]. Equations (4.21) follow easily from the single-mode expectation value and variance of the number of quanta [Eq. (4.10)], but they can also be obtained from the following rules. Let $\underline{M}$ be any two-dimensional matrix

$$\underline{M}\equiv\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}. \tag{4.22}$$

One wants to evaluate the expectation value and variance of the quadratic form

$$\underset{\sim}{\mathbf{a}}^\dagger\underline{M}\underset{\sim}{\mathbf{a}}=M_{11}a_+^\dagger a_+ +M_{22}a_-^\dagger a_- +M_{12}a_+^\dagger a_-^\dagger +M_{21}a_+a_- +M_{22}\ . \tag{4.23}$$

It is an easy task to show that

$${}_{\rm coh}\langle\underset{\sim}{\mu}\,|\,\underset{\sim}{\mathbf{a}}^\dagger\underline{M}\underset{\sim}{\mathbf{a}}\,|\,\underset{\sim}{\mu}\rangle_{\rm coh}=\underset{\sim}{\mu}^\dagger\underline{M}\underset{\sim}{\mu}+M_{22}\ , \tag{4.24a}$$

$${}_{\rm coh}\langle\underset{\sim}{\mu}\,|\,[\Delta(\underset{\sim}{\mathbf{a}}^\dagger\underline{M}\underset{\sim}{\mathbf{a}})]^2\,|\,\underset{\sim}{\mu}\rangle_{\rm coh}=\underset{\sim}{\mu}^\dagger\underline{M}^2\underset{\sim}{\mu}+M_{12}M_{21}\ . \tag{4.24b}$$

By using Eq. (2.8), one can obtain Eqs. (4.21) directly from Eqs. (4.24).

The symmetrically ordered characteristic function for a two-mode coherent state $|\underset{\sim}{\mu}\rangle_{\rm coh}$ is obtained easily from the analogous single-mode quantity [Eq. (4.11)]:

$$\begin{aligned}{}_{\rm coh}\langle\underset{\sim}{\mu}\,|\,\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})\,|\,\underset{\sim}{\mu}\rangle_{\rm coh}&=e^{-\underset{\sim}{\nu}^\dagger\underset{\sim}{\nu}/2}\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu})\\&=e^{-\underset{\sim}{\eta}^\dagger\Delta\underset{\sim}{\eta}/2}\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta})\\&={}_{\rm coh}\langle\underset{\sim}{\mu}\,|\,\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})\,|\,\underset{\sim}{\mu}\rangle_{\rm coh}\end{aligned} \tag{4.25}$$

[Eqs. (2.33), (3.50), and (3.57)]. In Eq. (4.25) we write the characteristic function first in terms of the vector variable $\underset{\sim}{\nu}$ and then in terms of the vector variable $\underset{\sim}{\eta}$.

The two-mode coherent states are not orthonormal:

$${}_{\rm coh}\langle\underset{\sim}{\mu}'\,|\,\underset{\sim}{\mu}\rangle_{\rm coh}=\mathbf{D}(\tfrac{1}{2}\underset{\sim}{\mu}',\underset{\sim}{\mu})e^{-(\underset{\sim}{\mu}-\underset{\sim}{\mu}')^\dagger(\underset{\sim}{\mu}-\underset{\sim}{\mu}')/2} \tag{4.26}$$

[Eqs. (3.63b) and (4.25); cf. Eq. (4.13)]. They do, however, satisfy a completeness relation which follows trivially from the single-mode completeness relation (4.14):

$$1=\int\frac{d^4\mu}{\pi^2}\,|\,\underset{\sim}{\mu}\rangle_{\rm coh\,coh}\langle\underset{\sim}{\mu}\,|\ . \tag{4.27}$$

Hence the trace of an operator f is given by

$${\rm tr}f=\int\frac{d^4\mu}{\pi^2}\,{}_{\rm coh}\langle\underset{\sim}{\mu}\,|\,f\,|\,\underset{\sim}{\mu}\rangle_{\rm coh}\ , \tag{4.28}$$

and the trace of $\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})$ is

$${\rm tr}[\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})]=e^{-\underset{\sim}{\nu}^\dagger\underset{\sim}{\nu}/2}\int\frac{d^4\mu}{\pi^2}\mathbf{D}(\underset{\sim}{\mu},\underset{\sim}{\nu})=\pi^2\delta^4(\underset{\sim}{\nu}) \tag{4.29}$$

[Eqs. (4.25) and (3.59a)].

In the MP a two-mode coherent state evolves freely in the following way:

$$U_M(t)\,|\,\underset{\sim}{\mu}\rangle_{\rm coh}=|\,\underset{\sim}{\mu}e^{-i\epsilon t}\rangle_{\rm coh}=|\,\mu_+e^{-i\epsilon t},\mu_-e^{i\epsilon t}\rangle_{\rm coh} \tag{4.30}$$

[Eq. (3.60a)]. A rotation transforms a coherent state according to

$$R(\theta)\,|\,\underset{\sim}{\mu}\rangle_{\rm coh}=|\,e^{-i\theta\underset{\sim}{\sigma}_3}\underset{\sim}{\mu}\rangle_{\rm coh}=|\,\mu_+e^{-i\theta},\mu_-e^{-i\theta}\rangle_{\rm coh} \tag{4.31}$$

[Eq. (3.61a)]. Combining Eqs. (4.30) and (4.31) yields the SP free evolution

$$\begin{aligned}e^{-iH_0t}\,|\,\underset{\sim}{\mu}\rangle_{\rm coh}&=|\,e^{-i(\Omega\underset{\sim}{\sigma}_3+\epsilon\underset{\sim}{1})t}\underset{\sim}{\mu}\rangle_{\rm coh}\\&=|\,\mu_+e^{-i(\Omega+\epsilon)t},\mu_-e^{-i(\Omega-\epsilon)t}\rangle_{\rm coh}\ .\end{aligned} \tag{4.32}$$

In the degenerate limit a two-mode coherent state reduces to a single-mode coherent state:

$$|\,\underset{\sim}{\mu}\rangle_{\rm coh}\underset{p}{\rightarrow}|\,\mu\rangle_{\rm coh},\ \ \mu\equiv2^{-1/2}(\mu_++\mu_-) \tag{4.33}$$

[Eq. (I.8.22)].

B. Squeezed states

1. *Two-mode squeezed states*

The most important states in two-photon optics—the states produced by an ideal two-photon device (see Sec. IV A of I)—are the *two-mode squeezed states,* which can be defined by

$$\begin{aligned}|\,\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}&\equiv|\,\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r,\varphi)}\\&\equiv\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})S(r,\varphi)\,|\,0\rangle\end{aligned} \tag{4.34}$$

[Eq. (I.4.17)]. Using Eqs. (3.46) and (4.1), one can write $|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$ in the form

$$|\,\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}=\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})\,|\,0\rangle_{(r,\varphi)}=\mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha)\,|\,0\rangle_{(r,\varphi)}\ ; \tag{4.35}$$

hence, a two-mode squeezed state can be obtained by applying the "squeezed" displacement operator $\mathbf{D}(\underset{\sim}{\alpha},\underset{\sim}{\mu}_\alpha)=S(r,\varphi)\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}_\alpha)S^\dagger(r,\varphi)$ to the squeezed vacuum. Using Eq. (3.47) in the definition of $|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$, one shows that a two-mode squeezed state can be generated by applying $S(r,\varphi)$ to a two-mode coherent state:

$$\begin{aligned}|\,\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}&=S(r,\varphi)\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}_\alpha)\,|\,0\rangle\\&=S(r,\varphi)\,|\,\underset{\sim}{\mu}_\alpha\rangle_{\rm coh}\end{aligned} \tag{4.36}$$

[Eq. (I.4.15)]. Notice that $|\underset{\sim}{\mu}_\alpha\rangle_{(0,\varphi)}=|\underset{\sim}{\mu}_\alpha\rangle_{\rm coh}$. The unitary equivalence between squeezed states and coherent states is a powerful tool for generating properties of the two-mode squeezed states. For example, using Eqs. (4.18) and (4.36), one can tell immediately that $|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$ is an eigenstate of the squeezed annihilation operators (1.10):

$$\alpha_\pm(r,\varphi)\,|\,\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}=\mu_{\alpha_\pm}\,|\,\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}\ . \tag{4.37}$$

As another example, one can use Eqs. (4.19) and (4.36) to obtain an expansion of $|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$ in terms of the squeezed number eigenstates (4.4):

$$|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}=e^{-\boldsymbol{\mu}_\alpha^\dagger\boldsymbol{\mu}_\alpha/2}e^{\mu_{\alpha_+}a_+^\dagger}e^{\mu_{\alpha_-}a_-^\dagger}|0\rangle_{(r,\varphi)}$$
$$=e^{-\boldsymbol{\mu}_\alpha^\dagger\boldsymbol{\mu}_\alpha/2}\sum_{n_+,n_-}\frac{(\mu_{\alpha_+})^{n_+}(\mu_{\alpha_-})^{n_-}}{[(n_+!)(n_-!)]^{1/2}}|n_+,n_-\rangle_{(r,\varphi)}. \tag{4.38}$$

The expansion of $|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}$ in terms of the number eigenstates $|n_+,n_-\rangle$ is, in general, quite complicated, yet neither interesting nor enlightening. It does, however, have a simple form for the squeezed vacuum $|0\rangle_{(r,\varphi)}$—a form obtained by using the factorization (3.66):

$$S(r,\varphi)|0\rangle=(\cosh r)^{-1}e^{-a_+^\dagger a_-^\dagger e^{2i\varphi}\tanh r}|0\rangle$$
$$=(\cosh r)^{-1}\sum_{n=0}^{\infty}(-e^{2i\varphi}\tanh r)^n|n,n\rangle. \tag{4.39}$$

Thus the squeezed vacuum is a superposition of number eigenstates which have equal numbers of quanta in the a_+ mode and the a_- mode.

The two-mode squeezed state $|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}$ has TSQP noise [Eqs. (I.5.1)] and (I.5.6)]. Its nonzero first- and second-order moments are most easily obtained by noting that Eqs. (4.20a) and (4.36) imply

$$\langle\boldsymbol{\alpha}_{r,\varphi}\rangle=\boldsymbol{\mu}_\alpha,\quad \langle\Delta\boldsymbol{\alpha}_{r,\varphi}\Delta\boldsymbol{\alpha}_{r,\varphi}^\dagger\rangle_{\rm sym}=\tfrac{1}{2}\underline{1}. \tag{4.40}$$

One can obtain the moments of the creation and annihilation operators and the quadrature-phase amplitudes simply by making matrix transformations of Eqs. (4.40):

$$\langle\underline{\mathbf{a}}\rangle=\boldsymbol{\mu},\quad \overline{\underline{\Sigma}}_{r,\varphi}=\langle\Delta\underline{\mathbf{a}}\,\Delta\underline{\mathbf{a}}^\dagger\rangle_{\rm sym}=\tfrac{1}{2}\underline{C}_{-2r,\varphi}, \tag{4.41a}$$
$$\langle\boldsymbol{\mathscr{A}}\rangle=\boldsymbol{\xi},\quad \underline{\Sigma}_{r,\varphi}=\langle\Delta\boldsymbol{\mathscr{A}}\,\Delta\boldsymbol{\mathscr{A}}^\dagger\rangle_{\rm sym}=\tfrac{1}{2}\underline{A}\,\underline{\lambda}\,\underline{C}_{-2r,\varphi}\underline{\lambda}\,\underline{A}^\dagger \tag{4.41b}$$

[cf. Eqs. (4.20)]. Using Eqs. (A25), (A17)—(A19), and (A4)—(A6), one can expand $\overline{\underline{\Sigma}}_{r,\varphi}$ and $\underline{\Sigma}_{r,\varphi}$ as

$$\overline{\underline{\Sigma}}_{r,\varphi}=\tfrac{1}{2}\underline{1}\cosh(2r)$$
$$-\tfrac{1}{2}[\underline{\sigma}_1\cos(2\varphi)-\underline{\sigma}_2\sin(2\varphi)]\sinh(2r), \tag{4.42a}$$
$$\underline{\Sigma}_{r,\varphi}=\tfrac{1}{2}\underline{1}\cosh(2r)-\tfrac{1}{2}(1-\epsilon^2/\Omega^2)^{1/2}\underline{\sigma}_3\sinh(2r)\cos(2\varphi)$$
$$-\tfrac{1}{2}(1-\epsilon^2/\Omega^2)^{1/2}\underline{\sigma}_1\sinh(2r)\sin(2\varphi)$$
$$-\tfrac{1}{2}(\epsilon/\Omega)\underline{\sigma}_2\cosh(2r) \tag{4.42b}$$

[cf. Eqs. (I.7.8) and (I.7.9)].

The rules (4.24), which give the expectation value and variance of an arbitrary quadratic form $\underline{\mathbf{a}}^\dagger\underline{M}\,\underline{\mathbf{a}}$ with respect to a coherent state, can easily be generalized to squeezed states, once again by using Eq. (4.36):

$${}_{(r,\varphi)}\langle\boldsymbol{\mu}_\alpha|\underline{\mathbf{a}}^\dagger\underline{M}\,\underline{\mathbf{a}}|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}=\boldsymbol{\mu}_\alpha^\dagger\overline{\underline{M}}\boldsymbol{\mu}_\alpha+\overline{M}_{22}=\boldsymbol{\mu}^\dagger\underline{M}\boldsymbol{\mu}+\overline{M}_{22}, \tag{4.43a}$$

$${}_{(r,\varphi)}\langle\boldsymbol{\mu}_\alpha|[\Delta(\underline{\mathbf{a}}^\dagger\underline{M}\,\underline{\mathbf{a}})]^2|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}$$
$$=\boldsymbol{\mu}_\alpha^\dagger\overline{\underline{M}}^{\,2}\boldsymbol{\mu}_\alpha+\overline{M}_{12}\overline{M}_{21}$$
$$=\boldsymbol{\mu}^\dagger\underline{M}\,\underline{C}_{-2r,\varphi}\underline{M}\boldsymbol{\mu}+\overline{M}_{12}\overline{M}_{21}, \tag{4.43b}$$

$$\overline{\underline{M}}\equiv\underline{C}_{-r,\varphi}\underline{M}\,\underline{C}_{-r,\varphi}. \tag{4.44}$$

Applying Eqs. (4.43) to Eq. (2.8) yields the expectation value and variance of the free Hamiltonian with respect to $|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}$:

$$\langle H_0\rangle=\Omega\boldsymbol{\mu}^\dagger\underline{\lambda}^2\boldsymbol{\mu}+2\Omega\sinh^2 r=\Omega\boldsymbol{\xi}^\dagger\boldsymbol{\xi}+2\Omega\sinh^2 r, \tag{4.45a}$$
$$\langle(\Delta H_0)^2\rangle=\Omega^2\boldsymbol{\mu}^\dagger\underline{\lambda}^2\underline{C}_{-2r,\varphi}\underline{\lambda}^2\boldsymbol{\mu}+\Omega^2\sinh^2(2r)$$
$$=2\Omega^2\boldsymbol{\xi}^\dagger\underline{\Sigma}_{r,\varphi}\boldsymbol{\xi}+\Omega^2\sinh^2(2r) \tag{4.45b}$$

[Eq. (4.41b); cf. Eqs. (4.21)]. The reader should note the appearance of $\underline{\Sigma}_{r,\varphi}$ in Eq. (4.45b). Its presence there is no accident: for any state the highest-order term in the expression for $\langle(\Delta H_0)^2\rangle$ is quadratic in the mean quadrature-phase amplitudes; for any state with TSQP noise [Eq. (I.5.1)], it is easy to demonstrate that the highest-order term is given by $2\Omega^2\boldsymbol{\xi}^\dagger\underline{\Sigma}\boldsymbol{\xi}$, where $\boldsymbol{\xi}=\langle\boldsymbol{\mathscr{A}}\rangle$.

The symmetrically ordered characteristic function for a two-mode squeezed state can be obtained immediately from Eqs. (3.47), (4.25), and (4.36):

$${}_{(r,\varphi)}\langle\boldsymbol{\mu}_\alpha|\mathbf{D}(\underline{\mathbf{a}},\boldsymbol{\nu})|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}={}_{\rm coh}\langle\boldsymbol{\mu}_\alpha|\mathbf{D}(\underline{\mathbf{a}},\boldsymbol{\nu}_\alpha)|\boldsymbol{\mu}_\alpha\rangle_{\rm coh}$$
$$=e^{-\boldsymbol{\nu}_\alpha^\dagger\boldsymbol{\nu}_\alpha/2}\mathbf{D}(\boldsymbol{\mu}_\alpha,\boldsymbol{\nu}_\alpha). \tag{4.46}$$

This result can be transformed so that the characteristic function is written in terms of $\boldsymbol{\nu}=\underline{C}_{r,\varphi}^{-1}\boldsymbol{\nu}_\alpha$ or $\boldsymbol{\eta}=\underline{A}\,\underline{\lambda}^{-1}\underline{\sigma}_3\boldsymbol{\nu}$:

$${}_{(r,\varphi)}\langle\boldsymbol{\mu}_\alpha|\mathbf{D}(\underline{\mathbf{a}},\boldsymbol{\nu})|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}=e^{-\boldsymbol{\nu}^\dagger\underline{C}_{2r,\varphi}\boldsymbol{\nu}/2}\mathbf{D}(\boldsymbol{\mu},\boldsymbol{\nu})$$
$$=e^{-\boldsymbol{\eta}^\dagger\underline{\Sigma}_{r,\varphi}\boldsymbol{\eta}}\mathscr{D}(\boldsymbol{\xi},\boldsymbol{\eta})$$
$$={}_{(r,\varphi)}\langle\boldsymbol{\mu}_\alpha|\mathscr{D}(\boldsymbol{\mathscr{A}},\boldsymbol{\eta})|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)} \tag{4.47}$$

[Eqs. (3.50), (3.57), and (A24)]. Notice the presence of $\underline{\Sigma}_{r,\varphi}$ [Eq. (4.41b)] in the expression for the characteristic function. Its presence signals the fact that a two-mode squeezed state has Gaussian TSQP noise: the noise moments of arbitrary order are determined by the second-order noise moments contained in $\underline{\Sigma}_{r,\varphi}$.

The two-mode squeezed states are not orthonormal. The inner product of two squeezed states with the same r and φ is given by

$${}_{(r,\varphi)}\langle\boldsymbol{\mu}'_\alpha|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}={}_{\rm coh}\langle\boldsymbol{\mu}'_\alpha|\boldsymbol{\mu}_\alpha\rangle_{\rm coh}$$
$$=\mathbf{D}(\tfrac{1}{2}\boldsymbol{\mu}'_\alpha,\boldsymbol{\mu}_\alpha)e^{-(\boldsymbol{\mu}_\alpha-\boldsymbol{\mu}'_\alpha)^\dagger(\boldsymbol{\mu}_\alpha-\boldsymbol{\mu}'_\alpha)/2}$$
$$=\mathbf{D}(\tfrac{1}{2}\boldsymbol{\mu}',\boldsymbol{\mu})e^{-(\boldsymbol{\mu}-\boldsymbol{\mu}')^\dagger\underline{C}_{2r,\varphi}(\boldsymbol{\mu}-\boldsymbol{\mu}')/2} \tag{4.48}$$

[Eq. (4.26)]. For squeezed states with different r and/or φ, the inner product is considerably more complicated than Eq. (4.48); it is derived in Appendix C. The set of two-mode squeezed states with a particular r and φ does satisfy a completeness relation

$$1=\int\frac{d^4\mu_\alpha}{\pi^2}|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)\,(r,\varphi)}\langle\boldsymbol{\mu}_\alpha|, \tag{4.49}$$

which is just a unitary transformation of the completeness relation (4.27) for coherent states. Equation (4.49) allows us to write the trace of an operator f as an integral over squeezed states with the same r and φ:

$$\mathrm{tr} f = \int \frac{d^4\mu_\alpha}{\pi^2}\, {}_{(r,\varphi)}\langle \underset{\sim}{\mu}_\alpha | f | \underset{\sim}{\mu}_\alpha \rangle_{(r,\varphi)} \,. \quad (4.50)$$

The MP free evolution of $|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$ is given by

$$U_M(t)\,|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)} = |\underset{\sim}{\mu}_\alpha e^{-i\epsilon t}\rangle_{(r,\varphi)} = |\mu_{\alpha_+} e^{-i\epsilon t}, \mu_{\alpha_-} e^{i\epsilon t}\rangle_{(r,\varphi)} \quad (4.51)$$

[Eqs. (3.7) and (4.30)]; under a rotation $|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$ transforms according to

$$R(\theta)\,|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)} = |e^{-i\theta \underline{\sigma}_3}\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi-\theta)} = |\mu_{\alpha_+} e^{-i\theta}, \mu_{\alpha_-} e^{-i\theta}\rangle_{(r,\varphi-\theta)} \quad (4.52)$$

[Eqs. (3.16) and (4.31)]. Equations (4.51) and (4.52) together give the SP free evolution

$$e^{-iH_0 t}\,|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)} = |e^{-i(\Omega\underline{\sigma}_3+\epsilon\underline{1})t}\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi-\Omega t)} = |\mu_{\alpha_+} e^{-i(\Omega+\epsilon)t}, \mu_{\alpha_-} e^{-i(\Omega-\epsilon)t}\rangle_{(r,\varphi-\Omega t)} \quad (4.53)$$

[cf. Eq. (4.32)].

In a separate paper one of us (BLS) considers the wave functions for two-mode squeezed states in the usual coordinate and momentum representations.[20]

2. Degenerate squeezed states

In the degenerate limit a two-mode squeezed state $|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$ becomes a *degenerate squeezed state*[9,21,11] $|\mu_\alpha\rangle_{(r,\varphi)}$:

$$|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)} \underset{p}{\rightarrow} |\mu_\alpha\rangle_{(r,\varphi)} \equiv D(a,\mu) S_1(r,\varphi)\,|0\rangle \,, \quad (4.54a)$$

$$\mu_\alpha \equiv 2^{-1/2}(\mu_{\alpha_+} + \mu_{\alpha_-}) = \mu\cosh r + \mu^* e^{2i\varphi}\sinh r \quad (4.54b)$$

[Eqs. (I.8.23)]. Equation (3.27) can be used to show that $S_1(r,\varphi)$ transforms a single-mode coherent state into a degenerate squeezed state:

$$|\mu_\alpha\rangle_{(r,\varphi)} = S_1(r,\varphi) D(a,\mu_\alpha)\,|0\rangle = S_1(r,\varphi)\,|\mu_\alpha\rangle_{\mathrm{coh}} \,. \quad (4.55)$$

A degenerate squeezed state is an eigenstate of the squeezed annihilation operator (2.48):

$$\alpha(r,\varphi)\,|\mu_\alpha\rangle_{(r,\varphi)} = \mu_\alpha\,|\mu_\alpha\rangle_{(r,\varphi)} \quad (4.56)$$

[cf. Eq. (4.37)].

The properties of degenerate squeezed states can be derived in the same way as the properties of two-mode squeezed states. Here we content ourselves with providing a list of properties of the state $|\mu_\alpha\rangle_{(r,\varphi)}$. Above each equation in the list we give the equation number of the analogous two-mode property. All the results in the list, except Eqs. (4.62)–(4.65), can be found in Yuen's comprehensive paper[11] on "two-photon coherent states;" some of the results are also given in Refs. 9, 21, and 16.

Many of the properties are most conveniently stated in terms of the single-mode vector notation introduced in Table II. The list of properties is as follows:

Eq. (4.39):

$$S_1(r,\varphi)\,|0\rangle = (\cosh r)^{-1/2} e^{-(a^\dagger)^2 e^{2i\varphi}(\tanh r)/2}\,|0\rangle = (\cosh r)^{-1/2} \times \sum_{n=0}^{\infty} \frac{[(2n)!]^{1/2}}{n!}\left(-\tfrac{1}{2}e^{2i\varphi}\tanh r\right)^n |2n\rangle \,, \quad (4.57)$$

Eq. (4.40):

$$\langle \alpha(r,\varphi)\rangle = \mu_\alpha, \quad \langle \Delta\underset{\sim}{\underline{\alpha}}_{r,\varphi}\, \Delta\underset{\sim}{\underline{\alpha}}^\dagger_{r,\varphi}\rangle_{\mathrm{sym}} = \tfrac{1}{2}\underline{1} \,, \quad (4.58)$$

Eq. (4.41a):

$$\langle a\rangle = \mu, \quad \underline{\Sigma}_{r,\varphi} \equiv \langle \Delta\underset{\sim}{\underline{a}}\, \Delta\underset{\sim}{\underline{a}}^\dagger\rangle_{\mathrm{sym}} = \tfrac{1}{2}\underline{C}_{-2r,\varphi} \,, \quad (4.59a)$$

Eq. (4.41b):

$$\langle \underset{\sim}{x}\rangle = \underset{\sim}{\xi}, \quad \underline{\Sigma}_{r,\varphi} \equiv \langle \Delta\underset{\sim}{x}\, \Delta\underset{\sim}{x}^\dagger\rangle_{\mathrm{sym}} = \tfrac{1}{2}\underline{A}\,\underline{C}_{-2r,\varphi}\underline{A}^\dagger \,, \quad (4.59b)$$

Eqs. (I.7.8):

$$\langle |\Delta a|^2\rangle = \tfrac{1}{2}\cosh(2r) \,, \qquad \langle (\Delta a)^2\rangle = -\tfrac{1}{2}e^{2i\varphi}\sinh(2r) \,, \quad (4.60)$$

Eq. (4.42b):

$$\underline{\Sigma}_{r,\varphi} = \tfrac{1}{2}\underline{1}\cosh(2r) - \tfrac{1}{2}\underline{\sigma}_3 \sinh(2r)\cos(2\varphi) - \tfrac{1}{2}\underline{\sigma}_1 \sinh(2r)\sin(2\varphi) \,, \quad (4.61)$$

Eq. (4.22):

$$\underline{M} \equiv \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad M_{11} = M_{22} \,, \quad (4.62)$$

Eq. (4.44):

$$\overline{\underline{M}} \equiv \underline{C}_{-r,\varphi}\underline{M}\,\underline{C}_{-r,\varphi}, \quad \overline{M}_{11} = \overline{M}_{22} \,, \quad (4.63)$$

Eq. (4.23):

$$\tfrac{1}{2}\underset{\sim}{\underline{a}}^\dagger \underline{M}\underset{\sim}{\underline{a}} = M_{11}a^\dagger a + \tfrac{1}{2}M_{12}(a^\dagger)^2 + \tfrac{1}{2}M_{21}a^2 + \tfrac{1}{2}M_{11} \,, \quad (4.64)$$

Eq. (4.43a):

$${}_{(r,\varphi)}\langle \mu_\alpha |\, \tfrac{1}{2}\underset{\sim}{\underline{a}}^\dagger \underline{M}\underset{\sim}{\underline{a}}\, | \mu_\alpha\rangle_{(r,\varphi)} = \tfrac{1}{2}\underset{\sim}{\mu}^\dagger \underline{M}\,\underset{\sim}{\mu} + \tfrac{1}{2}\overline{M}_{11} \,, \quad (4.65a)$$

Eq. (4.43b):

$${}_{(r,\varphi)}\langle \mu_\alpha |\, [\Delta(\tfrac{1}{2}\underset{\sim}{\underline{a}}^\dagger \underline{M}\,\underset{\sim}{\underline{a}})]^2 \,| \mu_\alpha\rangle_{(r,\varphi)} = \tfrac{1}{2}\underset{\sim}{\mu}^\dagger \underline{M}\,\underline{C}_{-2r,\varphi}\underline{M}\,\underset{\sim}{\mu} + \tfrac{1}{2}\overline{M}_{12}\overline{M}_{21} \,, \quad (4.65b)$$

Eq. (4.45a):

$$\langle a^\dagger a\rangle = |\mu|^2 + \sinh^2 r = \tfrac{1}{2}\underset{\sim}{\xi}^\dagger \underset{\sim}{\xi} + \sinh^2 r \,, \quad (4.66a)$$

Eq. (4.45b):

$$\langle[\Delta(a^\dagger a)]^2\rangle=\tfrac{1}{2}\underset{\sim}{\mu}^\dagger \underset{\sim}{C}_{-2r,\varphi}\underset{\sim}{\mu}+\tfrac{1}{2}\sinh^2(2r)$$
$$=\underset{\sim}{\xi}^\dagger \underset{\approx}{\Sigma}_{r,\varphi}\underset{\sim}{\xi}+\tfrac{1}{2}\sinh^2(2r)\ , \tag{4.66b}$$

Eq. (4.46):

$${}_{(r,\varphi)}\langle\mu_\alpha|D(a,\nu)|\mu_\alpha\rangle_{(r,\varphi)}$$
$$={}_{\rm coh}\langle\mu_\alpha|D(a,\nu_\alpha)|\mu_\alpha\rangle_{\rm coh}$$
$$=e^{-|\nu_\alpha|^2/2}D(\mu_\alpha,\nu_\alpha)=e^{-\underset{\sim}{\nu}_\alpha^\dagger\underset{\sim}{\nu}_\alpha/4}D(\mu_\alpha,\nu_\alpha)\ , \tag{4.67}$$

Eq. (4.47):

$${}_{(r,\varphi)}\langle\mu_\alpha|D(a,\nu)|\mu_\alpha\rangle_{(r,\varphi)}=e^{-\underset{\sim}{\nu}^\dagger \underset{\sim}{C}_{2r,\varphi}\underset{\sim}{\nu}/4}D(\mu,\nu)$$
$$=e^{-\underset{\sim}{\eta}^\dagger \underset{\approx}{\Sigma}_{r,\varphi}\underset{\sim}{\eta}/2}\mathscr{D}(\underset{\sim}{\xi},\underset{\sim}{\eta})$$
$$={}_{(r,\varphi)}\langle\mu_\alpha|\mathscr{D}(\underset{\sim}{\mathbf{x}},\underset{\sim}{\eta})|\mu_\alpha\rangle_{(r,\varphi)}\ , \tag{4.68}$$

Eq. (4.48):

$${}_{(r,\varphi)}\langle\mu'_\alpha|\mu_\alpha\rangle_{(r,\varphi)}={}_{\rm coh}\langle\mu'_\alpha|\mu_\alpha\rangle_{\rm coh}$$
$$=D(\tfrac{1}{2}\mu'_\alpha,\mu_\alpha)e^{-|\mu_\alpha-\mu'_\alpha|^2/2}$$
$$=D(\tfrac{1}{2}\mu',\mu)e^{-(\underset{\sim}{\mu}-\underset{\sim}{\mu}')^\dagger \underset{\sim}{C}_{2r,\varphi}(\underset{\sim}{\mu}-\underset{\sim}{\mu}')/4}\ , \tag{4.69}$$

Eq. (4.49):

$$1=\int\frac{d^2\mu_\alpha}{\pi}|\mu_\alpha\rangle_{(r,\varphi)\,(r,\varphi)}\langle\mu_\alpha|\ . \tag{4.70}$$

V. CONCLUSION

This concluding section is a good place to recapitulate the key ideas behind papers I and II and to hint at what lies ahead. The goal of this series of papers is to develop a formalism suited to the analysis of two-photon devices. The crucial feature of a two-photon device is that its output consists of pairs of simultaneously emitted photons. Hence the starting point for our formalism is a pair of electromagnetic-field modes which are excited by emission of a pair of photons. The natural variables for describing the excitation of these two modes are the quadrature-phase amplitudes, and the natural quantum states are the two-mode squeezed states—the states generated by an ideal two-photon device. These basic building blocks were the focus of paper I, where our objective was to develop a physical understanding of the quadrature-phase amplitudes and the two-mode squeezed states. In the present paper we have described the mathematical structure of the formalism and developed techniques for manipulating its fundamental components. We introduced a vector notation which simplifies the mathematical description and makes it easy to learn and use the language of the quadrature-phase amplitudes. The vector notation also provides quick translation into the conventional language of creation and annihilation operators.

An important feature of the vector notation—built into it right at the start—is that it recognizes the quadrature-phase amplitudes as the fundamental variables and, hence, it naturally associates a_+ with $a_-^\dagger$. This feature has profound consequences for the operator orderings that are preferred in two-photon optics. One natural ordering for the quadrature-phase amplitudes and their Hermitian conjugates places $\alpha_1^\dagger$ and $\alpha_2^\dagger$ to the left of α_1 and α_2 (recall that $[\alpha_1,\alpha_2]=0$); this kind of ordering, which we call *quadrature-phase normal ordering,* is equivalent to normal ordering of the a_+ mode and antinormal ordering of the a_- mode. Another natural ordering, which we call *quadrature-phase antinormal ordering,* places α_1 and α_2 to the left of $\alpha_1^\dagger$ and $\alpha_2^\dagger$; it is equivalent to antinormal ordering of the a_+ mode and normal ordering of the a_- mode. Using the commutators (2.36) and (2.38), one can write the two-mode displacement operator in terms of these two orderings:

$$\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})=e^{-\underset{\sim}{\nu}^\dagger \underset{\sim}{\sigma}_3\underset{\sim}{\nu}/2}e^{\underset{\sim}{\mathbf{a}}^\dagger \underset{\sim}{\sigma}_3\underset{\sim}{\nu}}e^{-\underset{\sim}{\nu}^\dagger \underset{\sim}{\sigma}_3\underset{\sim}{\mathbf{a}}}$$
$$=e^{-\underset{\sim}{\eta}^\dagger \underset{\sim}{\Pi}\underset{\sim}{\eta}/2}e^{\underset{\sim}{\mathscr{A}}^\dagger\underset{\sim}{\eta}}e^{-\underset{\sim}{\eta}^\dagger\underset{\sim}{\mathscr{A}}}=\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})\ , \tag{5.1a}$$

$$\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\nu})=e^{\underset{\sim}{\nu}^\dagger \underset{\sim}{\sigma}_3\underset{\sim}{\nu}/2}e^{-\underset{\sim}{\nu}^\dagger \underset{\sim}{\sigma}_3\underset{\sim}{\mathbf{a}}}e^{\underset{\sim}{\mathbf{a}}^\dagger \underset{\sim}{\sigma}_3\underset{\sim}{\nu}}$$
$$=e^{\underset{\sim}{\eta}^\dagger \underset{\sim}{\Pi}\underset{\sim}{\eta}/2}e^{-\underset{\sim}{\eta}^\dagger\underset{\sim}{\mathscr{A}}}e^{\underset{\sim}{\mathscr{A}}^\dagger\underset{\sim}{\eta}}=\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta}) \tag{5.1b}$$

[cf. Eq. (3.38)]. The lesson is that the natural orderings for two-photon optics, which are based on the quadrature-phase amplitudes, require opposite ordering of the two modes.

These operator orderings will play a prominent role in paper III, where the focus will be on characteristic functions and their complex Fourier transforms, quasiprobability distributions. The expectation value $\langle\mathscr{D}(\underset{\sim}{\mathscr{A}},\underset{\sim}{\eta})\rangle\equiv\Phi(\underset{\sim}{\eta})$ is a characteristic function whose Taylor expansion yields the symmetrically ordered moments of α_1, α_2, $\alpha_1^\dagger$, and $\alpha_2^\dagger$; its complex Fourier transform is a two-mode version of the well-known Wigner distribution function.[22] The expectation values $\langle e^{\underset{\sim}{\mathscr{A}}^\dagger\underset{\sim}{\eta}}e^{-\underset{\sim}{\eta}^\dagger\underset{\sim}{\mathscr{A}}}\rangle=e^{\underset{\sim}{\eta}^\dagger\underset{\sim}{\Pi}\underset{\sim}{\eta}/2}\Phi(\underset{\sim}{\eta})$ and $\langle e^{-\underset{\sim}{\eta}^\dagger\underset{\sim}{\mathscr{A}}}e^{\underset{\sim}{\mathscr{A}}^\dagger\underset{\sim}{\eta}}\rangle=e^{-\underset{\sim}{\eta}^\dagger\underset{\sim}{\Pi}\underset{\sim}{\eta}/2}\Phi(\underset{\sim}{\eta})$ are characteristic functions whose Taylor expansions yield moments of α_1, α_2, $\alpha_1^\dagger$, and $\alpha_2^\dagger$ that are, respectively, quadrature-phase normally ordered and quadrature-phase antinormally ordered. The complex Fourier transforms of these characteristic functions are new two-photon quasiprobability distributions, whose definitions build into them the association of a_+ with $a_-^\dagger$ which is responsible for the squeezing of the output of two-photon devices. Paper III will generalize quadrature-phase normal and antinormal orderings to a continuum of intermediate orderings and will explore the characteristic functions and quasiprobability distributions that arise from these general operator orderings.

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APPENDIX A: PROPERTIES OF TRANSFORMATION MATRICES

1. $\underline{A} \equiv 2^{-1/2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$

$$\underline{A}^{-1} = \underline{A}^{\dagger} = 2^{-1/2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} , \tag{A1}$$

$$\det \underline{A} = i , \tag{A2}$$

$$\underline{A} = \tfrac{1}{2} e^{i\pi/4} [\underline{1} - i(\underline{\sigma}_1 - \underline{\sigma}_2 + \underline{\sigma}_3)] = 2^{-1/2} e^{i(\pi/4)(\underline{1} - \underline{\sigma}_1 + \underline{\sigma}_2 - \underline{\sigma}_3)} , \tag{A3}$$

$$\underline{A}\, \underline{\sigma}_1 \underline{A}^{\dagger} = \underline{\sigma}_3 , \tag{A4}$$

$$\underline{A}\, \underline{\sigma}_2 \underline{A}^{\dagger} = -\underline{\sigma}_1 , \tag{A5}$$

$$\underline{A}\, \underline{\sigma}_3 \underline{A}^{\dagger} = -\underline{\sigma}_2 , \tag{A6}$$

$$\underline{A}^{\dagger} \underline{\sigma}_1 \underline{A} = -\underline{\sigma}_2 , \tag{A7}$$

$$\underline{A}^{\dagger} \underline{\sigma}_2 \underline{A} = -\underline{\sigma}_3 , \tag{A8}$$

$$\underline{A}^{\dagger} \underline{\sigma}_3 \underline{A} = \underline{A}^{T} \underline{A} = \underline{\sigma}_1 \tag{A9}$$

(superscript T denotes a transpose).

2. $\underline{\lambda} \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$

$$\underline{\lambda}^{\dagger} = \underline{\lambda} , \tag{A10}$$

$$\det \underline{\lambda} = \lambda_+ \lambda_- = (1 - \epsilon^2/\Omega^2)^{1/2} , \tag{A11}$$

$$\underline{\lambda}^{-1} = (1 - \epsilon^2/\Omega^2)^{-1/2} \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{bmatrix} . \tag{A12}$$

Equation (A12) allows one to obtain properties of $\underline{\lambda}^{-1}$ from properties of $\underline{\lambda}$ by multiplying by factors of $(1-\epsilon^2/\Omega^2)^{-1/2}$ and reversing the sign of ϵ.

$$\underline{\lambda} = \tfrac{1}{2}(\lambda_+ + \lambda_-)\underline{1} + \tfrac{1}{2}(\lambda_+ - \lambda_-)\underline{\sigma}_3 , \tag{A13}$$

$$[\underline{\lambda}, \underline{\sigma}_1] = i(\lambda_+ - \lambda_-)\underline{\sigma}_2 , \tag{A14}$$

$$[\underline{\lambda}, \underline{\sigma}_2] = -i(\lambda_+ - \lambda_-)\underline{\sigma}_1 , \tag{A15}$$

$$[\underline{\lambda}, \underline{\sigma}_3] = \underline{0} , \tag{A16}$$

$$\underline{\lambda}^2 = \underline{1} + (\epsilon/\Omega)\underline{\sigma}_3 , \tag{A17}$$

$$\underline{\lambda}\, \underline{\sigma}_1 \underline{\lambda} = (1-\epsilon^2/\Omega^2)^{1/2} \underline{\sigma}_1 , \tag{A18}$$

$$\underline{\lambda}\, \underline{\sigma}_2 \underline{\lambda} = (1-\epsilon^2/\Omega^2)^{1/2} \underline{\sigma}_2 , \tag{A19}$$

$$\underline{\lambda}\, \underline{\sigma}_3 \underline{\lambda} = \underline{\sigma}_3 \underline{\lambda}^2 = (\epsilon/\Omega)\underline{1} + \underline{\sigma}_3 . \tag{A20}$$

3. $\underline{C}_{r,\varphi} \equiv \begin{bmatrix} \cosh r & e^{2i\varphi}\sinh r \\ e^{-2i\varphi}\sinh r & \cosh r \end{bmatrix}$

$$\underline{C}^{\dagger}_{r,\varphi} = \underline{C}_{r,\varphi} , \tag{A21}$$

$$\det \underline{C}_{r,\varphi} = 1 , \tag{A22}$$

$$\underline{C}^{\dagger}_{r,\varphi} \underline{\sigma}_3 \underline{C}_{r,\varphi} = \underline{C}_{r,\varphi} \underline{\sigma}_3 \underline{C}_{r,\varphi} = \underline{\sigma}_3 , \tag{A23}$$

$$\underline{C}^{-1}_{r,\varphi} = \underline{\sigma}_3 \underline{C}_{r,\varphi} \underline{\sigma}_3 = \underline{C}_{-r,\varphi} = \underline{C}_{r,\varphi+\pi/2} . \tag{A24}$$

The last two equalities in Eq. (A24) are the analogs of the last two equalities in Eq. (1.9).

$$\underline{C}_{r,\varphi} = \underline{1}\cosh r + [\underline{\sigma}_1 \cos(2\varphi) - \underline{\sigma}_2 \sin(2\varphi)]\sinh r = e^{r[\underline{\sigma}_1 \cos(2\varphi) - \underline{\sigma}_2 \sin(2\varphi)]} , \tag{A25}$$

$$\underline{C}_{r,\varphi} \underline{C}_{r',\varphi} = \underline{C}_{r+r',\varphi} , \tag{A26}$$

$$e^{i\theta\underline{\sigma}_3} \underline{C}_{r,\varphi} e^{-i\theta\underline{\sigma}_3} = \underline{C}_{r,\varphi+\theta} . \tag{A27}$$

APPENDIX B: FACTORIZATION AND PRODUCTS OF SQUEEZE OPERATORS

In this appendix we first factor the degenerate and the two-mode squeeze operators, $S_1(r,\varphi)$ and $S(r,\varphi)$ [Eqs. (2.46) and (1.8)], into products of exponentials; we then use these factored expressions to show explicitly that the product of two or more different squeeze operators (differing both in their magnitudes and directions of squeezing) is equal to the product of a rotation operator and a squeeze operator.

Factoring the two-mode squeeze operator turns out to be the same task as factoring the degenerate squeeze operator. In either case the problem reduces to factoring the expression

$$M(r,\varphi) \equiv \exp[r(Ae^{-2i\varphi} - A^{\dagger}e^{2i\varphi})] , \tag{B1}$$

where the operator A and its adjoint $A^{\dagger}$ obey the commutation relations

$$[A,A^{\dagger}] \equiv B = B^{\dagger}, \quad [A,B] = 2A, \quad [A^{\dagger},B] = -2A^{\dagger} . \tag{B2}$$

For the two-mode squeeze operator $M = S(r,\varphi)$, one sets $A = a_+ a_-$ and $B = 1 + a_+^{\dagger}a_+ + a_-^{\dagger}a_-$; for the degenerate squeeze operator $M = S_1(r,\varphi)$, one sets $A = \frac{1}{2}a^2$ and $B = \frac{1}{2} + a^{\dagger}a$. The commutation relations (B2) immediately imply the following useful relations (and their Hermitian conjugates):

$$e^{tA}A^{\dagger}e^{-tA} = A^{\dagger} + tB + t^2 A , \tag{B3a}$$

$$e^{tA}Be^{-tA} = B + 2tA , \tag{B3b}$$

$$e^{tB}Ae^{-tB} = e^{-2t}A , \tag{B3c}$$

where t is any complex number. The relations (B3) follow from the general relation

$$e^{tR}Se^{-tR} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\{R^n S\}, \qquad \{R^n S\} \equiv [R,\{R^{n-1}S\}], \quad \{R^0 S\} \equiv S \tag{B4}$$

[for a derivation see, for example, Eq. (8.105) of Ref. 23].

There are many approaches one can take to factor the operator $M(r,\varphi)$ into products of exponentials involving the operators A, $A^{\dagger}$, and B; here we briefly describe two approaches. In the "differential equation" approach[24] one multiplies the exponent of $M(r,\varphi)$ by a parameter t and sets the resulting expression equal to the desired product of exponentials involving the operators A, $A^\dagger$, and B, with coefficients in the exponents which are functions (to be determined) of the parameter t. One then takes the derivative with respect to t of both expressions, equates the two expressions by using the relations (B3) to put them in the same form, and solves the resulting coupled first-order differential equations (subject to boundary conditions at $t=0$) to find the coefficients as functions of t; the task is completed by setting the parameter t equal to one. This procedure is straightforward and, for this particular problem, not difficult.

A more elegant and more versatile approach,[25,15,17] however, makes use of the fact that the factored forms for $M(r,\varphi)$ are consequences only of the commutation relations (B2); this means that the operators A, $A^\dagger$, and B can be replaced by matrices which obey the commutation relations (B2), and the problem of factoring $M(r,\varphi)$ can be reduced to factoring an exponential of a sum of matrices. The problem becomes particularly simple if one uses for this purpose the Pauli spin matrices (2.7), which have the following properties:

$$\underline{\sigma}_i\underline{\sigma}_j=\delta_{ij}\underline{1}+i\epsilon_{ijk}\underline{\sigma}_k,\quad \underline{1}\equiv\begin{bmatrix}1&0\\0&1\end{bmatrix}, \tag{B5a}$$

$$e^{\gamma_j\underline{\sigma}_j}=\underline{1}\cosh\gamma+\underline{\sigma}_j\gamma_j\gamma^{-1}\sinh\gamma\ ,\quad \gamma\equiv(\gamma_1^2+\gamma_2^2+\gamma_3^2)^{1/2} \tag{B5b}$$

where $i,j,k=1,2$, or 3, γ_j are arbitrary complex numbers, and a summation over repeated indices is implied. For the matrices $\underline{\sigma}_+$ and $\underline{\sigma}_-$ defined by

$$\begin{aligned}\underline{\sigma}_+&\equiv\tfrac{1}{2}(\underline{\sigma}_1+i\underline{\sigma}_2)=\begin{bmatrix}0&1\\0&0\end{bmatrix},\\ \underline{\sigma}_-&\equiv\tfrac{1}{2}(\underline{\sigma}_1-i\underline{\sigma}_2)=\begin{bmatrix}0&0\\1&0\end{bmatrix}=\underline{\sigma}_+^\dagger\ ,\end{aligned} \tag{B6a}$$

the properties (B5) have the following important consequences:

$$\begin{aligned}&[\underline{\sigma}_+,\underline{\sigma}_-]=\underline{\sigma}_3,\quad [\underline{\sigma}_\pm,\underline{\sigma}_3]=\mp2\underline{\sigma}_\pm\ ,\\ &\underline{\sigma}_+^2=\underline{\sigma}_-^2=\underline{0},\quad \underline{\sigma}_\pm\underline{\sigma}_\mp=\tfrac{1}{2}(\underline{1}\pm\underline{\sigma}_3)\ ,\end{aligned} \tag{B6b}$$

$$\underline{\sigma}_\pm\underline{\sigma}_3=-\underline{\sigma}_3\underline{\sigma}_\pm=\mp\underline{\sigma}_\pm\ , \tag{B6c}$$

$$e^{t\underline{\sigma}_\pm}=\underline{1}+t\underline{\sigma}_\pm\ . \tag{B6d}$$

The commutation relations (B2) and (B6b) admit the formal correspondence

$$A\rightarrow-\underline{\sigma}_-,\quad A^\dagger\rightarrow\underline{\sigma}_+,\quad B\rightarrow\underline{\sigma}_3\ . \tag{B7}$$

This correspondence implies that

$$\begin{aligned}M(r,\varphi)&\rightarrow e^{-r(\underline{\sigma}_+e^{2i\varphi}+\underline{\sigma}_-e^{-2i\varphi})}\\ &=\underline{1}\cosh r-(\underline{\sigma}_+e^{2i\varphi}+\underline{\sigma}_-e^{-2i\varphi})\sinh r=\underline{C}_{r,\varphi}^{-1}\end{aligned} \tag{B8}$$

[Eqs. (B5b) and (B6); cf. Eq. (A25)]. The correspondence (B7) is not unique; we choose it because it leads to the correspondence (B8), which is the correspondence induced naturally by the definition (2.22) of $\underline{C}_{r,\varphi}$. Note that while the operator $M(r,\varphi)$ is unitary, the matrix $\underline{C}_{r,\varphi}^{-1}$ is not.

The matrix $\underline{C}_{r,\varphi}^{-1}$ is easily factored into exponentials of $\underline{\sigma}_+$, $\underline{\sigma}_-$, and $\underline{\sigma}_3$. For example, one such factorization is

$$\begin{aligned}&\underline{C}_{r,\varphi}^{-1}=e^{-\Gamma\underline{\sigma}_+}e^{-g\underline{\sigma}_3}e^{-\Gamma^*\underline{\sigma}_-}\ ,\\ &\Gamma\equiv e^{2i\varphi}\tanh r,\quad g\equiv\ln(\cosh r)\end{aligned} \tag{B9a}$$

[Eqs. (B5b), (B6), and (B8)]. This implies, through the correspondence (B7), that[14–17]

$$M(r,\varphi)=e^{-\Gamma A^\dagger}e^{-gB}e^{\Gamma^*A}\ . \tag{B9b}$$

The five other factored forms for $M(r,\varphi)$, which correspond to all other orderings of the operators A, $A^\dagger$, and B, are easily derived from Eq. (B9b) with the help of the following rules:

$$e^{s\underline{\sigma}_\pm}e^{t\underline{\sigma}_3}=e^{t\underline{\sigma}_3}e^{se^{\mp2t}\underline{\sigma}_\pm}\ , \tag{B10a}$$

$$\begin{aligned}e^{s\underline{\sigma}_+}e^{t\underline{\sigma}_-}&=e^{te^{-w}\underline{\sigma}_-}e^{se^{w}\underline{\sigma}_+}e^{w\underline{\sigma}_3}\\ &=e^{w\underline{\sigma}_3}e^{te^{w}\underline{\sigma}_-}e^{se^{-w}\underline{\sigma}_+}\ ,\quad w\equiv\ln(1+st)\end{aligned} \tag{B10b}$$

[Eqs. (B5b) and (B6)]. Equation (B10a) follows from the matrix version of Eq. (B3c), and Eq. (B10b) corresponds to the rule for interchanging the order of exponentials of A and $A^\dagger$:

$$\begin{aligned}e^{sA^\dagger}e^{-tA}&=e^{-te^{-w}A}e^{se^{w}A^\dagger}e^{wB}\\ &=e^{wB}e^{-te^{w}A}e^{se^{-w}A^\dagger}\ .\end{aligned} \tag{B11}$$

The final result is that the operator $M(r,\varphi)$ defined by Eqs. (B1) and (B2), i.e., the squeeze operator, has the following six equivalent factored forms:

$$\begin{aligned}M(r,\varphi)&=e^{-\Gamma A^\dagger}e^{-gB}e^{\Gamma^*A}=e^{-\Gamma A^\dagger}e^{\Gamma^*e^{2g}A}e^{-gB}=e^{-gB}e^{-\Gamma e^{2g}A^\dagger}e^{\Gamma^*A}\\ &=e^{\Gamma^*A}e^{-\Gamma e^{2g}A^\dagger}e^{gB}=e^{\Gamma^*A}e^{gB}e^{-\Gamma A^\dagger}=e^{gB}e^{\Gamma^*e^{2g}A}e^{-\Gamma A^\dagger}\ .\end{aligned} \tag{B12}$$

The above rules and factored forms for the degenerate and two-mode squeeze operators allow us to prove what one would expect intuitively: the product of two different squeeze operators is equal to the product of a rotation operator and a squeeze operator. Equation (B8) implies the correspondence

$$M^\dagger(r',\varphi')M(r,\varphi)\rightarrow\underline{C}_{r',\varphi'}\underline{C}_{r,\varphi}^{-1}=(\underline{C}_{r,\varphi}\underline{C}_{r',\varphi'}^{-1})^{-1}\ . \tag{B13}$$

The product $\underline{C}_{r,\varphi}\underline{C}_{r',\varphi'}^{-1}$ can be written as the product of another C matrix, $\underline{C}_{R,\Phi}$, and $e^{i\Theta\underline{\sigma}_3}$, giving

$$\underline{C}_{r,\varphi}\underline{C}^{-1}_{r',\varphi'}=\underline{C}_{R,\Phi}e^{i\Theta\underline{\sigma}_3}=e^{i\Theta\underline{\sigma}_3}\underline{C}_{R,\Phi-\Theta}=\begin{bmatrix} e^{i\Theta}\cosh R & e^{i(2\Phi-\Theta)}\sinh R \\ e^{-i(2\Phi-\Theta)}\sinh R & e^{-i\Theta}\cosh R \end{bmatrix}, \tag{B14}$$

where R, Φ, and Θ are defined explicitly by

$$e^{i\Theta}\cosh R=\cosh r\cosh r'-e^{2i(\varphi-\varphi')}\sinh r\sinh r', \tag{B15a}$$

$$e^{i(2\Phi-\Theta)}\sinh R=e^{2i\varphi}\sinh r\cosh r'-e^{2i\varphi'}\sinh r'\cosh r. \tag{B15b}$$

The correspondences (B7) and (B8) then imply that

$$M^\dagger(r',\varphi')M(r,\varphi)=e^{-i\Theta B}M(R,\Phi)=M(R,\Phi-\Theta)e^{-i\Theta B}. \tag{B16}$$

By using the fact that $\underline{C}^{-1}_{r,\varphi}=\underline{\sigma}_3\underline{C}_{r,\varphi}\underline{\sigma}_3$ [Eq. (A24)] and noting that $\underline{C}_{r,\varphi}\underline{\sigma}_3\underline{C}_{r',\varphi'}$ is the matrix of commutators defined by $\underline{\alpha}_{r,\varphi}$ and $\underline{\alpha}_{r',\varphi'}$, one can write the defining relation for R, Φ, and Θ [Eq. (B14)] in terms of the nonvanishing commutators of the squeezed annihilation operators:

$$\underline{C}_{R,\Phi}e^{i\Theta\underline{\sigma}_3}=\underline{C}_{r,\varphi}\underline{\sigma}_3\underline{C}_{r',\varphi'}\underline{\sigma}_3=\begin{bmatrix} [\alpha_+(r,\varphi),\alpha_+^\dagger(r',\varphi')] & -[\alpha_+(r,\varphi),\alpha_-(r',\varphi')] \\ [\alpha_-^\dagger(r,\varphi),\alpha_+^\dagger(r',\varphi')] & -[\alpha_-^\dagger(r,\varphi),\alpha_-(r',\varphi')] \end{bmatrix}. \tag{B17}$$

APPENDIX C: INNER PRODUCTS OF SQUEEZED STATES

In this appendix we derive the inner product of arbitrary squeezed states. The derivation is sketched for two-mode squeezed states. The same derivation works for degenerate squeezed states, so for them we merely list the main result.

One way to derive the general inner product is to begin with the matrix element $\langle 0|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$. This matrix element is easily obtained by using the number-eigenstate expansions of a two-mode coherent state [Eq. (4.19)] and the squeezed vacuum state [Eq. (4.39)]:

$$\langle 0|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}=\langle 0|\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})S(r,\varphi)|0\rangle$$
$$=(\cosh r)^{-1}e^{-\underset{\sim}{\mu}^\dagger\underset{\sim}{\mu}/2}e^{-\mu_+^*\mu_-^*e^{2i\varphi}\tanh r}. \tag{C1}$$

It is instructive to write the exponents in Eq. (C1) in terms of the vector notation:

$$\langle 0|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}=(\cosh r)^{-1}\exp[-\tfrac{1}{2}(\cosh r)^{-1}\underset{\sim}{\mu}^\dagger(\underline{C}_{r,\varphi}+i\underline{D}_{r,\varphi})\underset{\sim}{\mu}]$$
$$=(\cosh r)^{-1}\exp[-\tfrac{1}{2}(\cosh r)^{-1}\underset{\sim}{\mu}_\alpha^\dagger(\underline{C}^{-1}_{r,\varphi}+i\underline{D}_{r,\varphi})\underset{\sim}{\mu}_\alpha]. \tag{C2}$$

The matrix $\underline{D}_{r,\varphi}$ is defined by

$$\underline{D}_{r,\varphi}\equiv\begin{bmatrix} 0 & -ie^{2i\varphi}\sinh r \\ ie^{-2i\varphi}\sinh r & 0 \end{bmatrix}$$
$$=\underline{D}^\dagger_{r,\varphi}=-\underline{D}_{-r,\varphi}; \tag{C3}$$

it has the following easily verified properties:

$$\underline{D}_{r,\varphi}=-\frac{1}{2}\frac{\partial\underline{C}_{r,\varphi}}{\partial\varphi}=[\underline{\sigma}_1\sin(2\varphi)+\underline{\sigma}_2\cos(2\varphi)]\sinh r, \tag{C4a}$$

$$\underline{D}_{r,\varphi}e^{i\theta\underline{\sigma}_3}=e^{-i\theta\underline{\sigma}_3}\underline{D}_{r,\varphi}, \tag{C4b}$$

$$\underline{D}_{r,\varphi}=\underline{C}_{r',\varphi}\underline{D}_{r,\varphi}\underline{C}_{r',\varphi}. \tag{C4c}$$

Equation (C2) decomposes $\langle 0|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$ neatly into a magnitude times a phase factor.

Consider now the general inner product

$${}_{(r',\varphi')}\langle\underset{\sim}{\mu}'_\alpha|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$$
$$=\langle 0|S^\dagger(r',\varphi')\mathbf{D}^\dagger(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu}')\mathbf{D}(\underset{\sim}{\mathbf{a}},\underset{\sim}{\mu})S(r,\varphi)|0\rangle, \tag{C5}$$

where

$$\underset{\sim}{\mu}_\alpha=\underline{C}_{r,\varphi}\underset{\sim}{\mu},\quad \underset{\sim}{\mu}'_\alpha=\underline{C}_{r',\varphi'}\underset{\sim}{\mu}'. \tag{C6}$$

Equations (3.63b), (3.47), and (3.67) imply

$${}_{(r',\varphi')}\langle\underset{\sim}{\mu}'_\alpha|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$$
$$=e^{-i\Theta}\mathbf{D}(\tfrac{1}{2}\underset{\sim}{\mu}',\underset{\sim}{\mu})\langle 0|\underline{C}_{r,\varphi}(\underset{\sim}{\mu}-\underset{\sim}{\mu}')\rangle_{(R,\Phi)}, \tag{C7}$$

where R, Φ, and Θ are defined by Eq. (3.68) [see also Eqs. (B15)]:

$$\underline{C}_{R,\Phi}e^{i\Theta\underline{\sigma}_3}=\underline{C}_{r,\varphi}\underline{C}^{-1}_{r',\varphi'}. \tag{C8a}$$

The Hermitian conjugate of Eq. (C8a) is the useful relation

$$e^{-i\Theta\underline{\sigma}_3}\underline{C}_{R,\Phi}=\underline{C}^{-1}_{r',\varphi'}\underline{C}_{r,\varphi}. \tag{C8b}$$

Note that

$$\underline{D}_{R,\Phi}e^{i\Theta\underline{\sigma}_3}=e^{-i\Theta\underline{\sigma}_3}\underline{D}_{R,\Phi}=\underline{D}_{r,\varphi}\cosh r'-\underline{D}_{r',\varphi'}\cosh r \tag{C9}$$

[Eq. (B15b)]. The matrix element (C2) can now be used in Eq. (C7) to give the desired result,

$$ {}_{(r',\varphi')}\langle \underset{\sim}{\mu}'_\alpha | \underset{\sim}{\mu}_\alpha \rangle_{(r,\varphi)} = N\mathbf{D}(\tfrac{1}{2}\underset{\sim}{\mu}',\underset{\sim}{\mu})\exp[-\tfrac{1}{2}(\underset{\sim}{\mu}-\underset{\sim}{\mu}')^{\mathrm{T}}(\underline{F}+i\underline{G})(\underset{\sim}{\mu}-\underset{\sim}{\mu}')] , \tag{C10} $$

where

$$ N \equiv (e^{i\Theta}\cosh R)^{-1} = (\cosh r \cosh r' - e^{2i(\varphi-\varphi')}\sinh r \sinh r')^{-1} \tag{C11a} $$

[Eq. (B15a)], and

$$ \underline{F}\cosh R \equiv \underline{C}_{r,\varphi}\underline{C}^{-1}_{R,\Phi}\underline{C}_{r,\varphi} = \underline{C}_{r',\varphi'}e^{-i\Theta\underline{\sigma}_3}\underline{C}_{r,\varphi} = \underline{C}_{r,\varphi}e^{i\Theta\underline{\sigma}_3}\underline{C}_{r',\varphi'} , \tag{C11b} $$

$$ \underline{G}\cosh R \equiv \underline{C}_{r,\varphi}\underline{D}_{R,\Phi}\underline{C}_{r,\varphi} = \underline{C}_{r,\varphi}(\underline{D}_{r,\varphi}\cosh r' - \underline{D}_{r',\varphi'}\cosh r)e^{-i\Theta\underline{\sigma}_3}\underline{C}_{r,\varphi} = \underline{C}_{r',\varphi'}(\underline{D}_{r,\varphi}\cosh r' - \underline{D}_{r',\varphi'}\cosh r)e^{i\Theta\underline{\sigma}_3}\underline{C}_{r',\varphi'} \tag{C11c} $$

[$\underline{F}=\underline{F}^\dagger$, $\underline{G}=\underline{G}^\dagger$; Eqs. (C8) and (C9)]. Note the relation

$$ \frac{e^{-i\Theta\underline{\sigma}_3}}{\cosh R} = \begin{bmatrix} N & 0 \\ 0 & N^* \end{bmatrix} . \tag{C11d} $$

Equations (C11) allow one to write the inner product (C10) in terms of the primary variables r, r', φ, and φ'. Three special cases deserve attention: (i) if $\underset{\sim}{\mu}=\underset{\sim}{\mu}'$, then Eq. (C10) reduces to

$$ {}_{(r',\varphi')}\langle \underset{\sim}{\mu}'_\alpha | \underset{\sim}{\mu}_\alpha \rangle_{(r,\varphi)} = N = (\cosh r \cosh r' - e^{2i(\varphi-\varphi')}\sinh r \sinh r')^{-1} \quad (\underset{\sim}{\mu}=\underset{\sim}{\mu}') ; \tag{C12} $$

(ii) if $\varphi=\varphi'$, then $\Theta=0$, $\Phi=\varphi$, and $R=r-r'$, so

$$ {}_{(r',\varphi)}\langle \underset{\sim}{\mu}'_\alpha | \underset{\sim}{\mu}_\alpha \rangle_{(r,\varphi)} = [\cosh(r-r')]^{-1}\mathbf{D}(\tfrac{1}{2}\underset{\sim}{\mu}',\underset{\sim}{\mu})\exp\{-\tfrac{1}{2}[\cosh(r-r')]^{-1}(\underset{\sim}{\mu}-\underset{\sim}{\mu}')^\dagger(\underline{C}_{r+r',\varphi}+i\underline{D}_{r-r',\varphi})(\underset{\sim}{\mu}-\underset{\sim}{\mu}')\} \tag{C13} $$

[cf. Eq. (4.48)]; (iii) if $r'=0$, then $\Theta=0$, $\Phi=\varphi$, and $R=r$, and Eq. (C10) gives the inner product of the two-mode coherent state $|\underset{\sim}{\mu}'\rangle_{\mathrm{coh}}$ with the two-mode squeezed state $|\underset{\sim}{\mu}_\alpha\rangle_{(r,\varphi)}$,

$$ {}_{\mathrm{coh}}\langle \underset{\sim}{\mu}' | \underset{\sim}{\mu}_\alpha \rangle_{(r,\varphi)} = (\cosh r)^{-1}\mathbf{D}(\tfrac{1}{2}\underset{\sim}{\mu}',\underset{\sim}{\mu})\exp[-\tfrac{1}{2}(\cosh r)^{-1}(\underset{\sim}{\mu}-\underset{\sim}{\mu}')^\dagger(\underline{C}_{r,\varphi}+i\underline{D}_{r,\varphi})(\underset{\sim}{\mu}-\underset{\sim}{\mu}')] \tag{C14} $$

[cf. Eq. (C2)].

For degenerate squeezed states the general inner product is given by[11]

$$ {}_{(r',\varphi')}\langle \mu'_\alpha | \mu_\alpha \rangle_{(r,\varphi)} = N^{1/2}D(\tfrac{1}{2}\mu',\mu)\exp[-\tfrac{1}{4}(\underset{\sim}{\mu}-\underset{\sim}{\mu}')^\dagger(\underline{F}+i\underline{G})(\underset{\sim}{\mu}-\underset{\sim}{\mu}')] , \tag{C15} $$

where we use the single-mode vector notation introduced in Table II, with $\underset{\sim}{\mu}_\alpha=\underline{C}_{r,\varphi}\underset{\sim}{\mu}$ and $\underset{\sim}{\mu}'_\alpha=\underline{C}_{r',\varphi'}\underset{\sim}{\mu}'$.

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